Approaching Cesàro's inequality through GeoGebra Discovery

The "discovery function" of proof

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De Villiers' inspiration: The "discovery function" of proof

- ✓ "… a heuristic description of some of my personal experiences of the explanatory and discovery functions of proof with a geometric conjecture made by a Grade 11 student: Clough's conjecture"

- ✓ ". . . at least acquaint students with the idea that a deductive argument can provide additional insight and some form of novel discovery . . . Problem posing and generalisation through the utilisation of the 'discovery' function of proof is as important and creative as problem-solving itself, and ways of encouraging this kind of thinking in students need to be further explored."

De Villiers, M. (2012) An illustration of the explanatory and discovery functions of proof. Pythagoras, 33(3), 193.
https://doi.org/10.4102/pythagoras.v33i3.193

Our aim

Describe some personal experiences of the discovery function of proof by the interaction with our “symbolic geometry calculator", the program **GeoGebra Discovery**

- But, GeoGebra Discovery’s main feature is, precisely, the automated verification of geometric statements, without bringing any human readable argument for their truth or falsity. There is not proof at all!
  - Why do we regard the possibility to follow with GeoGebra Discovery a parallel path to the one established in de Villiers' cited work?

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Our “discovery function” of proof experience

- **The *discovery function* for GeoGebra Discovery**, i.e. what the researchers and programmers involved in its debugging and improvement have to discover.
- ***The extended discovery opportunity,*** for standard users, to allow them discovering geometric properties having an 'oracle' at hand.
- **The *discovery function* of automated proof**, for advanced users, when analyzing the problems and difficulties of the performance of GeoGebra Discovery commands while dealing with different statements, yielding to discover new algebro-geometric properties.

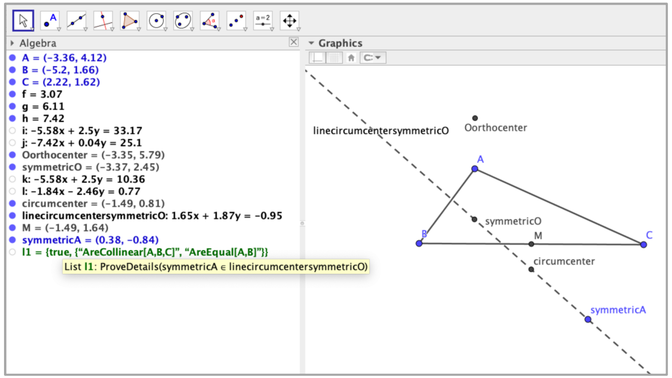
The 'discovery function' for GeoGebra Discovery

Our project...

From

GeoGebra Automated reasoning tools in Geometry

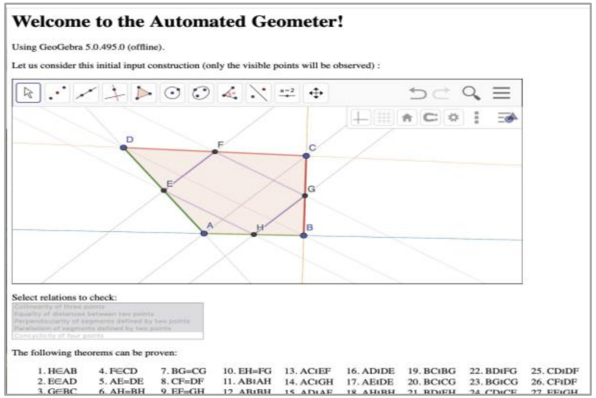
Exploration, discovery or verification of some guessed or conjectured property in a figure



to

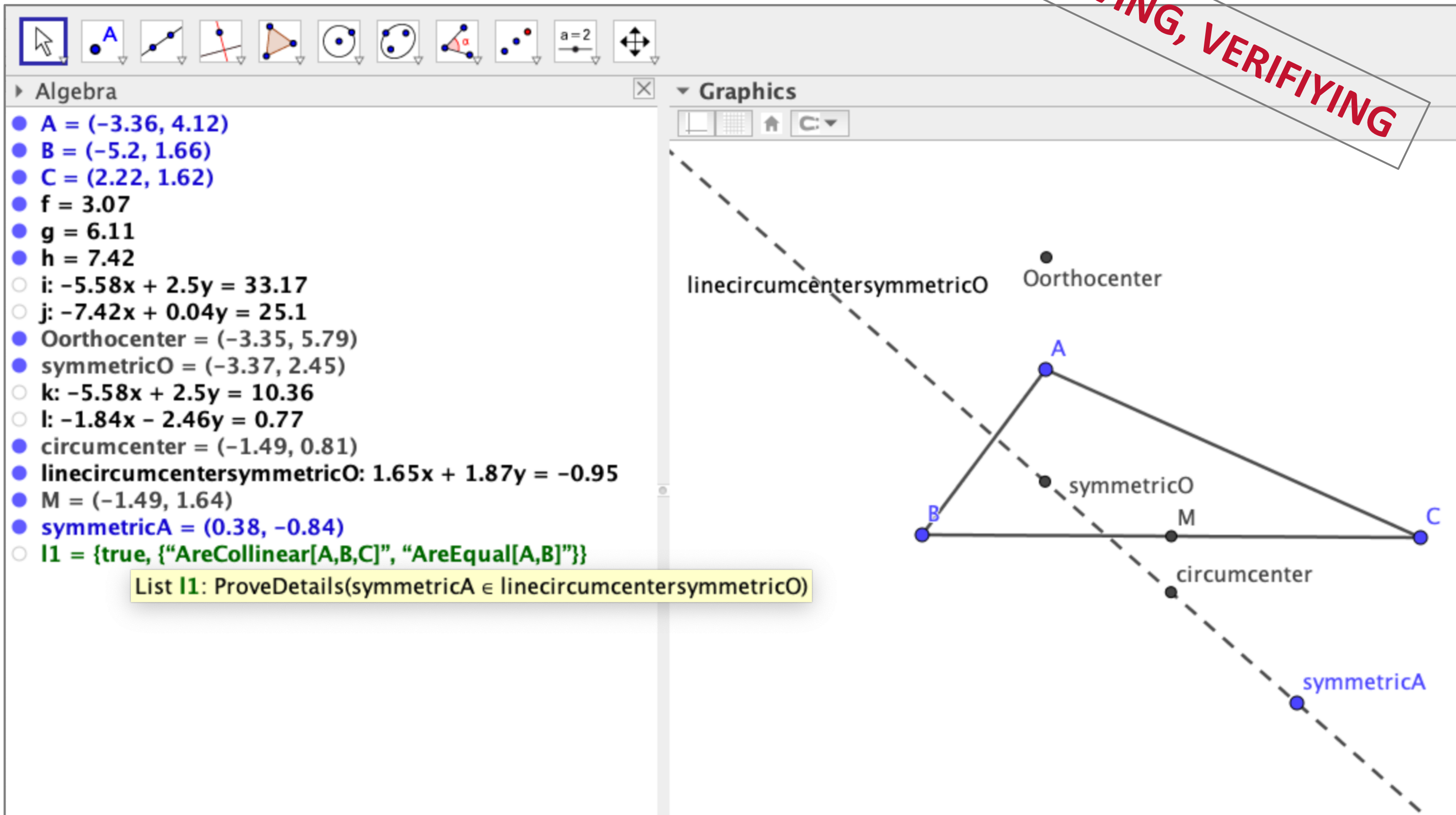
Mechanical geometer program

Finding a large collection of properties in a figure without previously guessing any property



