On the locus related to chords of conic sections

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Abstract

The paper deals with the locus of points related to chords of conic sections. Firstly the locus is explored using dynamic geometry software, particularly for displaying it, secondly, using elimination in computer algebra software, the locus equation is derived. However this elimination leads to the zero elimination ideal. It is shown how to compute the searched equation in such a case. Further the locus is applied in the proof of the theorem which is related to the Frgier point. Finally, connection between the original formulation of the locus and the formulation by an envelope is demonstrated.
By solving the problem we mainly use dynamic geometry software GeoGebra and computer algebra program CoCoA and Singular.

1 Introduction

In the paper we investigate the locus of points related to chords of conic sections. Firstly, the locus is explored using dynamic geometry program GeoGebra [3], particularly for displaying it, secondly, using elimination in computer algebra program CoCoA [1] and Singular [6], the locus equation is derived, see Theorem 1. However this elimination leads to the zero elimination ideal. It is shown how to compute the searched locus equation in such a case.

Next, we will show a connection of the Theorem 1 to the theorem on Frégier point [7]. Finally a correspondence between the original formulation of the locus and a formulation by an envelope is discussed, see Theorem 2.
Let us start with a few examples of the locus in various types of conics.

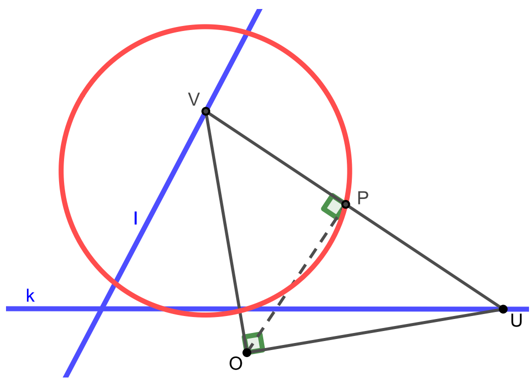

Figure 1: Determine the locus of P when U moves along the line k

Consider two lines k, l, a point U in k and an arbitrary point O. Construct the point V in l such that $OV \perp OU$. Determine the locus of the foot P of the perpendicular from O to the chord UV when U moves along the line k.

To display the locus we use GeoGebra command `Locus`, which works on numerical basis. It seems that the point P lies in a circle, Fig. 1.

Similarly, let us show another example:

Given a hyperbola centred at its centre O and a point U in it. Construct the point V in the hyperbola such that $OV \perp OU$. Determine the locus of the foot P of the perpendicular from O to the chord UV when U moves along the hyperbola.

Using the command `Locus` it seems that the point P lies in a circle as well, see Fig. 2.

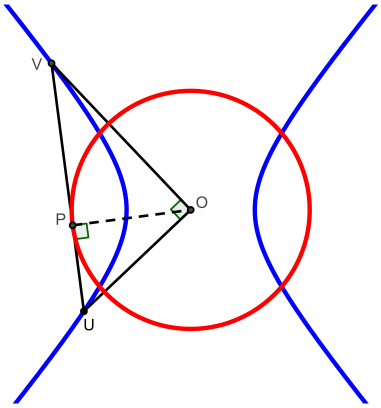

Figure 2: Determine the locus of P when U moves along the hyperbola

The construction is also valid when the point O is not at the centre of a conic. It may happen

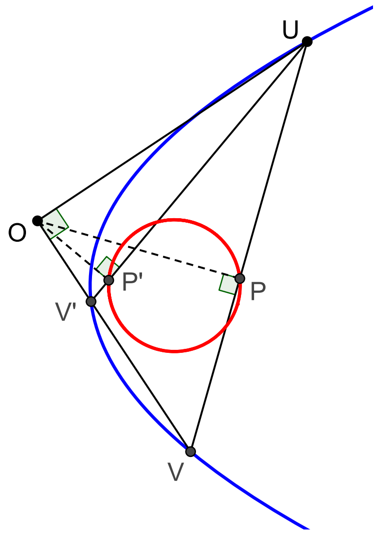

Figure 3: The locus of P is a circle

that sometimes we get only a part of the circle. To get the whole circle, realize that there are two points V, V' in the conic such that $OU \perp OV$ and $OU \perp OV'$. This leads to two points P, P' that together trace the circle, see Fig. 3 in the case of a parabola. This enables us to use this construction for all conics.

