Blending Knowledge and Technology to Construct Steiner Chains

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Abstract: *The aim of this paper is to show the crucial role of knowledge to tackle some geometric problems. We have chosen to show in detail how to tackle the problem of construction of Steiner chains, that is to say, closed chains of tangent circles also tangent to two given nested circles (to simplify the presentation and get understandable figures). Accordingly, a great part of the paper is a reminder of the necessary knowledge used for the techniques of construction of such chains: notion of harmonic division, notion of bundle of circles, notion of polar of a point with respect to a circle and above all the notion of inversion and the link between this transformation and the previous notions. The paper shows how these techniques can be used with some dynamic geometry software such as the New Cabri (containing the tool "macro" but not the tool "inversion"), Cabri 3D (containing the tool "inversion" with respect to spheres) or TI-NSpire (less efficient for complex figures because the tools "macro" and "inversion" are not available). Finally, we will perform very precise figures mostly with the New Cabri where Steiner chains associated to two given nested circles are generated and can be animated when their conditions of existence are satisfied.*

1. Link between bundle of circles at base points and bundle of circles at Poncelet points ([4])

1.1. Definitions (figure 1 left):

1.1.1. The bundle of circles at base points *A* and *A'* with $A \neq A'$, is the set of circles centered on the perpendicular bisector of [*AA'*] and passing through *A* and *A'*.

1.1.2. The bundle of circles at Poncelet points *A* and *A'* with $A \neq A'$ is the set of circles centered on (*AA'*) cutting this line in two points *C* and *D* , these points cutting the segment [*AA'*] under the same ratio or such that (*A*, *A'*, *C*, *D*) is an harmonic division (see §2) or such that *D* is the image of *C* with the inversion (see §4) of circle the circle with diameter [*AA'*]

1.2. Orthogonality of such bundles (Figure 1 left)

Let us prove the following theorem

Theorem: Each circle of the bundle of circles at base points *A* and *A'* is orthogonal to every circle of the bundle at Poncelet points *A* and *A'*.

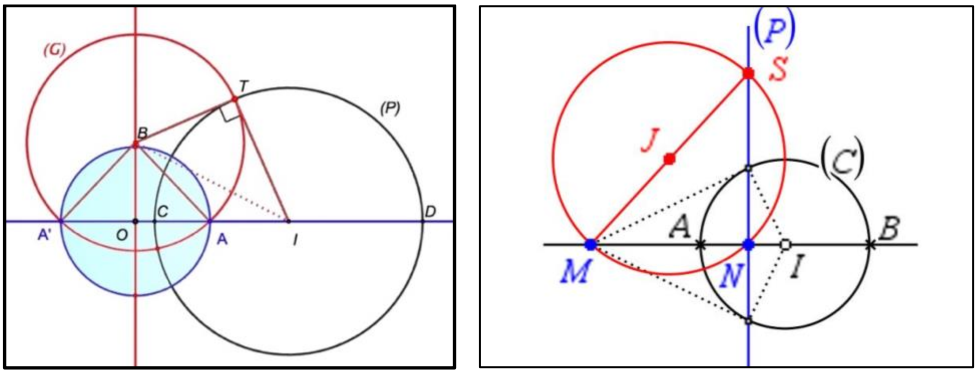

Figure 1: Orthogonality of bundles and construction of a polar

Let us give *A* and *A'* two points symmetric with respect to point *O* on the horizontal blue line. (*G*) is one of the circles of the bundle at base points *A* and *A'*, centered at *B* on the red line (perpendicular bisector of [*AA'*]). In the system of axes supported by the blue and the red lines, *A*(*a*,0), *A'*(-*a*,0) and *B*(0,β). Let us find all the circles centered on (*AA'*) (center called *I*), orthogonal to (*G*) and passing through a point *T* of (*G*). Let us consider one of these circles (*P*). The radius of (*G*) is equal to *BA'* or *BT* or *BA*: we know that $BA^2 = a^2 + \beta^2$. Let us evaluate BI^2 in two different ways:
$BI^2 = BT^2 + TI^2 = BT^2 + CI^2 = BA^2 + CI^2 = a^2 + \beta^2 + CI^2$ and also $BI^2 = BO^2 + OI^2 = \beta^2 + OI^2$.
From these we get $a^2 = OI^2 - CI^2$ which is the power of *O* for (*P*) ([9]) that can also be written *OC.OD*. So, *D* is the image of *C* by the inversion centered at *O* which circle is the blue circle (center *O* and radius *OA* = *a*). Another consequence is that (*A'*, *A*, *C*, *D*) is an harmonic division and circle (*P*) is one of the circles of the bundle at Poncelet points *A* and *A'*.
When *T* is moving on (*G*), we get all the centers *I* on the blue line and therefore all the circles of the Poncelet bundle. That completes the proof.

2. Harmonic division

Definition: if *A*, *B*, *C* and *D* are four points belonging to the same line, (*A*, *B*, *C*, *D*) is an harmonic division if *C* and *D* cut segment [*AB*] under the same ratio. In particular: $\frac{\overline{CA}}{\overline{CB}} = -\frac{\overline{DA}}{\overline{DB}}$

Property 1: If *a*, *b*, *c* and *d* are the abscissa of *A*, *B*, *C* and *D* with respect to an axis supported by (*AB*) and where the origin is located at the midpoint of [*AB*], the previous condition can be written: $c.d = a^2$ or $\overline{OC}.\overline{OD} = OA^2 = a^2$ where *D* (respectively *C*) is the image of *C* (respectively *D*) by the **inversion centered at *O* which ratio is a^2**.

Property 2: If the origin of the axis is located at *A*, this relation becomes:

$$\frac{2}{b} = \frac{1}{c} + \frac{1}{d} \text{ or } \frac{2}{\overline{OB}} = \frac{1}{\overline{OC}} + \frac{1}{\overline{OD}}$$

3. Polar of a point for a circle (Figure 1 right) ([2])

3.1. Définition: The polar of point *M* for a circle (*C*) (center *I* and radius *r*), is the set of points *S* such that the circle of diameter [*MS*] is orthogonal to (*C*).

3.2. How to characterize the polar of a point for a circle

3.2.1 This polar is not an empty set because point *N* inverse of *M* by the inversion centered at *I* and which circle is (*C*) belongs to this polar. In fact, $IM.IN = r^2$ means that the power ([9]) of *I* for this circle is equal the square of the radius of (*C*), which means also that the circle with diameter [*MN*] is orthogonal to circle (*C*). With respect to a system of axes with origin *I*, supported by (*IA*) and its perpendicular at *I*, if $M(x_M,0)$ then, $N(\frac{r^2}{x_M},0)$.

3.2.2. Let us find now the set of points $S\ (x_S, y_S)$ such that the circle of diameter [*MS*] is orthogonal to (*C*). If *J* is the midpoint of [*SM*], which means the center of such a circle, the necessary and sufficient condition for this property is $IJ^2 = JM^2 + r^2$.
With $J(\frac{x_M+x_S}{2}, \frac{y_S}{2})$, $IJ^2 = (\frac{x_M+x_S}{2})^2 + (\frac{y_S}{2})^2$ and as $JM^2 = (\frac{x_M-x_S}{2})^2 + (\frac{y_S}{2})^2$, the necessary and sufficient condition can be written: $x_M.x_S = r^2$ or $x_S = \frac{r^2}{x_M}$ which is the equation of the line passing through *N* (inverse of *M* with respect to (*C*)) and perpendicular to (*IM*). So:

Theorem: The polar of a point *M* of a circle (*C*) centered at *I* is the perpendicular at *N* to (*IN*) where *N* is the image of *M* by the inversion of circle (*C*).