Going from …

PROVING, VERIFIYING



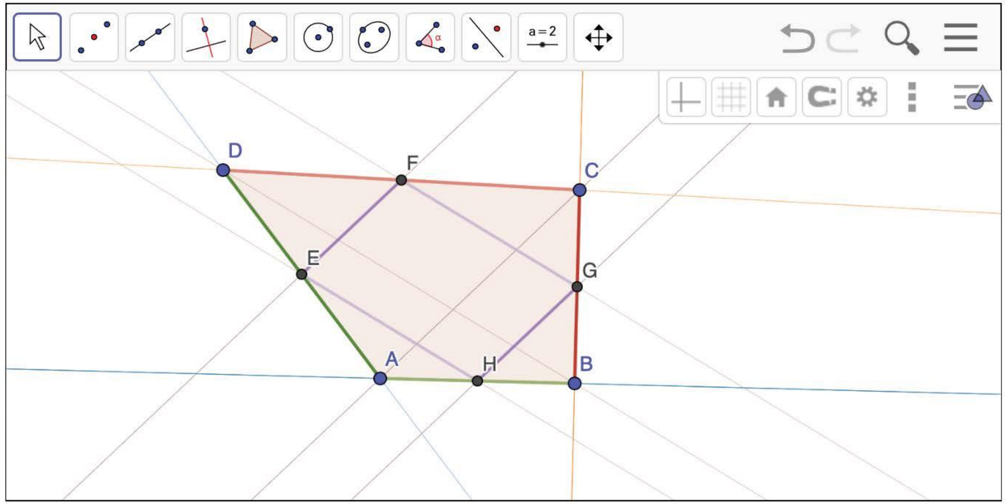
to...

Welcome to the Automated Geometer!

DISCOVERY

Using GeoGebra 5.0.495.0 (offline).

Let us consider this initial input construction (only the visible points will be observed) :



Select relations to check:

- Collinearity of three points
- Equality of distances between two points
- Perpendicularity of segments defined by two points
- Parallelism of segments defined by two points
- Concyclicity of four points

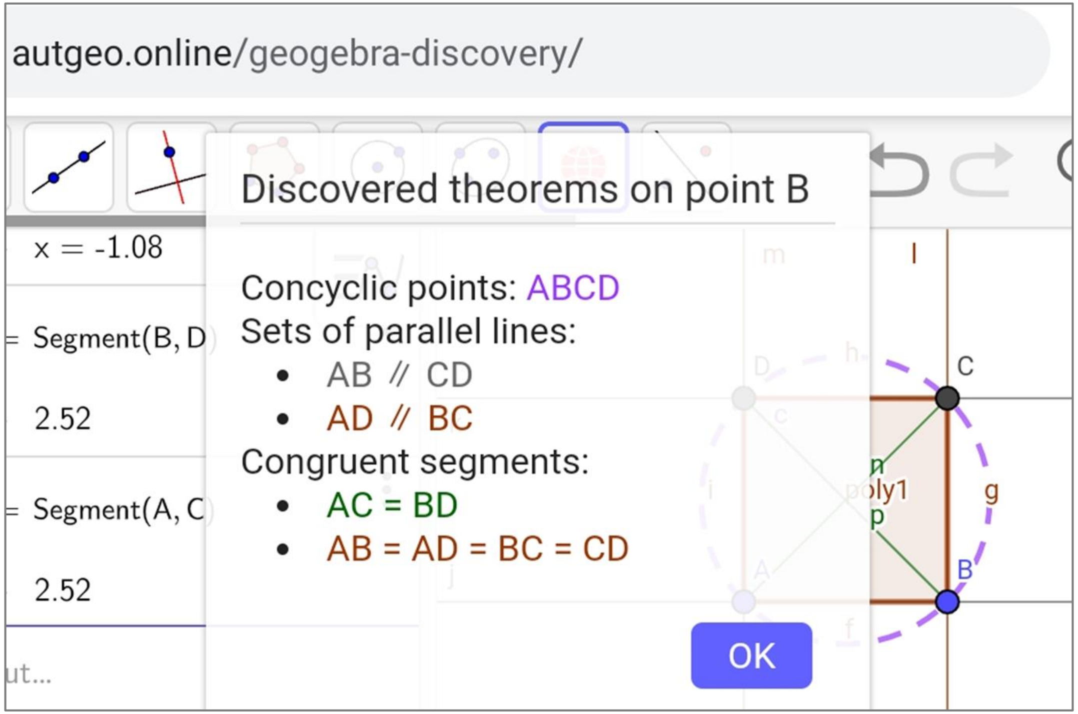
The following theorems can be proven:

1. H∈AB	4. F∈CD	7. BG=CG	10. EH=FG	13. AC∥EF	16. AD∥DE	19. BC∥BG	22. BD∥FG	25. CD∥DF	28. EH∥FG
2. E∈AD	5. AE=DE	8. CF=DF	11. AB∥AH	14. AC∥GH	17. AE∥DE	20. BC∥CG	23. BG∥CG	26. CF∥DF	
3. G∈BC	6. AH=BH	9. EF=GH	12. AB∥BH	15. AD∥AE	18. AH∥BH	21. BD∥EH	24. CD∥CF	27. EF∥GH	

Finished, found 28 theorems among 1190 possible statements.
Elapsed time: 0h 0m 5s

Automated Geometer: http://www.autgeo.online/ag/automated-geometer.html?offline=1

Through improvements in GeoGebra ART



➢ **GeoGebra Discovery**

What’s GeoGebra Discovery?

GeoGebra Discovery is an experimental version of GeoGebra

- ✓ a “symbolic geometry calculator“ with some new GeoGebra features to conjecture, discover and prove statements based on complex and real algebraic geometry
- ✓ under development
- ✓ not yet included in the official GeoGebra version

Online version (GeoGebra 6)
http://autgeo.online/geogebra-discovery/

Desktop version (GeoGebra 5)
https://github.com/kovzol/geogebra/releases

GeoGebra Discovery's website
https://github.com/kovzol/geogebra-discovery#geogebra-discovery

GeoGebra Discovery tools and commands

- `Prove` and `ProveDetails`: proving the truth or failure of a given statement (improved, inequalities).
- `LocusEquation`: discovering how to modify a given figure so that a wrong statement becomes true (improved).
- `Envelope`: computing the equation of a curve which is tangent to a family of objects while a certain parent of the family moves on a path (improved).
- `Relation`: discovering the relation holding among some concrete elements of the given figure (improved and new features, inequalities).
- `Discover`: discovering all statements holding true involving one element in the figure selected by the user (**new**, inequalities).
- `Compare`: comparison between segment lengths (**new**, inequalities).

The 'extended discovery opportunity‘

- ICMI-Kuwait example
- Botema’s Inequalities
- A eJMT Problem Corner
- The treasure island problem

ICMI Study "School Maths in the 90's"

Consider, for example, the following question (to other aspects of which we shall wish to refer later):

Two lines are drawn from one vertex of a square to the midpoints of the two non-adjacent sides. They divide the diagonal into three segments (see Figure 5.2).

(a) Are those three segments equal?

(b) Suggest several ways in which the problem can be generalised.

(c) Does your answer to (a) generalise?

(d) Can the argument you used in (a) be used in the more general cases?

(e) If your answer to (d) is 'No', can you find an argument which does generalise?