In all examples above we get the same locus. In the next section we will formulate the related theorem and prove it.

2 Chord of conics

In accordance with the previous constructions we formulate the theorem:

Theorem 1: *Given a conic κ, a point U in κ and an arbitrary point O. Let V be a point in κ such that $OV \perp OU$. Then the foot P of the perpendicular from O to the line UV when U moves along the conic lies on:*

a) a circle if κ is not an equilateral hyperbola or a pair of mutually orthogonal lines,

b) a line if κ is an equilateral hyperbola or a pair of mutually orthogonal lines.

Proof: We will find the locus equation. Consider a conic

$$\kappa : ax^2 + cy^2 + dx + ey + f = 0. \tag{1}$$

Adopt a rectangular system such that $O = [r, s]$, $U = [u_1, u_2]$, $V = [v_1, v_2]$ and $P = [p, q]$,

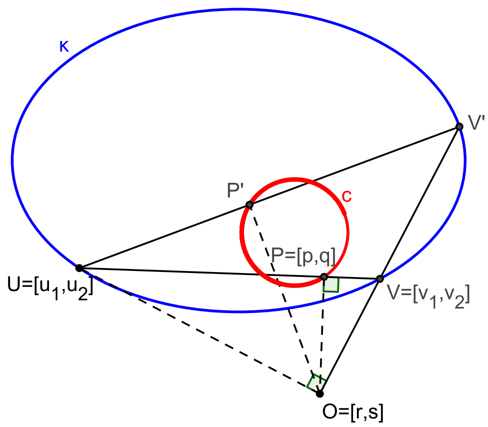

Figure 4: Determine the locus of P when U moves along the conic κ

Fig. 4. Then:

$U \in \kappa \Leftrightarrow h_1 := au_1^2 + cu_2^2 + du_1 + eu_2 + f = 0,$

$V \in \kappa \Leftrightarrow h_2 := av_1^2 + cv_2^2 + dv_1 + ev_2 + f = 0,$

$OV \perp OU \Leftrightarrow h_3 := (u_1 - r)(v_1 - r) + (u_2 - s)(v_2 - s) = 0,$

$OP \perp UV \Leftrightarrow h_4 := (p - r)(u_1 - v_1) + (q - s)(u_2 - v_2) = 0,$

$P \in UV \Leftrightarrow h_5 := pu_2 + u_1v_2 + qv_1 - u_2v_1 - pv_2 - qu_1 = 0.$

Elimination of u_1, u_2, v_1, v_2 in the system $h_1 = 0, h_2 = 0, \dots h_5 = 0$ yields

```
Use R::=Q[a,c,d,e,f,p,q,r,s,u[1..2],v[1..2]];
I:=Ideal(au[1]^2+cu[2]^2+du[1]+eu[2]+f,av[1]^2+cv[2]^2+dv[1]+ev[2]+f,
(u[1]-r)(v[1]-r)+(u[2]-s)(v[2]-s),(p-r)(u[1]-v[1])+(q-s)(u[2]-v[2]),
pu[2]+u[1]v[2]+qv[1]-u[2]v[1]-pv[2]-qu[1]);
Elim(u[1]..v[2],I);
Ideal(0);
```

the zero elimination ideal, see [2]. This could be a problem. I am not sure whether a general solution of this problem (in the case of zero elimination ideal) is known. We could tackle the problem in the following way.