Remark: if *M'* is a point of the polar of *M* for (*C*) and if (*MM'*) cuts circle (*C*) at points *U* and *V*, it is equivalent to say that (*M*, *M'*, *U*, *V*) is an harmonic division (Figure 2 left).

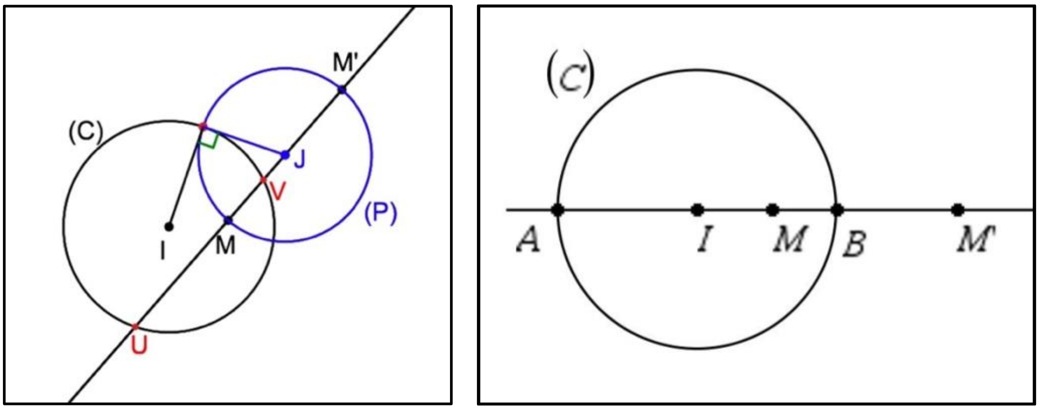

Figure 2: Property of polars, definition of inversion and construction of polar

4. Inversion ([7], [8]) (Figure 2 right)

4.1. Definition: If (*C*) is a circle centered at *I* with radius *r*, we call inversion with respect to this circle, the plane transformation mapping each point *M* different from *I* onto a point *M'* belonging to the ray [*IM*) such as $IM.IM' = r^2$ and letting *I* be invariant.

As *I* is the midpoint of [*AB*], the equality of definition can be interpreted differently:
(*A*, *B*, *M*, *M'*) is an harmonic division or *M'* is the orthogonal projection of *M* on its polar for (*C*).

4.2. Images of lines (Figure 3)
Let us consider the inversion of circle (*C*) (center *I*, radius *r*) and a line (*D*).
The image of a line passing (*D*) through *I* is globally invariant by such an inversion.
The image by such an inversion of a line (*D*) which does not contain *I* is a circle containing *I*:
We give a proof for the two possible cases. The first case is when the line cuts circle (*C*) and the second one is when it does not. In these two cases, we call *m* the orthogonal projection of *I* on (*D*) and *m'* its image by the given inversion.

First case "*(D)* cuts circle *(C)*" (Figure 3 left): point *m'* belongs to the image of *(D)*. For each other point *n* of *(D)* (different from *m*) with image *n'*, we have: *Im.Im'* = *In.In'* or $\frac{Im}{In} = \frac{In'}{Im'}$. Triangles *Imn* and *In'm'* have a common angle; the previous equality establishes that these triangles are similar. Or *Imn* is a right-angle triangle at *m*, so *In'm'* is a right-angle triangle at *n'*. therefore, *n'* belongs to the circle of diameter [*Im'*]. Conducting the same reasoning would prove that each point of the circle is the image of a point of *(D)* by this inversion.

Second case "*(D)* does not cut circle *(C)*" (Figure 3 right): same reasoning.

Theorem: The image of a line (*D*) by an inversion (when (*D*) does not contain the center *I* of the inversion) is a circle of diameter [*Im'*] where *m* is the orthogonal projection of *I* on *(D)* and *m'* is the image of *m* by this inversion.

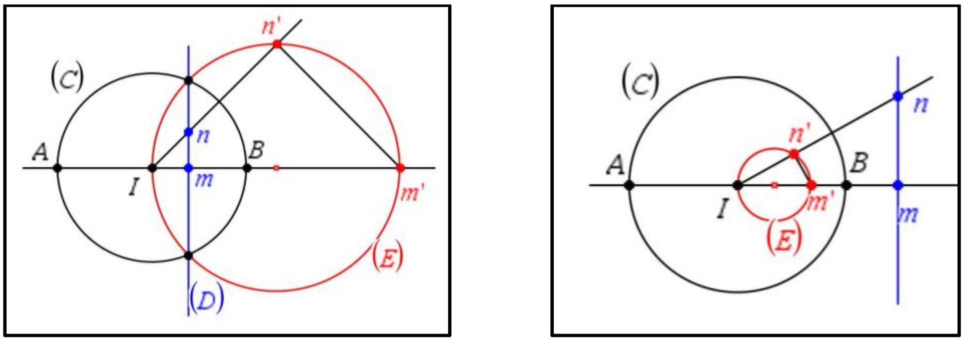

Figure 3: Images of lines by an inversion

4.3. Images of circles

First case "Image of a circle passing through the center of the inversion": as an inversion is an involution, the image of a circle is a line (use the previous result).

Second case "Image of a circle which does not contain the center of the inversion" (Figure 4): Let us consider here the inversion **I** centered at *O* and whose circle is (*C*). We try to find the image of circle (*G*) whose center is *I* and diameter [*MN*] and for which *O*, *M* and *N* are collinear. *M'* and *N'* are the images of *M* and *N* by the inversion **I**. Point *P* is a point generating circle (*G*) ; its image *P'* generates the image (*G')* of (*G)* by **I**. *Q* is the second intersection point between (*OP*) and circle (*G*): this point generates (*G*) exactly when *P* generates (*G*) and therefore *Q'* its image by **I** generates (*G'*) exactly when *P* generates (*G*).

If *r* is the radius of circle (*C*), thanks to the definition of the inversion, we obtain:

$OQ' = \frac{r^2}{OQ}$ and $OM' = \frac{r^2}{OM}$. Let us evaluate now the ratio $\frac{OQ'}{OP}$:

$\frac{OQ'}{OP} = \frac{\frac{r^2}{OQ}}{OP} = \frac{r^2}{OP.OQ}$ or $OP.OQ = OM.ON$ (power of point *O* for circle (*G*)).

Therefore, $\frac{OQ'}{OP} = \frac{r^2}{OM.ON} = \frac{\frac{r^2}{OM}}{ON} = \frac{OM'}{ON}$.

The equality $\frac{OQ'}{OP} = \frac{OM'}{ON}$ reflects the fact that *Q'* is the image of *P* by the dilation centered at *O* and whose scale is $\frac{OM'}{ON}$. So, (*G'*) is a circle of diameter [*M'N'*].

Remark: be careful! The center of (*G'*) is not the image of *I* by the inversion **I.** It is the image *J* of *I* by the previous dilation. We will show a simple way to construct this center in using the composition of two inversions (it will allow constructions of centers of a Steiner chain easyly: see 5.5.).

Position of *J*: *J* is the midpoint of [*M'N'*], so, $OJ = \frac{OM'+ON'}{2} = \frac{\frac{r^2}{OM}+\frac{r^2}{ON}}{2} = \frac{r^2}{2}.\left(\frac{1}{OM} + \frac{1}{ON}\right)$.

If *H* is the harmonic conjugate of *O* with respect to *M* and *N*, we can write the known equality $\frac{2}{OH} = \frac{1}{OM} + \frac{1}{ON}$. Let us note that *H* can be considered as the image of *O* by the inversion of circle (*G*) or the orthogonal projection of *I* on the polar of *O* for this circle.