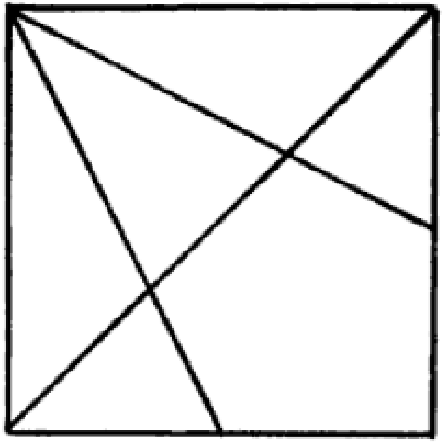
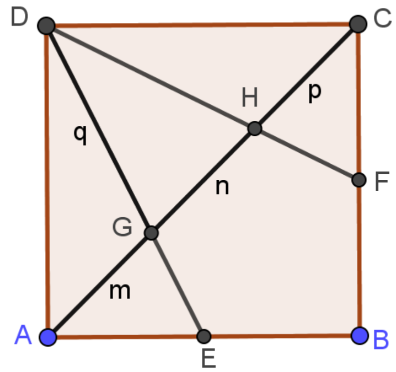


Fig. 5.2

Howson, G., Wilson, B. (1986) ICMI Study series: School mathematics in the 1990's. Cambridge University Press. Kuwait.

… verifying and discovering



Relation(m,n)

Relation

It is generally true that:

- **m** and **n** are parallel

under the condition:

- **A** and **B** are not equal

It is generally true that:

- **m** has the same length as **n**

under the condition:

- **A** and **B** are not equal

OK

Relation(m,q)

Relation

m does not have the same length as **q**
(checked numerically)

More...

OK

Relation

It is generally true that:

- **m** = $(1/5 \cdot \sqrt{10}) \cdot$ **q**

under the condition:

- the construction is not degenerate

OK

Relation detects lenghts ratio

Dealing with inequalities

GEOMETRIC INEQUALITIES

BY

O. BOTTEMA
Delft, The Netherlands

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THE SIDES AND THE RADII OF A TRIANGLE

5.3 $a+b+c \leqslant 3R\sqrt{3}.$

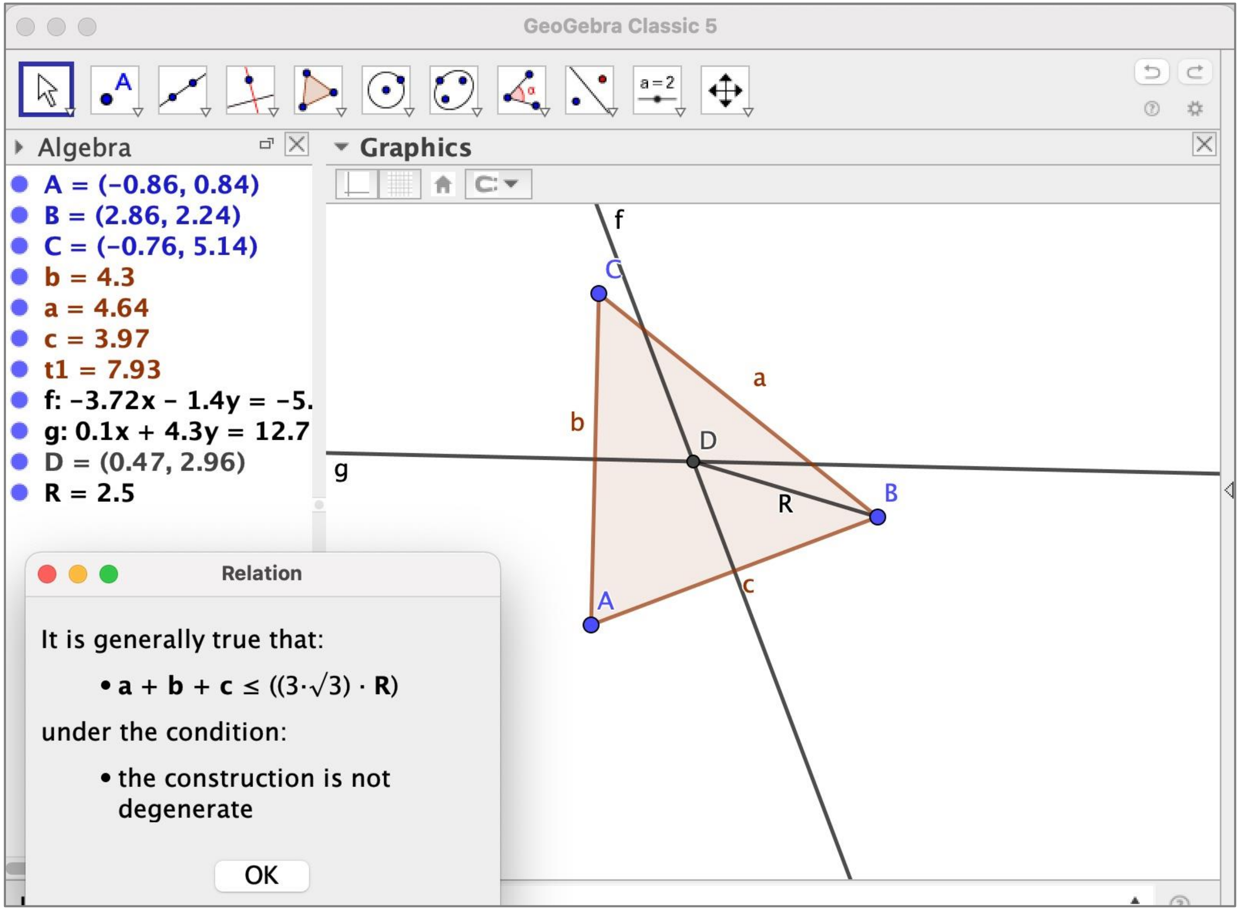
Equality holds if and only if $a=b=c$.

S. Nakajima, Tôhoku Math. J. 25 (1925), 115–121.

A. Padoa, Period. Mat. (4) 5 (1925), 80–85.

https://www.isinj.com/mt-usamo/Geometric%20Inequalities%20-%20Bottema,%20et.%20al.%20(1968).pdf

Relation gives also inequalities between lengths

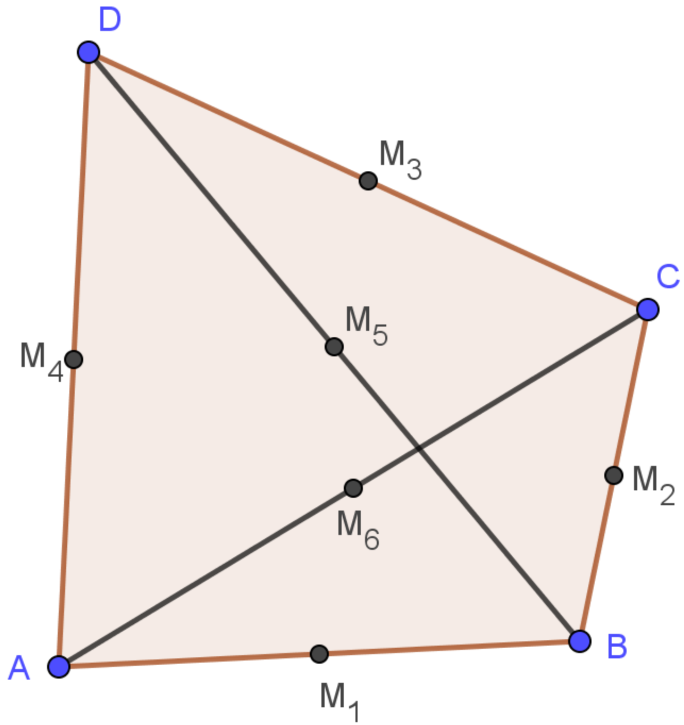


Solving some "Problem Corner"

eJMT Problem Corner, October 2020

Problem 1. Let M1, M2, M3, M4, M5, M6 be the midpoints of the edges AB, BC, CD, DA, AC, BD. Prove that the segments M1M3, M2M4, M5M6 are concurrent in a point E that bisects them all.