First compute the Hilbert dimension of I (cardinality of the maximal independent set of variables for I) in CoCoA. We get

```
Dim(R/I)=9;
```

But in standard cases we would expect that `Dim(R/I)=8` since we have 13 variables and 5 equations. Then there must be a component of dimension 9 that is degenerate.
Further we will proceed by a heuristic approach. Let us suppose, that $U \neq O$, i.e. $((u_1 - r)^2 + (u_2 - s)^2)t - 1 = 0$, where t is a slack variable. Realize that if $U = O$ then the line OU is not defined. We add this condition to the ideal I and eliminate variables u_1, u_2, v_1, v_2, t. One obtains

$$(a + c)(p^2 + q^2) + (d - 2cr)p + (e - 2as)q + cr^2 + as^2 + f = 0 \qquad (2)$$

which is a desired locus equation. In (2) we distinguish two cases:

a) If $a + c \neq 0$ then the locus is a circle.

b) If $c = -a$, i.e. if κ is an equilateral hyperbola or two orthogonal lines, the locus equation

$$(d - 2cr)p + (e - 2as)q + cr^2 + as^2 + f = 0,$$

represents a line, Fig. 5. □

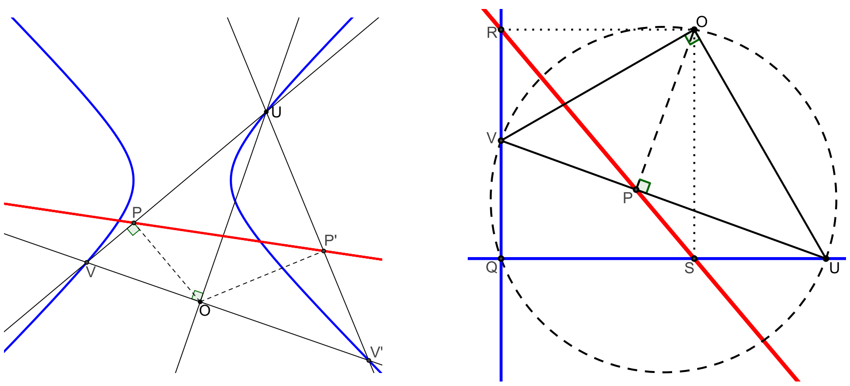

Figure 5: For equilateral hyperbola or two orthogonal lines the locus of the point P is a line

Remark: 1) We could also compute the characteristic series of the system of equations I [5]

using the command `char_series` in Singular [6].

2) One can easily check using the command `NF(I)` (normal form of I) in CoCoA that the product

$((a+c)(p^2+q^2)+(d-2cr)p+(e-2as)q+cr^2+as^2+f)((u_1-r)^2+(u_2-s)^2)$

really belongs to the ideal I, whereas the polynomial in (2) not.

3) The case of two orthogonal lines can also be proved classically. Applying the Simson–Wallace theorem on the triangle QUV and O in its circumcircle, the points P, S, R are collinear (the Simson line), Fig. 5 right.

2.1 Connection to the Frégier's theorem

About in 1815 M. Frégier published the following theorem [7], [9]:

Given a conic κ and a point O on κ, then the hypotenuses of right-angled triangles inscribed to κ and having common right-angle vertex O intersect at one point F, the Frégier point to O with respect to κ.

We will prove the theorem using the Theorem 1 when the point O lies in the conic, Fig. 6. The hypotenuse UV of the right triangle UOV intersects the locus circle c at the points P and F.

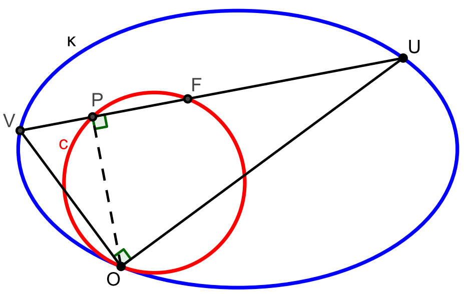

Figure 6: The point F is fixed for all positions of U

To show that the point F is fixed for all positions of U, realize that P lies in the circle c and hence the segment OF must be its diameter. Since O and c are fixed the Frégier theorem follows.