Eventually: $OJ = \frac{r^2}{OH}$ which means that *J* is the image of *H* by the inversion **I**.

Remark: this proof can be generalized to all circles of the plane which do not contain *O* in replacing distances by algebraic distances (used in the definition of harmonic division in §2).

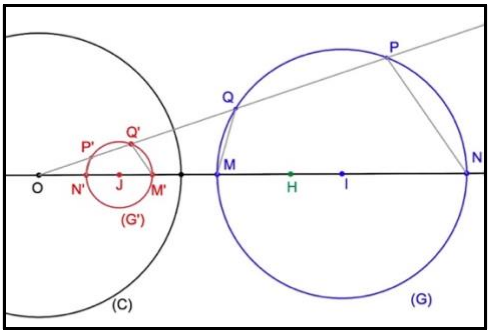

Figure 4: Image of a circle by an inversion

Theorem: The image of a circle (*G*) centered at *I* by an inversion I (when this circle does not contain the center *O* of the inversion) is another circle (*G'*) centered at *J* where *J* is the image by I of *H*, *H* being the image of *O* by the inversion of circle (*G*).

4.4. Angular property of the tangent lines at a point of two intersecting curves

4.4.1. An angular property of an inscribed quadrilateral (Figure 5 left)

AA'B'B is a quadrilateral inscribed in a circle centered at *O*. Suppose that (*AA'*) and (*BB'*) intersect at *M* and (*AB*) and (*A'B'*) intersect at *N*. We know that ∠ *MB'N* is equal to ∠*MAB*. If (*MI*) is the angle bisector of ∠*AMB* cutting (*A'B'*) at *J*, then ∠*BMI* = ∠*IMA*.
Let us consider the two similar triangles *MIA* and *MJB'*.
In triangle *MJB'*, ∠*M* + ∠ *B'* = ∠ *MJN* (green); in triangle *IMA*, ∠ *M* + ∠ *A* = ∠ *NIJ* (magenta).
As triangles *MIA* and *MJB'* are similar, these sums are equal and so:
∠ *IJN* (green) = ∠ *NIJ* (magenta), which means that triangle *NIJ* is isosceles (base [*IJ]*), and also that lines (*AB*) and (*A'B'*) are symmetric with respect to the perpendicular bisector of [*IJ*]. Eventually:

Theorem: If *AA'B'B* is a quadrilateral inscribed in a circle, if the angle bisector of ∠*BMA* cuts (*AB*) at *I* and (*A'B'*) at *J*, therefore (*AB*) and (*A'B'*) are symmetric with respect to the perpendicular bisector of [*IJ*].

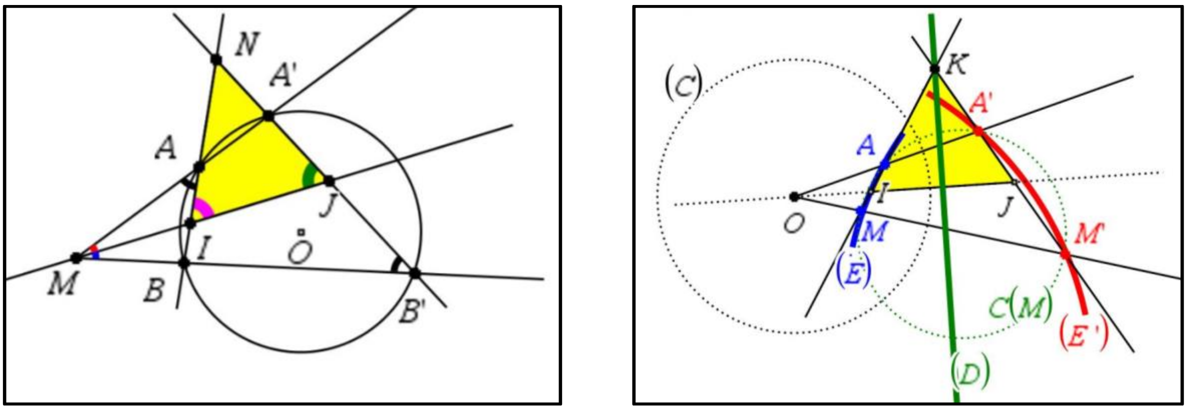

Figure 5: Angular property of inversions

4.4.2. A consequence about the angles of tangent lines to two secant curves and to their images (Figure 5 right)

Let us give the inversion of circle (*C*) centered at *O*; (*E'*) is the image of the blue curve (*E*) by this inversion; *A'* is the image on (*E'*) of point *A* belonging to (*E*) and *M'* is the image on (*E'*) of another point *M* belonging to (*E*).

By definition of the inversion, *OA.OA' = OM.OM'* and therefore points *A*, *A'*, *M'* and *M* belong to the same circle *C(M)*. Thanks to the previous result, triangle *KIJ* is isosceles (base [*IJ*]) and then lines (*AM*) and (*A'M'*) are symmetric with respect to the perpendicular bisector (*D*) of [*IJ*].
If *M* approaches *A* along (*E*), the limit of (*AM*) is the tangent (*T*) to (*C*) at *A* and then, as *M'* approaches *A'* along (*E'*), the limit of (*A'M'*) is the tangent (*T'*) to (*C'*) at *A'*. In those conditions as *I* approaches *A* and *J* approaches *A'*, the tangent lines (*T*) and (*T'*) are symmetric with respect to the perpendicular bisector of [*AA'*] (limit of (*D*) when *M* approaches *A*).

Theorem: If the curve (*E*) has (*E'*) for image by an inversion, if *A* is a point of (*E*) and *A'* its image by this inversion, therefore, the tangent lines at *A* and *A'* respectively to (*E*) and (*E'*) are symmetric with respect to the perpendicular bisector of [*AA'*].

So, we can deduce from this theorem

Corollary: If two curves intersect at A and if α is the angle of the tangent lines to these curves at this point, the angle between the tangent lines to the curves images of the previous ones by an inversion, at A' image of A by this inversion, is equal to -α.

Remark: if the tangent lines to the given intersecting curves are perpendicular, the tangent lines to the curves images of these curves are also perpendicular

4.5. Metric properties

Before the second part of this paper, let us recall how distances and radii of circles are modified by an inversion.
Let us give **I** the inversion of circle (*C*) (center *I*, radius *r*):

If [*A'B'*] is the image of [*AB*] by **I**: $A'B' = \frac{r^2.AB}{IA.IB}$ (a proof uses the fact that triangle *IAB* is similar to triangle *IB'A'*)

If a circle C_1 of radius r_1 has for image a circle C'_1 of radius r'_1, $r'_1 = \frac{r^2.r_1}{Im.Im'}$ where *m* and *m'* are the intersection points between the circle C_1 and the line joining *I* to the center of C_1 (with the notations of Figure 3 right).
Eventually, in a "certain way", if two circles are transformed onto two other circles by an inversion, the ratio between the radii of the image circles is "independent" of the scale of the inversion.