Provided by D. Ferrarello, M. F. Mammana, M. Pennisi, E. Taranto (University of Catania, Italy)



Kovacs, Z., Recio, T. (2021) Alternative solutions and comments to the Problem Corner, October 2020 issue. The Electronic Journal of Mathematics and Technology.
https://php.radford.edu/~ejmt/ProblemCornerDocs/eJMT_Alternative_Solutions_to_Oct2020.pdf

Solving some “Problem Corner”

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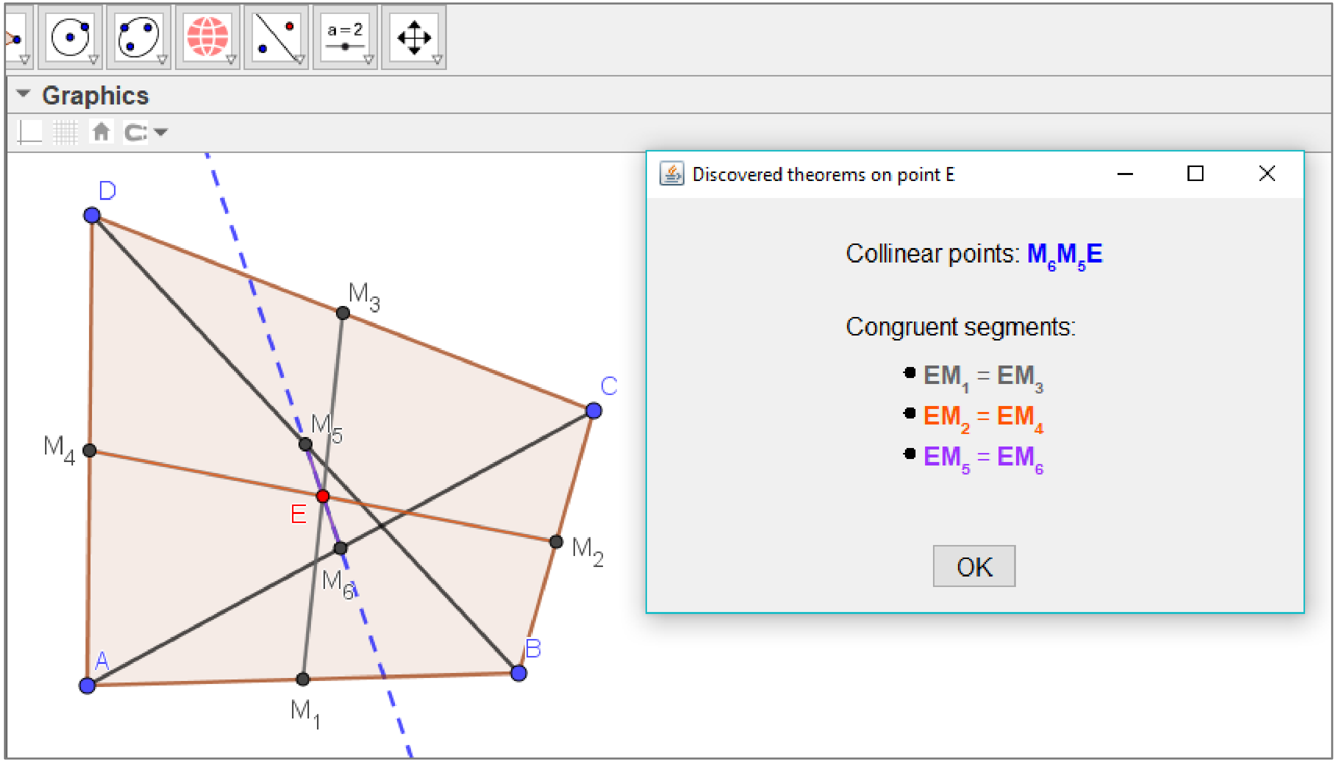
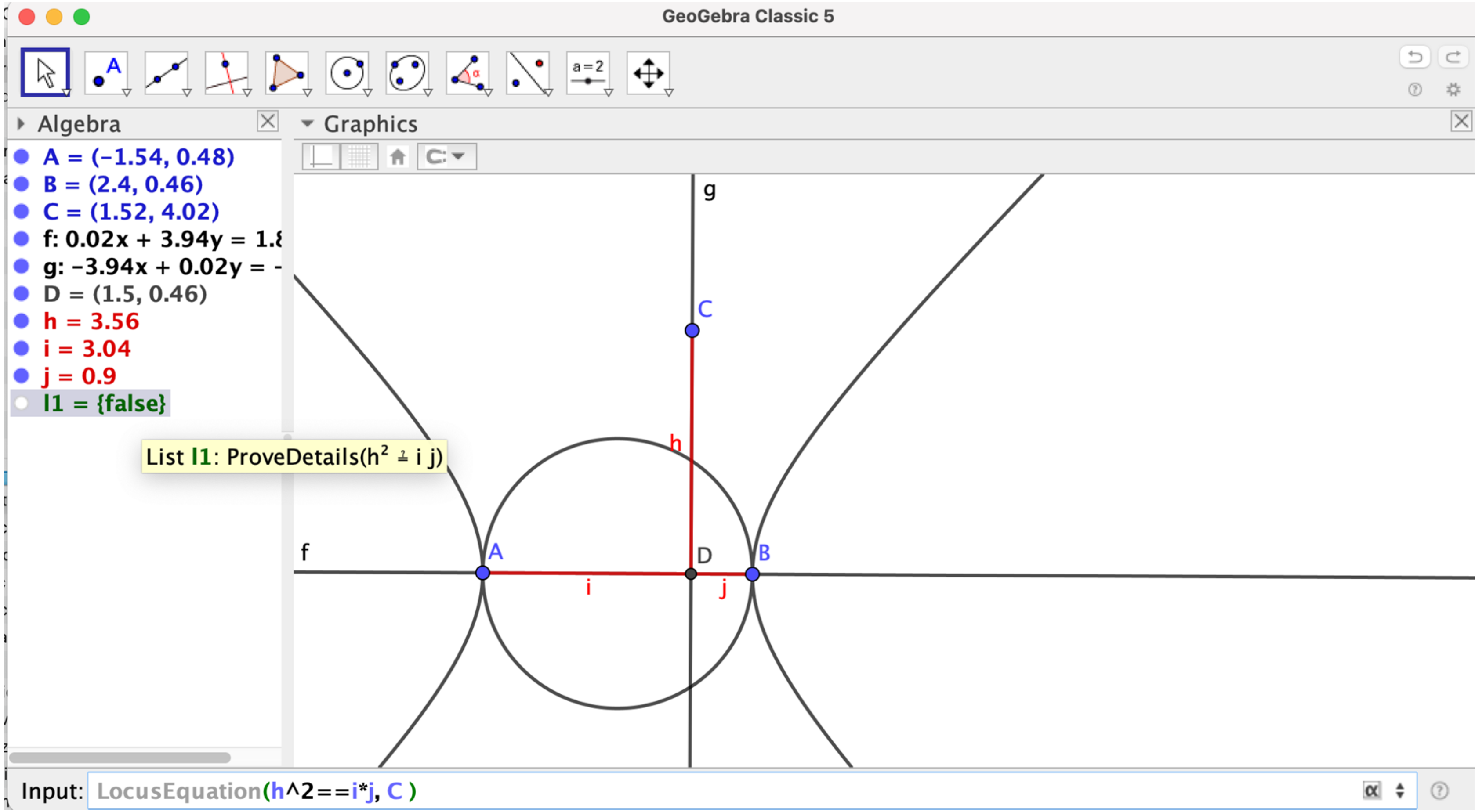