5. Images of bundles of circles by an inversion

5.1. Image of a bundle at base points

Figure 6 left: let us consider a bundle of circles at base points *A'* and *A*. *D* is the perpendicular bisector of [*AA'*] passing through the midpoint o_1 of this segment. Every circle C_n of this bundle is centered on *D* at o_n (in Figure 6 left, we have displayed C_1 centered at o_1, C_2 centered at o_2 and C_3 centered at o_3). Let us consider now an inversion **I** centered at *A* and which circle *C* passes through a point *v* of (*AA'*).
Remark: all the circles of the bundle pass through the center of the inversion.
As a consequence, the images of the circles of this bundle are lines. And, as every circle of the bundle passes through a common point A', all these lines pass through the image of *A'* by **I**. In Figure 6 (left and right), we can see that the images D_1, D_2 and D_3 of C_1, C_2 and C_3 pass really through a common point. This point belongs to the polar of *A'* with respect to *C* (circle of the inversion). The direction of each of these lines is respectively the direction of the perpendicular to the line joining *A* to each center of the given circles of the bundle.

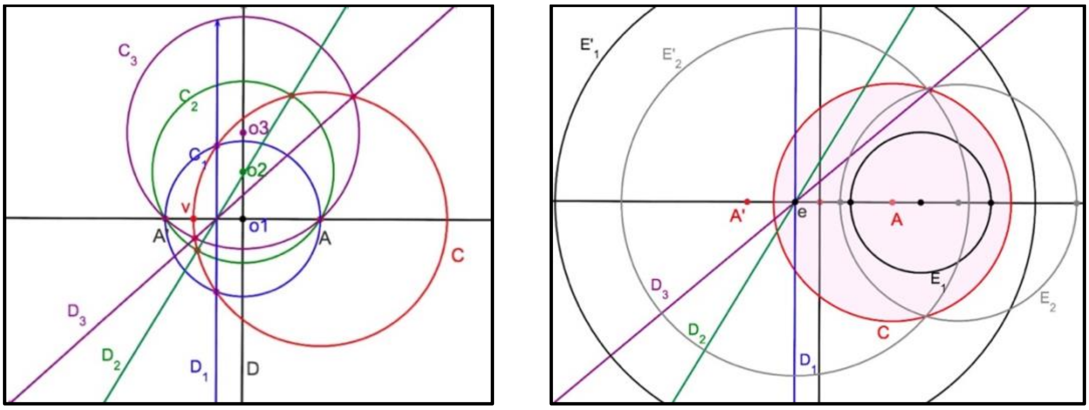

Figure 6: Image of a bundle of circles at base points

Let us consider now the bundle of Poncelet points *A'* and *A* whose circles are E_n (in Figure 6 right, only two circles are displayed: E_1 and E_2). We know that these circles are orthogonal to all circles of the previous bundle based at *A'* and *A*. As the inversion keeps the orthogonality of tangent lines, the images E'_n by I of the circles E_n are orthogonal to all the images of the circles C_n which means, to all lines D_n. Eventually, as all lines D_n pass through a common point *e*, E'_n is a circle centered at *e*.

Theorem: If F is a bundle at Poncelet points *A'* and *A*, the image of this bundle by any inversion centered at *A* is a bundle of concentric circles centered at *e* image of *A'* by this inversion.

5.2. Steiner chains of two concentric circles (existence conditions)

From now, our aim is to construct chains of tangent circles but also tangent to two given nested circles. We start by the case when the two given circles are concentric and necessarily the circles of the chain have the same radius.

In Figure 7, (*I*) and (*E*) are two concentric circles centered at *O* with radius respectively *r* and *k.r* where $k > 1$. T_1 is a point on (*E*) and (C_1) is the circle tangent to (*I*) and (*E*) respectively at t_1 and T_1, centered at O_1.

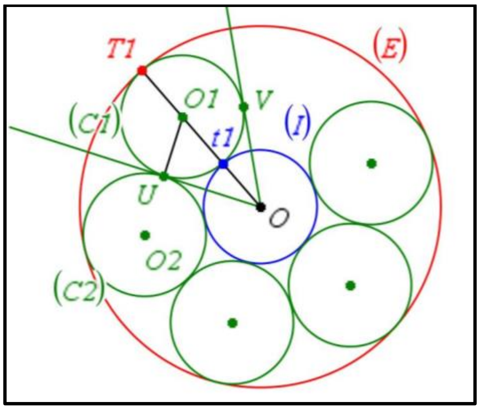

Figure 7: Chain included between two concentric circles

The radius of (*C1*) is equal to $(k-1).\frac{r}{2}$. *U* and *V* are the two contact points of the two tangent lines to (C_1) constructed from *O*.

The angle $\angle VOU$ is the angle of the rotation centered at *O* which allows by iteration of the images of (C_1), a chain of circles (C_n) isometric to (C_1), respecting our initial constraints.

Let's see now under what conditions for k and the number n of circles of the chain, this chain is closed, which means that the circle (C_n) touches for the first time (C_1) and is tangent to this circle at V.

If $\alpha = \angle UOV$ which is the double of $\angle UOO_1$, we have:

$sin(\frac{\alpha}{2}) = \frac{UO1}{OO1} = \frac{(k-1).\frac{r}{2}}{r+(k-1).\frac{r}{2}} = \frac{k-1}{k+1}$ and then: $\alpha = 2.sin^{-1}(\frac{k-1}{k+1})$.

So, the condition expressing that n circles exactly can be inscribed in one turn is:

$n.\alpha = 2.\pi$ or: $\boldsymbol{n.sin^{-1}\left(\frac{k-1}{k+1}\right) = \pi}$ **(Eq1)**

Similarly, the condition expressing that n circles exactly can be inscribed in m turns is:

$n.\alpha = 2.m.\pi$ or: $\boldsymbol{n.sin^{-1}\left(\frac{k-1}{k+1}\right) = m.\pi.}$ **(Eq2)**

Particular cases:

Chain of 2 circles in 1 turn: $n = 2$ in **Eq1** returns $k = 1$, which means that the two given circles are the same circle. This case is excluded.

Chain of 3 circles in 1 turn: $n = 3$ in **Eq1** returns $k = 7 + 4.\sqrt{3}$ (Checked in Figure 8 left)

Chain of 4 circles in 1 turn: $n = 4$ in **Eq1** returns $k = 3 + 2.\sqrt{2}$ (Checked in Figure 8 center)

Chain of 5 circles in 1 turn: $n = 5$ in **Eq1** returns $k = \frac{4+2.\sqrt{2.(5-\sqrt{5})}}{4-2.\sqrt{2.(5-\sqrt{5})}}$ (Checked in Figure 8 right).

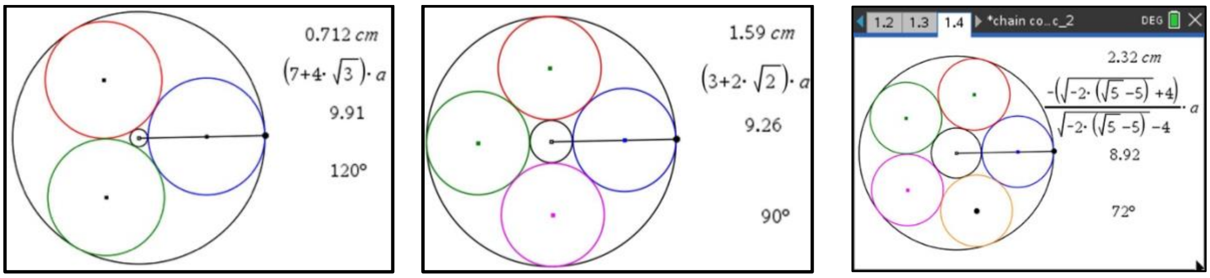

Figure 8: Closed chains in concentric circles

Chain of 6 circles in 1 turn: $n = 6$ in **Eq1** returns $k = 3$ (Figure 9 left)

General case for n circles in m turns: $k = \frac{1+\sin(\frac{m\pi}{n})}{1-\sin(\frac{m\pi}{n})}$ (particular case $n = 7$ and $m = 2$: Figure 9 right)