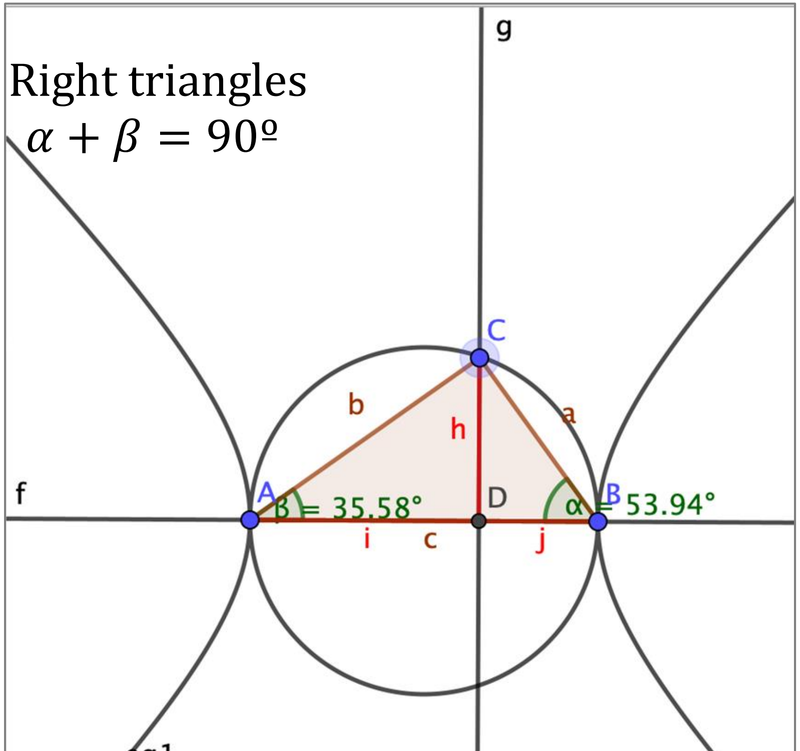
The 'discovery function' of automated proof:

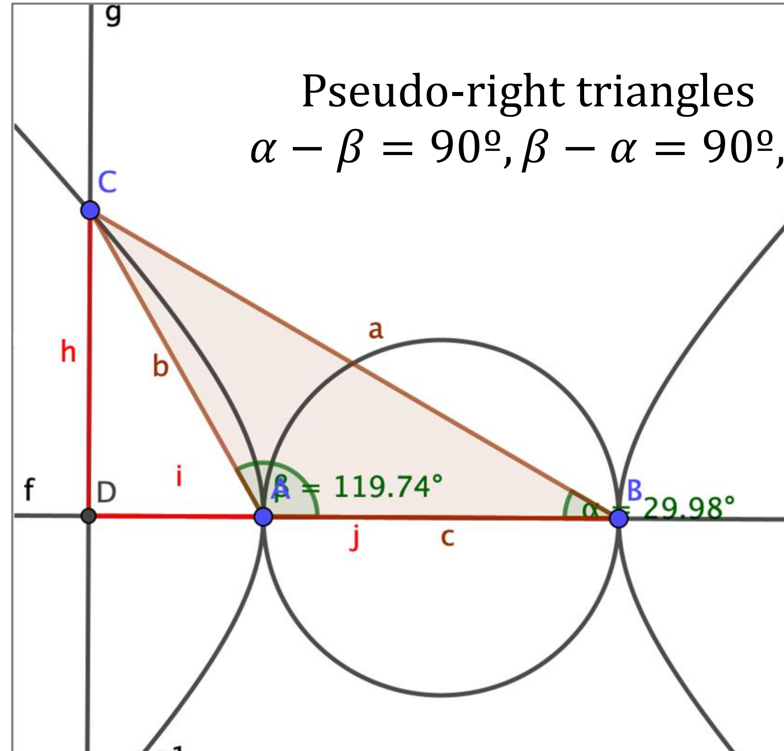
- The Altitude's theorem
- Cesàro's inequality

Discovering the Altitude’s theorem



Re-discovering the Altitude's theorem





Etayo-Gordejuela, F., de Lucas-Sanz, N., Recio, T., Velez, M.P. (2021) Inventando teoremas con GeoGebra: un nuevo teorema de la altura. Boletín de la Soc. Puig Adam, 111, 8-27

Cesàro's inequality

Original formulation of E. Cesaro's inequality.

Question 529.

THÉORÈME. — *Dans tout triangle, le produit des rapports de chaque côté à la somme des deux autres, ne surpasse pas* $\frac{1}{8}$.

(E. CESARO.)

Le théorème revient à prouver l'inégalité

$$\frac{abc}{(a+b)(b+c)(c+a)} < \frac{1}{8}. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Or, la moyenne géométrique de deux nombres est plus petite que leur moyenne arithmétique; ainsi :

$$\sqrt{ab} < \frac{a+b}{2}, \quad \sqrt{bc} < \frac{b+c}{2}, \quad \sqrt{ac} < \frac{a+c}{2}.$$

On conclut, de ces inégalités,

$$abc < \frac{(a+b)(b+c)(c+a)}{8} \ (^*).$$

(E. FAUQUEMBERGUE.)

Autres solutions par MM. Torrès, élève du Lycée de Bordeaux, Leinekugel, étudiant (Paris) et Cocheteux, élève de l'École des Mines (Liége).

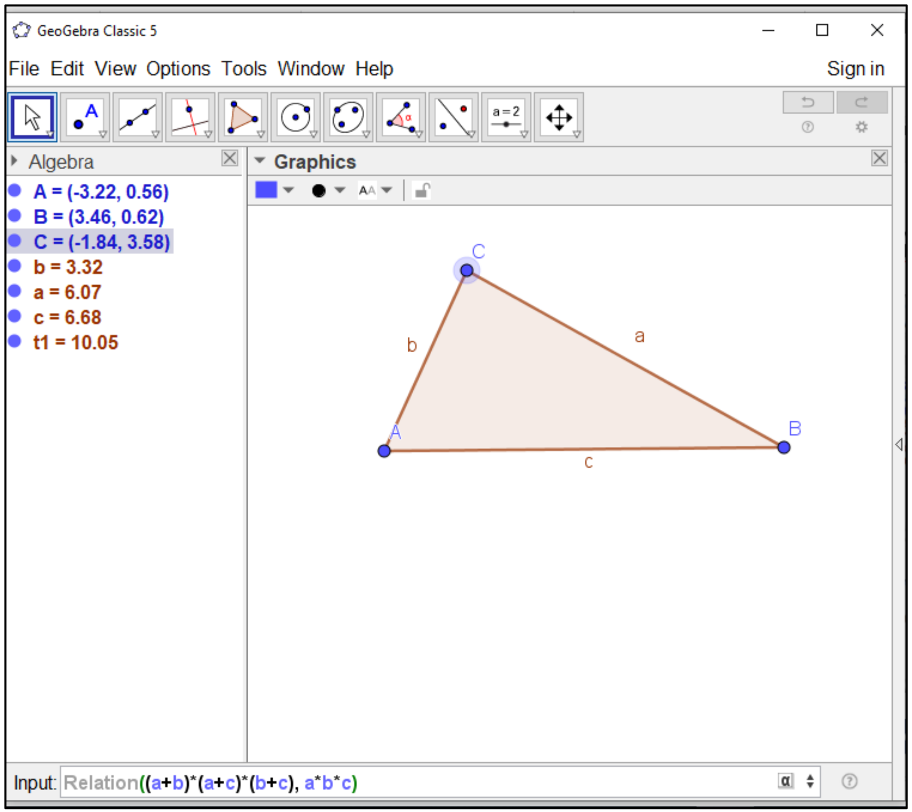
(*) Il est bon d'observer que, si le triangle est équilatéral,

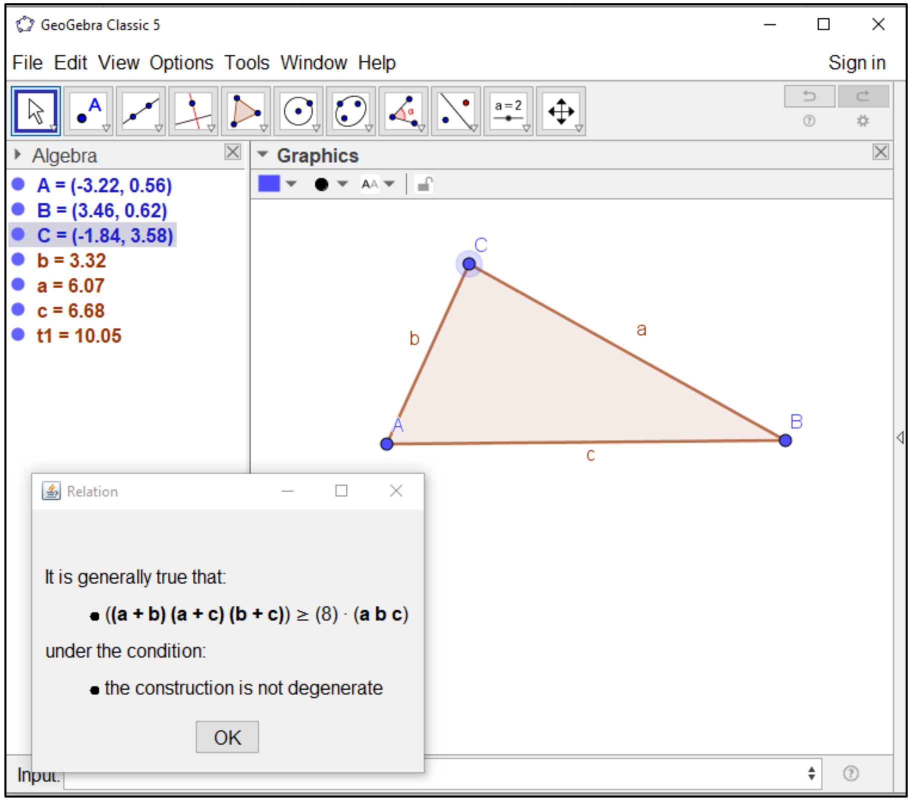
$$\frac{abc}{(a+b)(b+c)(c+a)} = \frac{1}{8}.$$

(E. C.)

Let's provide some geometric reasons that justifies Cesàro's

Cesàro's with GeoGebra Discovery

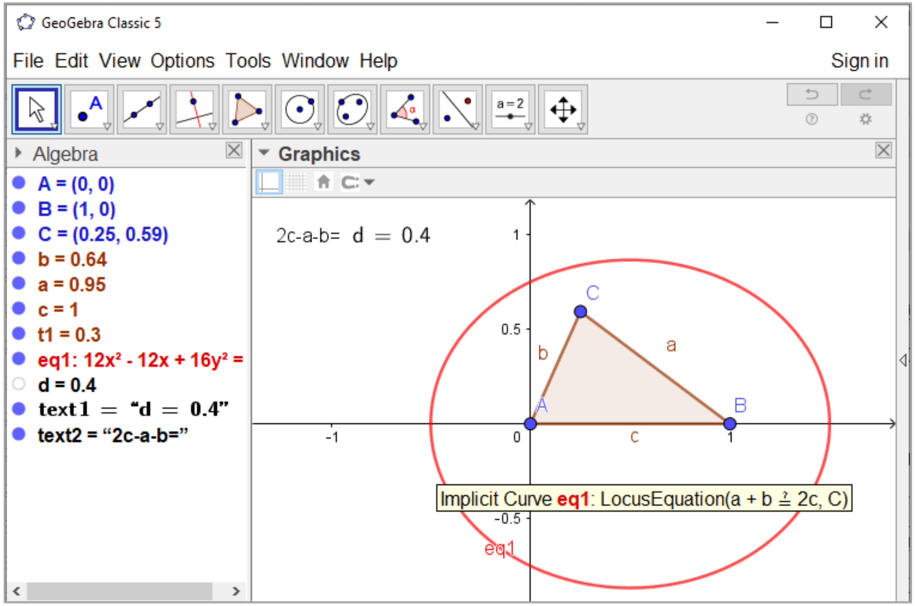




Learning path on Cesàro’s inequality I

- The triangle inequality → less accurate lower bound
  $(a+b) \geq c, (b+c) \geq a, (a+c) \geq b \implies (a+b)(a+c)(b+c) \geq a \cdot b \cdot c$

- Is it true that $(a+b) \geq 2c, (b+c) \geq 2a, (a+c) \geq 2b$?
  NO. Take the Pythagorean triple $a = 3, b = 4, c = 5$