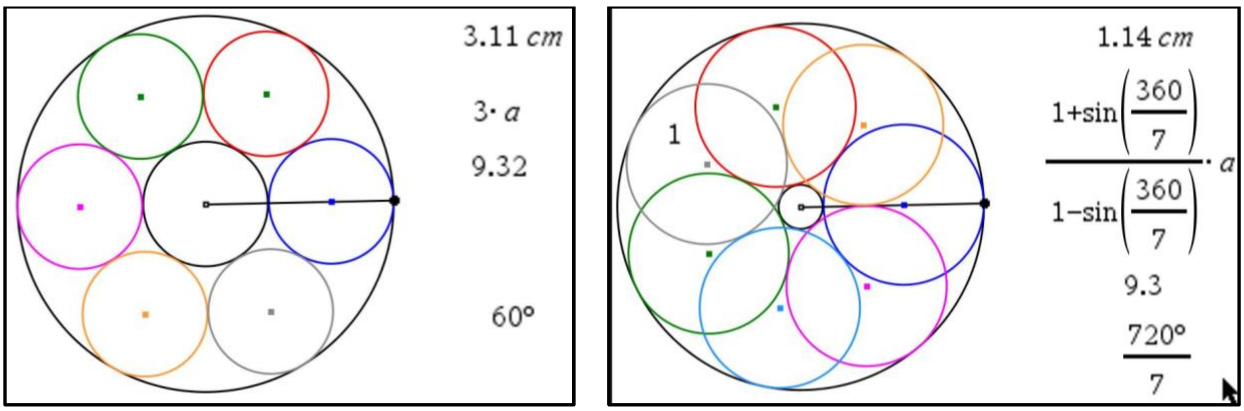

Figure 9: Other closed chains in concentric circles

5.3. Steiner chains: an approach of the general case

We start now from two concentric circles *INT* and *EXT* (same center *e*) which radii are respectively *r* and *k.r* where *k* can be chosen as we want (Figure 10 left). **I** is an inversion centered at *A* and of circle *C*. We know (see 4.2.) that the images of these two circles by this inversion are two circles *int* and *ext* belonging to the bundle at Poncelet points *A* and *A'* where *A'* is the image of *e* by the inversion **I** (Figure 10 right).

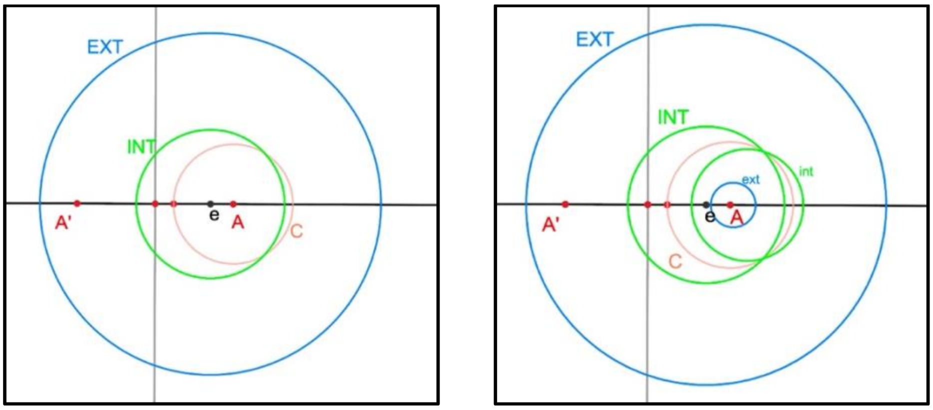

Figure 10: For our first chains in nested circles

From a point T_1 of *EXT*, let us construct (Figure 11 left), a chain of black circles (a sequence of tangent circles) tangent to *EXT* and *INT* (we can see three of these circles in Figure 11 left). The images of these circles by inversion **I** centered at *A*, is a chain of circles tangent to the two circles *ext* and *int* (here *ext* is inside of *int*: due to properties of inversions in this case of figure). If point *T*1 is dragged along *EXT* turning clockwise or anticlockwise inside the concentric circles, we obtain other chains between *EXT* and *INT* and by the way chains between *int* and *ext*.
Remark: if the value of *k* is correctly chosen, we can obtain closed chains of circles between the concentric circles and by the way also between their images *ext* and *int*. That is displayed un Figure 11 right where we have constructed such a chain of six circles called Steiner chain of circles between *ext* and *int*.

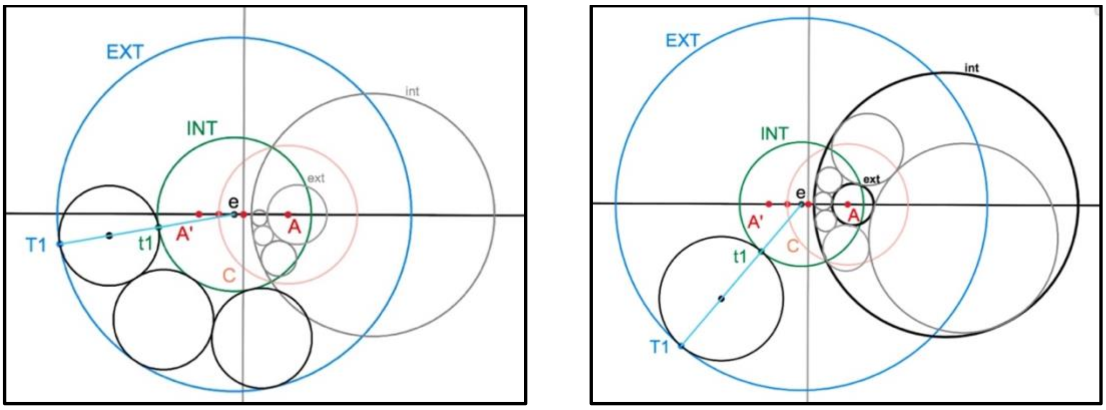

Figure 11: Construction of a Steiner chain

5.4. Steiner chains: general case

5.4.1. Two nested circles belong to the same Poncelet bundle
As shown in Figure 12 left, we consider two nested circles C_1 and C_2 centered respectively at $\boldsymbol{i_1}$ and $\boldsymbol{i_2}$. We show now how to construct the radical axis of these two circles (line containing all the points

having the same power for C_1 and C_2). As this radical axis is perpendicular to ($\boldsymbol{i_1 i_2}$), we need only to find and construct the point I on ($\boldsymbol{i_1 i_2}$) having the same power for the two circles, which means the point I verifying $Im.Im' = In.In'$ or which is equivalent:
$\frac{In}{Im} = \frac{Im'}{In'}$. This equality means that I is the center of the dilation transforming m onto n and n' onto m' and finally the segment [mn'] onto the segment [nm']. That justifies the elementary construction of I proposed in Figure 12 left.