- Could it be true in some cases?
  - Ask Geogebra Discovery

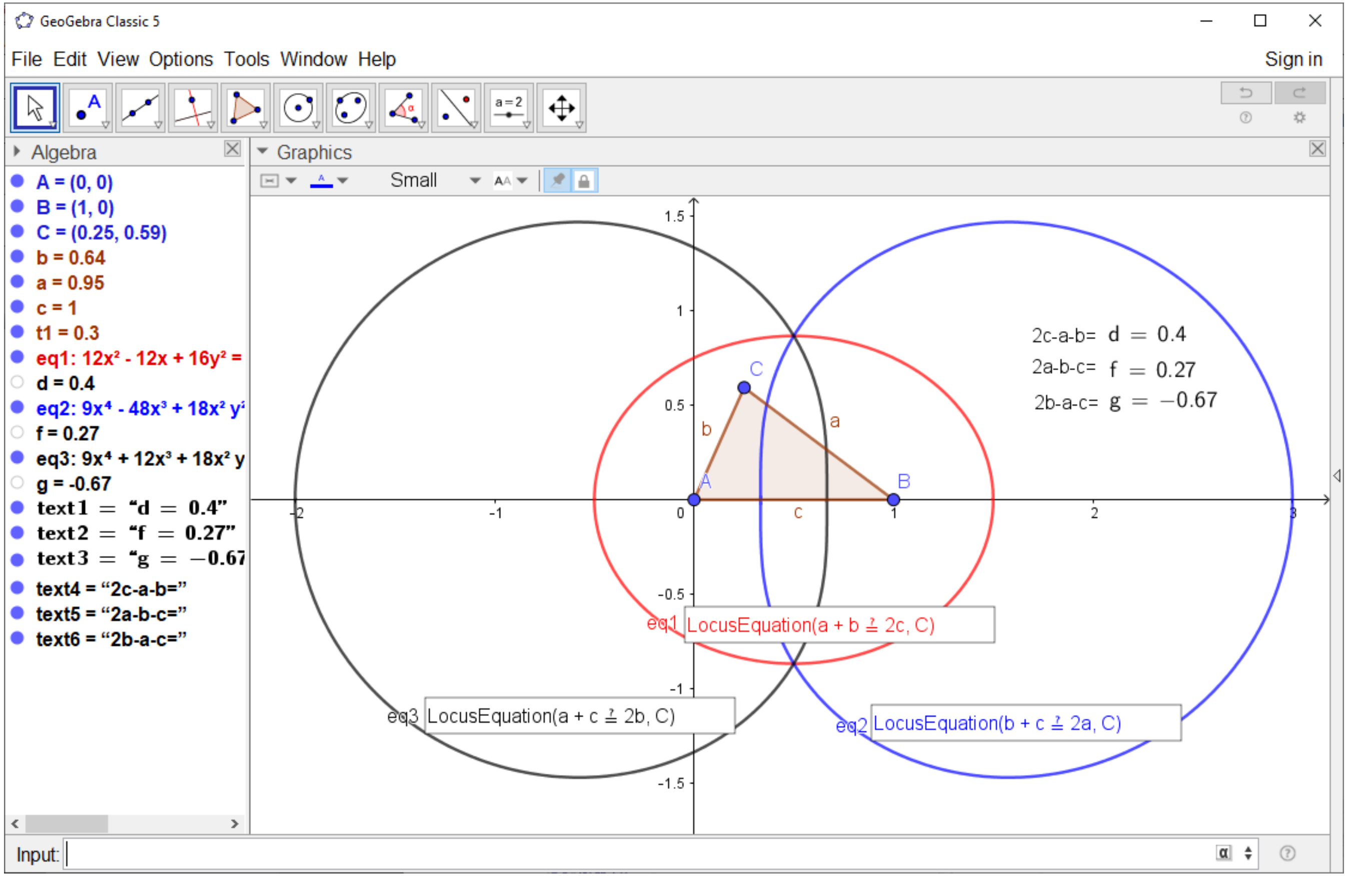
```
LocusEquation(2c==a+b,C)
```

  - Drag point C

$(a+b) \leq 2c$ inside the conic

$(a+b) \geq 2c$ outside the conic

Learning path on Cesàro's inequality II



Curves intersection:

- Not posible with GeoGebra CAS
- But, easy to deduce $a = b = c$

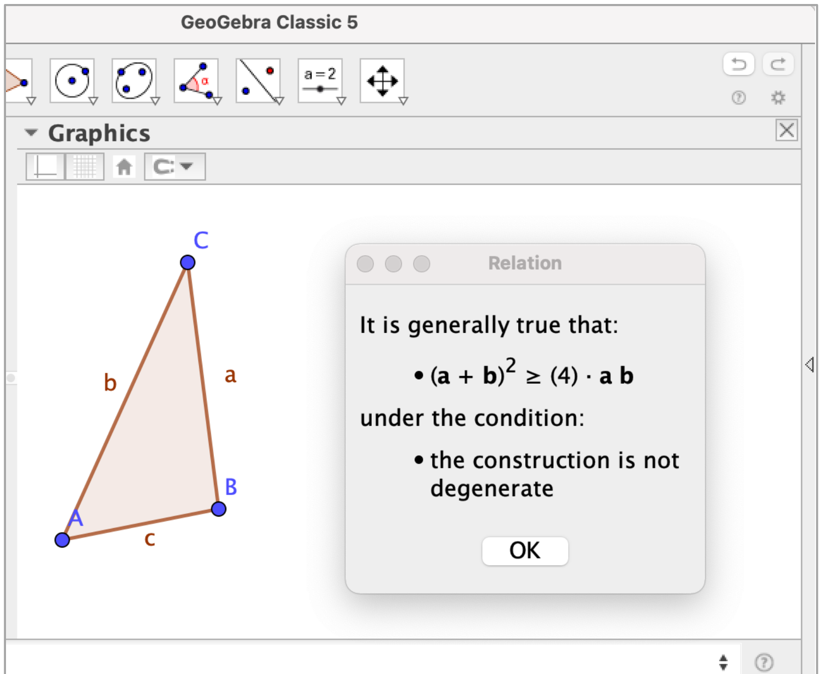
$$\Downarrow$$

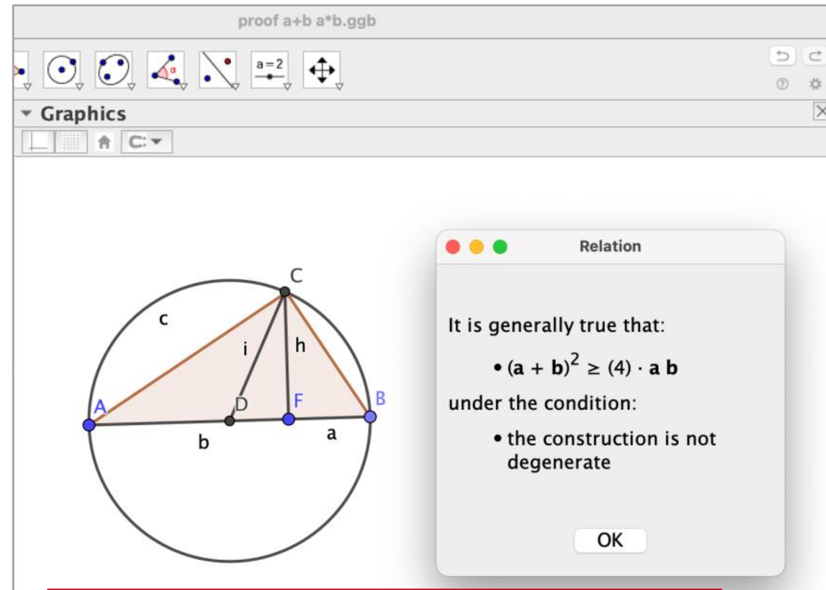
Cesàros equality holds over equilateral triangles

Learning path on Cesàro's inequality III

$(a+b) \geq 2c, (b+c) \geq 2a, (a+c) \geq 2b$ is false over general triangles

- Ask GGb Discovery about some relation





$$h^2 = \mathrm{a} \cdot \mathrm{b} \leq i^2 = \left(\frac{a+b}{2}\right)^2$$

Cesàro’s equality

Are equilateral triangles the only ones where Cesàro’s equality holds?

```
LocusEquation((a+b)*(a+c)*(b+c)==8*a*b*c, C)
```

```
(252 * x^(10)) + (256 * y^(10)) + ((1276 * x^(2)) * y^(8)) + ((2544 * x^(4)) * y^(6)) + ((2536 * x^(6)) * y^(4)) + ((1264 * x^(8)) * y^(2)) - (1260 * x^(9)) - ((1276 * x) * y^(8)) - ((5088 * x^(3)) * y^(6)) - ((7608 * x^(5)) * y^(4)) - ((5056 * x^(7)) * y^(2)) + (2151 * x^(8)) + (127 * y^(8)) + ((2548 * x^(2)) * y^(6)) + ((6866 * x^(4)) * y^(4)) + ((6596 * x^(6)) * y^(2)) - (1044 * x^(7)) - ((4 * x) * y^(6)) - ((1052 * x^(3)) * y^(4)) - ((2092 * x^(5)) * y^(2)) - (198 * x^(6)) - (254 * y^(6)) - ((738 * x^(2)) * y^(4)) - ((682 * x^(4)) * y^(2)) - (1044 * x^(5)) - ((4 * x) * y^(4)) - ((1048 * x^(3)) * y^(2)) + (2151 * x^(4)) + (127 * y^(4)) + ((2294 * x^(2)) * y^(2)) - (1260 * x^(3)) - ((1276 * x) * y^(2)) + (252 * x^(2)) + (256 * y^(2)) = 0, (252 * x^(10)) + (256 * y^(10)) + ((1276 * x^(2)) * y^(8)) + ((2544 * x^(4)) * y^(6)) + ((2536 * x^(6)) * y^(4)) + ((1264 * x^(8)) * y^(2)) - (1260 * x^(9)) - ((1276 * x) * y^(8)) - ((5088 * x^(3)) * y^(6)) - ((7608 * x^(5)) * y^(4)) - ((5056 * x^(7)) * y^(2)) + (2151 * x^(8)) + (127 * y^(8)) + ((2548 * x^(2)) * y^(6)) + ((6866 * x^(4)) * y^(4)) + ((6596 * x^(6)) * y^(2)) - (1044 * x^(7)) - ((4 * x) * y^(6)) - ((1052 * x^(3)) * y^(4)) - ((2092 * x^(5)) * y^(2)) - (198 * x^(6)) - (254 * y^(6)) - ((738 * x^(2)) * y^(4)) - ((682 * x^(4)) * y^(2)) - (1044 * x^(5)) - ((4 * x) * y^(4)) - ((1048 * x^(3)) * y^(2)) + (2151 * x^(4)) + (127 * y^(4)) + ((2294 * x^(2)) * y^(2)) - (1260 * x^(3)) - ((1276 * x) * y^(2)) + (252 * x^(2)) + (256 * y^(2))
```

Empty graph

Conflict on real points when computing over the complex!!

Cesàro's equality

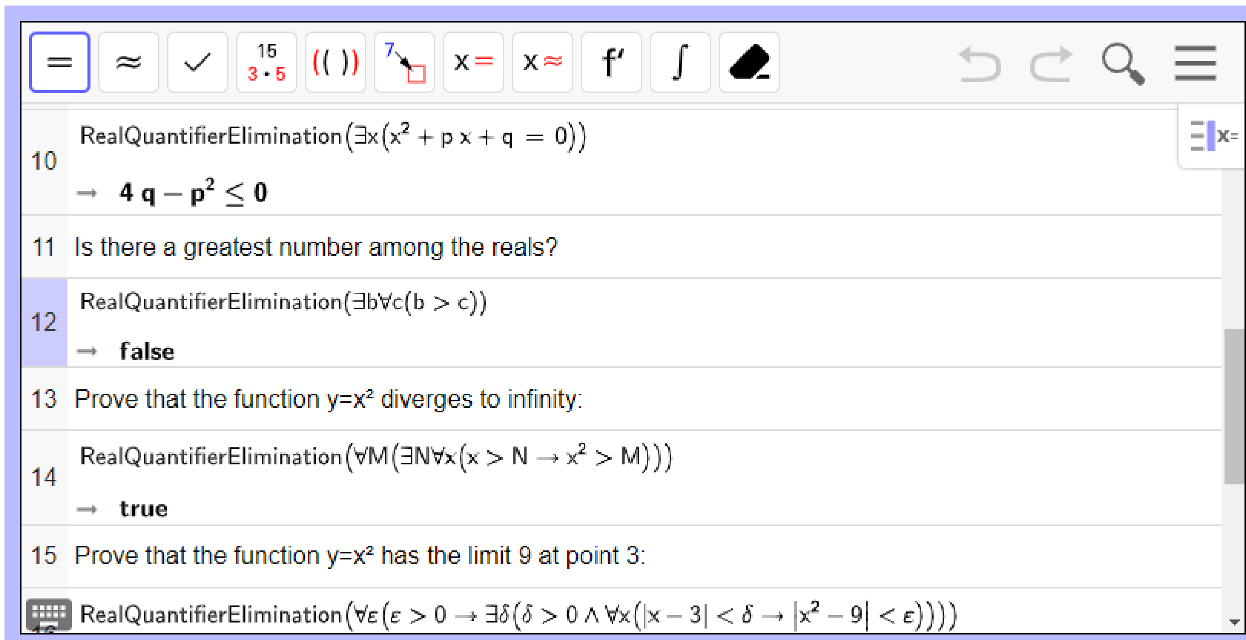
Are equilateral triangles the only where Cesàro's equality holds?

```
LocusEquation((a+b)*(a+c)*(b+c)==8*a*b*c, C)
```

- Points $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$ are in this locus, but are there more points?
- GeoGebra can not answer → Use MAPLE to detect real roots through
    - Discriminants
    - Real root isolation
  - The only points in the curve are $\{(0,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), (1,0)\}$
- Confirmation of Bottema's assertion that Cesaro's inequality holds as an equality only in the equilateral case

RealQuantifierElimination in GeoGebra Discovery

A command proposal: RealQuantifierElimination



Reflection

Gila Hanna and Xiaoheng (Kitty) Yan (2021) **Opening a discussion on teaching proof with automated theorem provers,** For the Learning of Mathematics, Nov. 2021.

GeoGebra's automated proving tools

GeoGebra …has gained in popularity over the last twenty years and is now widely used… GeoGebra has recently added an Automated Reasoning Tool (ART) to help students conjecture … that their conjecture is true. If that is not the …, ART can also help students make the necessary changes to the original conjecture (Hohenwarter, Kovács, & Recio, 2019, p. 216).

It is perhaps too early for empirical studies of classroom experience using the enhancements to GeoGebra… While it is reasonable to expect proof technology to foster students' proving abilities, and there is certainly supporting anecdotal evidence, its potential advantages have not yet been systematically assessed

Conclusions

- ❑ GGb Discovery is a powerful tool to deal with **learning paths to Discovery**
- ❑ GGb Discovery leads us to **new geometric challenges**
- ❑ GGb Discovery gives a very rich context for developing **human reasoning skills**
- ❑ GGb Discovery reveals needed **improvements** in the **user interface** and in **dealing with inequalities** (Real Algebraic Geometry).

But …

- ✓ What is the purpose of developing more and more performing ADG programs?
- ✓ In what context are we interested in having software that finds, e.g. the inequality between the sum of the sides of a triangle and the radius of the circumcircle?

“The key is to make a start, beginning with exploratory studies of the potential of these new tools at both the secondary and post-secondary levels.”

https://en.wikipedia.org/wiki/Gila_Hanna

Thank you!

Gracias!

Tomás Recio, Zoltán Kovács, Francisco Botana, Róbert Vajda, Antonio Montes, Pavel Pech, Philippe Richard, Steven Van Vaerenberg , Noah Dana-Picard, Chris Brown, PV, …