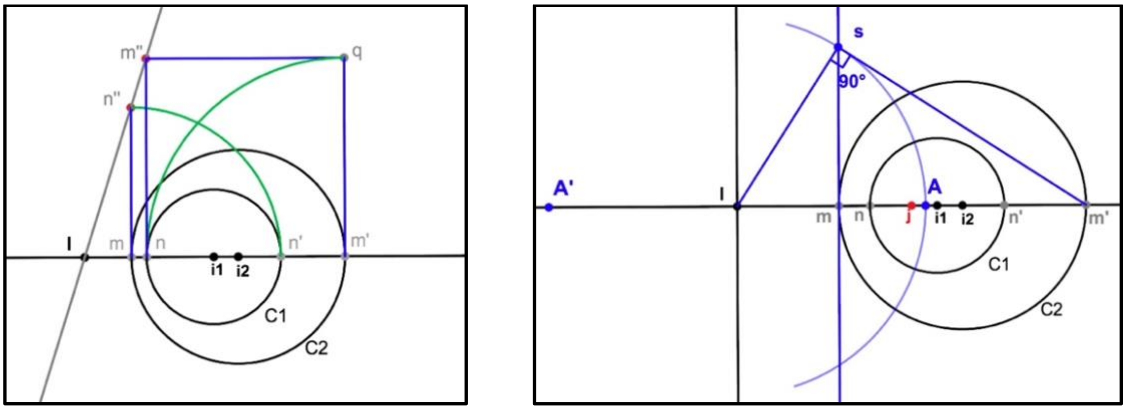

Figure 12: Poncelet bundle including two given nested circles

Figure 12 right shows precisely the construction of point A of the ray [Im) verifying $IA^2 = Im.Im'$ (this construction is based on a known property of right-angle triangles). As I is a point of the radical axis of C_1 and C_2, $IA^2 = In.In'$ where n and n' are the intersection points between C_1 and line ($\boldsymbol{i_1 i_2}$). If A' is the symmetric of A with respect to I, the previous relations means that (A', A, m, m') and (A', A, n, n') are harmonic divisions or better that circles C_1 et C_2 belong to the bundle at Poncelet points A' and A. We already know that the images of these two circles by any inversion centered at A or A' are two concentric circles (j is the midpoint of [Im'] in the construction of Figure 12 right).

5.4.2. How to construct a chain of circles tangent to two nested circles
Starting from C_1 and C_2, we construct as we did before point A' and A such that C_1 and C_2 belong to the bundle of points of Poncelet A' and A. We chose an inversion centered at A, for example. We transform these two circles by this inversion to get two concentric circles and their center. Then we construct a chain of tangent circles between the two concentric circles. The image of this chain by the inversion provides a chain of tangent circles between the circles we started from.
Now, the question is: what is the inversion that will generate a value of k compatible with the construction of a closed chain?

5.4.2.1. Complexity of the choice of the parameters of our problem

For such circles C_1 and C_2 whose radii are respectively r_1 and r_2 and centers i_1 and i_2 (where $r_1 < r_2$), let us evaluate the ratio of the radii of the circles images by an inversion I centered at A where A is one of the two points of the Poncelet bundle containing C_1 and C_2. Recall that this ratio is "independent" of the circle defining the inversion. All the following calculations are conducted in the axis system supported by the line ($i_1 i_2$) oriented from i_1 to i_2 *and* which origin is i_2. In this system, point i_1 is given by its abscissa α_1, with $-r_2 < \alpha_1 < r_2$.
■ We evaluate first the abscissa x_I of point I having the same power with respect to C_1 and C_2, which means, verifying $\overrightarrow{Im}.\overrightarrow{Im'} = \overrightarrow{In}.\overrightarrow{In'}$ or $(-r_2 - x_I)(r_2 - x_I) = (\alpha_1 - r_1 - x_I).(\alpha_1 + r_1 - x_I)$ that can be written as a quadratic relation: $2\,\boldsymbol{x_I}^2 + 2\,\alpha_1.\boldsymbol{x_I} + r_1^2 - r_2^2 - \alpha_1^2$ (where $r_1^2 - r_2^2 - \alpha_1^2 < 0$)

If $\Delta = \alpha_1^2 - 2.(r_1^2 - r_2^2 - \alpha_1^2) = 3\alpha_1^2 - 2r_1^2 + 2r_2^2$, we know that Δ is positive, so our quadratic equation has two solutions with opposite signs. If we add the constraint $\alpha_1 < 0$, then:

$$x_I = \frac{-\alpha_1 - \sqrt{\Delta}}{2}$$

■ The abscissa of *A*, x_A is evaluated thanks to the condition $IA^2 = \overrightarrow{Im}.\overrightarrow{Im'}$ which is the power ℘ of *I* with respect to circle C_2. We know that this power is also equal to $Ii_2^2 - r_2^2$ or $x_I^2 - r_2^2$. After computation, we get:

$IA^2 = \frac{1}{4}(4\,\alpha_1^{\,2} - 2r_1^2 - 2r_2^2 + 2\alpha_1\sqrt{\Delta})$ and as $IA^2 = (x_A - x_I)^2$, we obtain the abscissa of *A* and *A'*:

$x_A = x_I + \sqrt{℘}$ and $x_{A'} = x_I - \sqrt{℘}$.

■ Finally, if r'_1 and r'_2 are the respective radii of circles C'_1 and C'_2 images of C_1 and C_2 by **I**, we have (*r* is the radius of the circle of the inversion):

$$\frac{r'_1}{r'_2} = \frac{r^2.r_1}{\overrightarrow{Am}.\overrightarrow{Am'}} : \frac{r^2.r_2}{\overrightarrow{An}.\overrightarrow{An'}} = \frac{r_1}{r_2}.\frac{\overrightarrow{An}.\overrightarrow{An'}}{\overrightarrow{Am}.\overrightarrow{Am'}} \text{ or } \frac{r_1}{r_2}\,\frac{Ai_1^{\,2} - r_1^{\,2}}{Ai_2^{\,2} - r_2^{\,2}}.$$

We already know that to construct a Steiner chain with a given number of circles between two nested circles, the previous ratio must be equal to a specific number *k* (shown in 5.2.).

■ It is easy to imagine the complexity of the evaluation of one parameter with respect of the two other two parameters to obtain a particular value of *k*. If C_2, is given, the parameters we can modify are α_1 and r_1. For example, from the previous equation $\frac{r_1}{r_2}\,\frac{Ai_1^{\,2} - r_1^{\,2}}{Ai_2^{\,2} - r_2^{\,2}} = k$, we would have to find a formula giving $\alpha_1$with respect to r_1, r_2 and *k*, which was unsurmountable at an elementary level. If it had been possible, we would have been able to locate the center of C_1 with respect to the center of C_2 (if the radii of the two circles are fixed to obtain one circle inside the other compatible with the construction of a Steiner chain associated with *k*). For this reason, we will conclude with a technique of dichotomic approximation of the value of r_1 allowing the best approximation of the given *k* compatible with the construction of the associated Steiner chain.

5.4.2.2. Solution by successive approximations (figure 13)

On a given line, locate i_2 center of circle C_2, then i_1 center of circle C_1 with **radius** $\boldsymbol{r_1}$ which is a **displayed number** that can be **modified**. The number ***k*** is also a displayed number, evaluated by the software: it is the number which is the ratio of the radii of the concentric circles allowing the construction of a chain of *n* circles tangent to these concentric circles and by the way the construction of a Steiner chain of *n* tangent circles to the given circles C_1 and C_2.

We construct, as done before, point *A* (and point *A'*) which is the center of the inversion transforming C_1 and C_2 onto two concentric circles (the circle of this inversion is commanded with a point of the initial line).

We construct the images C'_1 and C'_2 of the circles C_1 and C_2 by the inversion and their common center *e* (which is the image of *A'* by the inversion of circle *C*), which allows us to evaluate (with the software we use: the New Cabri) their respective radii as well as the ratio of these radii. Considering the case of our figure, we evaluate r'_1/r'_2 which is displayed.

We construct a point t_2 of C_2 and its image t'_2 on C'_2 by the inversion. Thanks to the ray [et'_2), we construct the first circle tangent to C'_1 et C'_2 (tangent to C'_2 at t'_2) visible in grey in Figure 13 left. Then we construct the image of this circle by the inversion to obtain the first circle tangent to C_1 and C_2 (tangent to C_2 at t_2) visible in the same figure in black.

We construct now a chain of *n* grey circles tangent to C'_1 and C'_2 starting from the previous grey circle which images by the inversion provide a chain of black circles tangent to C_1 and C_2 starting from the previous black circle (Figure 13 right).

In Figure 13, we have chosen *n* = 6. But we can state that since r'_1/r'_2 is not equal to 3 which is the value of *k* associated with a closed chain of 6 circles, our chain of 6 grey circles is not closed between C'_1 and C'_2 and so, the black chain we have constructed by inversion is not a Steiner chain of 6 black circles.

The final technique consists in modifying the value of r_1 in a dichotomic way until the displayed value of r'_1/r'_2 is as close as possible to number 3 when using the maximum of digits allowed by the software. If necessary, we can also change the position of i_1.

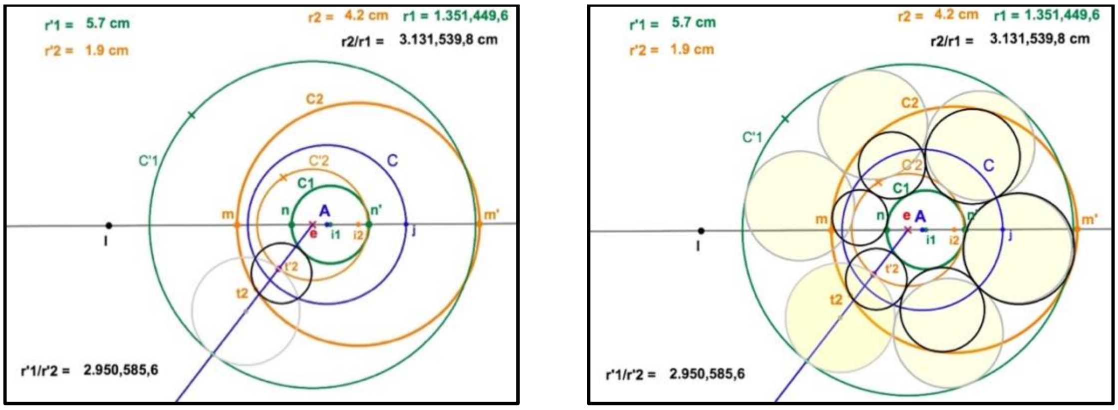

Figure 13: Starting constructions of a Steiner chain in two nested circles

After such an operation we reach a situation visible in Figure 14 left where the Steiner chain of 6 circles is constructed between C_1 et C_2. The animation of point t_2 along C_2, allows us to visualize all the Steiner chains of 6 circles between C_1 et C_2. On the same figure, we can state that, if we modify r_1 until the value underlined in green, we have succeeded to reach for the displayed value of r'_1/r'_2, 3.000,000,0. In Figure 14 right, we can appreciate the final result after hiding the constructions.

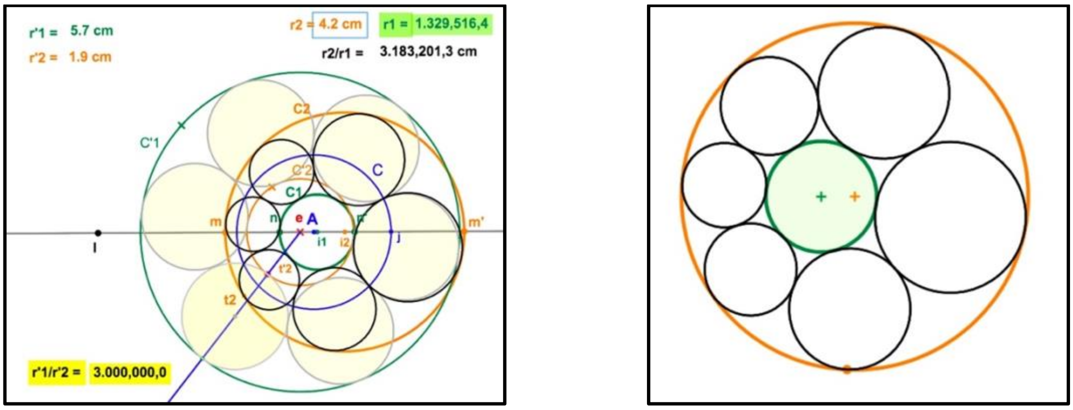

Figure 14: Final construction of a Steiner chain between two nested circles

In Figure 15 left, we show the construction of a Steiner chain of 11 circles closed after one turn (now in red). The value of k that $r'1/r'2$ must reach after an adequate modification of r_1 is equal to $\frac{1+\sin(\frac{\pi}{11})}{1-\sin(\frac{\pi}{11})}$ which value given by the software with its best approximation is 1.7844781. We have reached 1.7844785 that justifies the accuracy of our construction. We can appreciate the result when the constructions are hidden in Figure 15 right.

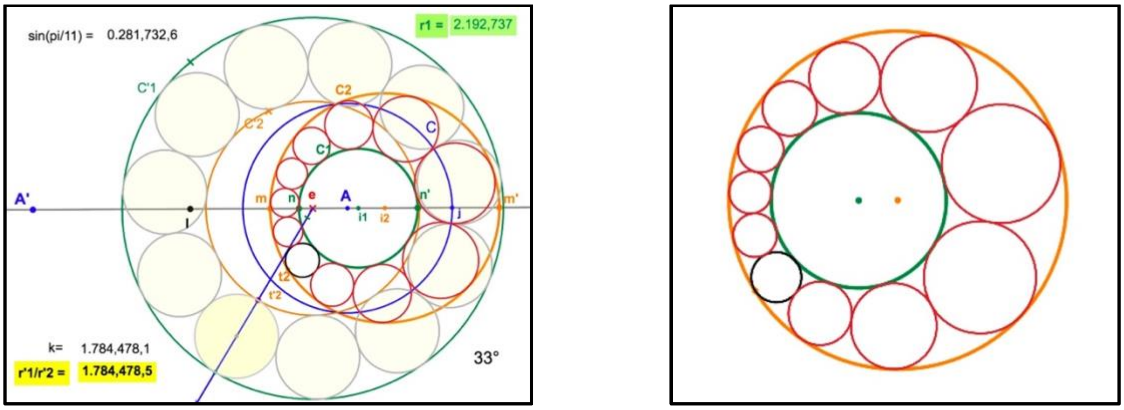

Figure 15: A Steiner chain of 11 circles

In order to construct all possible Steiner chains, we need such a figure in which we have constructed the number of circles corresponding to the expected chain, change the value of k corresponding to this expected chain by using the formula

$$\frac{1+\sin\left(\frac{m\pi}{n}\right)}{1-\sin\left(\frac{m\pi}{n}\right)}$$

where, n is the number of the circles of the chain and m is the number of turns necessary for the first closure of the chain

5.5. Steiner chains: some other properties ([6])

Only the use of Dynamic Geometry Software can help us to investigate the following properties.

■ The first one is about the centers of the circles of a Steiner chain and the contact points between them (Figure 16). In Figure 16 left, we have constructed the conic passing through five of the eleven centers of the corresponding Steiner chain: we can easyly conjecture that all the centers belong to this conic which is qualified as an ellipse (in red) by our software. In Figure 16 right, we have constructed the conic passing through five of the eleven contact points of the corresponding Steiner chain: we also can easyly conjecture that all these points belong to this conic which is qualified as a circle (in blue) by our software. These properties are known properties that can be investigated easyly in such figures with the appropriate software.

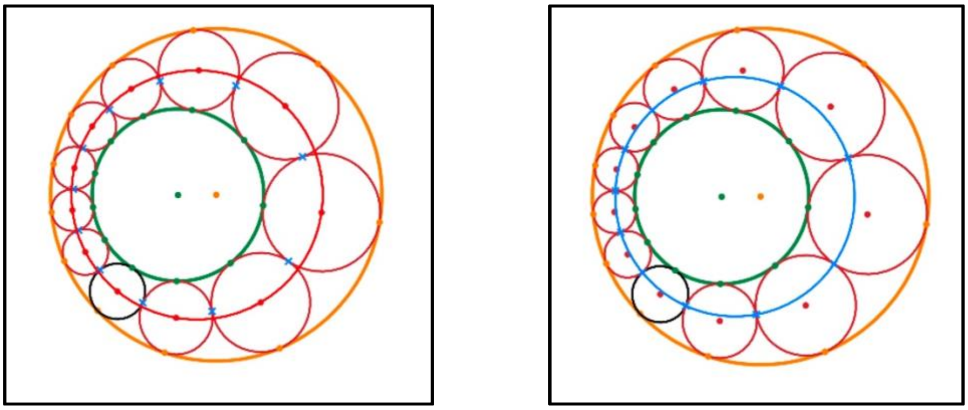

Figure 16: Some points of a chain on the same ellipse or the same circle

■ The second one, in the case of n = 6, is illustrated in Figure 17: here we can conjecture another known property stating that the three segments connecting the contact points of the Steiner chain with the exterior circle (Figure 17 left) pass through a common point belonging to the line of the centers of the two given circles. Same statement for the contact points of the interior circles (Figure 17 right).

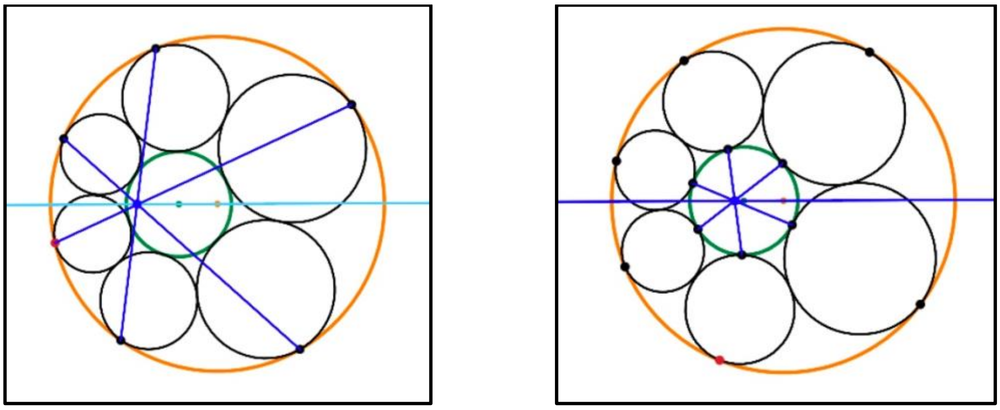

Figure 17: Segments passing through a common point on the line of the centers of the nested circles

6. Addendum

Here are some techological indications about the way we used the inversion of a point or the inversion of a circle under the **New Cabri environment**. This software is the one we used for all our figures even if most of them could have been realized with Cabri 3D (we did it but finally we have chosen the New Cabri for practical reasons). As we had to construct a great quantity of images of points or circles by an inversion, we have defined two macros for such a work:
Macro 1 returning the image of a given point by a given inversion.
Macro 2 returning the image of a given circle by a given inversion.
Creation of **Macro 1** (Figure 18 left): let us construct the circle (*C*) (centered at *O*) of an inversion, let us create a point *M*. Using the measurement tool, we display the radius r of (*C*) and the distance *OM*. We evaluate with the Calculator in Algebraic mode $\frac{r^2}{OM}$ which is *OM'*. We construct ray [*OM*): point M' is obtained with the tool Measurement Transfer in transferring $\frac{r^2}{OM}$ on ray [*OM*) (click on the number, click on the ray and click on *O*). Then we create Macro 1: chose **Initial Object(s)** and click on circle (C) and on point *M*, chose **Final Object(s)** and click on point *M'* and finally chose **Define Macro**. A square containing a wheel gear appears. It is possible to change this wheel gear onto another image: it is what I did in including number *1*. This square represents **Macro 1**. It is possible now to use it as as tool (inversion of a point): to check it we create a point *N*, then we select **Macro 1** in clicking on the square and the image of *N* is created after clicking on (*C*) first and on *N*.
Creation of **Macro 2** (Figure 18 right): we open another page of the same document, we copy **Macro 1** from the first page and paste it on page 2. We create circle (*C*) (circle of our inversion), a blue circle (*C1*) and a point *M* on (*C1*). Thanks to **Macro 1** we create the image *M'* of *M* (click on **Macro 1**, on circle (*C*) and on point *M*). Then ask the software to return the locus of *M'* when *M* describes (*C1*): it is circle (*C'1*) image of (*C1*) by the inversion of circle (*C*). We have now to define **Macro 2**: same algorithm than for **Macro 1**: Circle (*C*) and circle (*C1*) are the initial objects, (*C'1*) is the final object and clicking on **Define Macro** generates the button (square with a wheel gear) which is **Macro 2**. We have inserted number **2** to replace the wheel gear. At last we can test **Macro 2** in clicking on (*C*) and on a new blue circle (*C2*) to obtain its image by the inversion which is the orange circle (*C'2*).

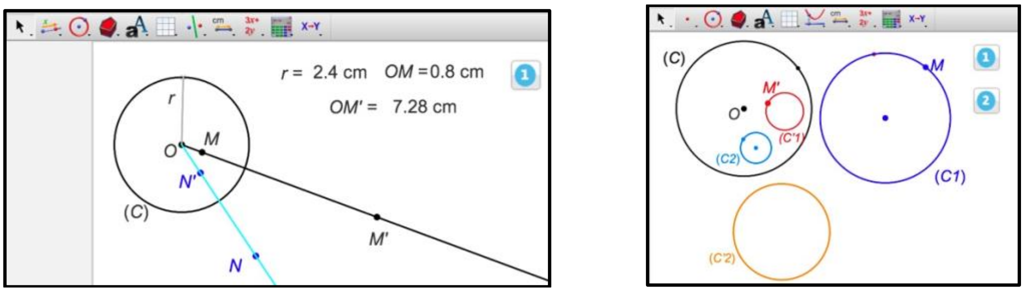

Figure 18: Creation of the two macros related to the tool "Inversion"

7. Conclusion

Thanks to dynamic geometry software allowing easy use of a tool for inversion, the paper has demonstrated how to solve the problem of the construction of Steiner chains as simply as possible. The paper provides to the reader with a wide range of the necessary knowledge to understand the techniques of construction and to show the power of these techniques in the figures created with the particular software used (here the New Cabri). As always in research of this kind, the work involved some questions which could not be solved with elementary tools (such as computations by hand or by CAS), and these are explored in the paper. It is hoped that this paper provides a clear view of the problems of construction of Steiner chains and will inspire teachers with some ideas for experimental investigations for their students even at a highschool level.

Acknowledgment

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WIKIPEDIA, Power of a point, article at https://en.wikipedia.org/wiki/Power_of_a_point

Software

Cabri "Standard", version 3.14 and *Cabri 3D* by Cabrilog S.A.S at http://www.cabri.com

TI-Nspire^{TM} *CX CAS Premium Teacher Software*, *version5.2.0.771* by Texas Instruments at https://education.ti.com/