3 Formulation of the locus by an envelope

In this section we arrive at the locus above using envelopes [8]. Let us briefly describe what is the envelope and how to obtain it [4].
The envelope of a one parameter family of curves $F(x, y, t) = 0$, is a curve which is tangent to every curve of the family.
The equation of the family may be given in an implicit form as $F(x, y, t) = 0$, where t is a parameter. To find the equation of the envelope, it is necessary to eliminate the parameter t both from the equation of the family and its partial derivative with respect to the

parameter $\partial F(x,y,t)/\partial t = 0$. This is guaranteed for those points for which $(\partial F(x,y,t)/\partial x)^2 + (\partial F(x,y,t)/\partial y)^2 \neq 0$. If both $\partial F(x,y,t)/\partial x$ and $\partial F(x,y,t)/\partial y$ are zero, then the envelope can have a singular point here.

We show a connection between the above formulation of the locus and its formulation by envelopes in the following Theorem 2, where a conic is for simplicity presented by two lines.

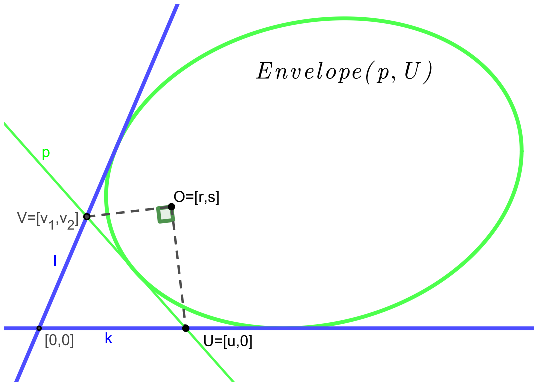

Figure 7: The envelope of lines p when U moves along the line k is a conic

Theorem 2: *Given two lines k, l containing a point U in k and an arbitrary point O. Let V be such a point in l that the lines OV and OU are orthogonal. For U moving along the line k the envelope of the family of lines $p = UV$ is a conic.*

Proof: Choose a rectangular coordinate system such that $k := y = 0$, $l := ax + by = 0$, $O = [r,s]$, $U = [u,0]$, $V = [v_1, v_2]$, Fig. 7. Then:

$V \in l \Leftrightarrow h_1 := av_1 + bv_2 = 0,$

$OV \perp OU \Leftrightarrow h_2 := (r - v_1, s - v_2) \cdot (r - u, s) = 0,$

$X \in UV \Leftrightarrow h_3 := xv_2 + uy - uv_2 - yv_1 = 0.$

Elimination of variables v_1, v_2 in the system $h_1 = 0, h_2 = 0$ and $h_3 = 0$ gives

```
Use R::=Q[a,b,r,s,x,y,u,v[1..2]];
J:=Ideal(av[1]+bv[2],(r-v[1])(r-u)+s(s-v[2]),xv[2+uy-uv[2]-yv[1]);
Elim(v[1..2],J);
```

a one parameter family of lines $p(x,y,u)$ with the parameter u

$p(u) := arxu - ar^2x - as^2x + 2bryu - asyu - br^2y - bs^2y + ar^2u + as^2u - aru^2 - byu^2 = 0.$

Partial derivative of $p(x,y,u)$ with respect to the parameter u yields

$\frac{\partial p}{\partial u} := ar^2 + as^2 + arx + 2bry - asy - 2aru - 2byu = 0.$

Finally, eliminating u in the system $p = 0, \partial p/\partial u = 0$ we get the equation of the envelope of the family of lines $p(x,y,u)$

$a^2r^2x^2 - 2as(ar+2bs)xy + s(a^2s - 4abr - 4b^2s)y^2 - 2a^2r(r^2+s^2)x - 2a^2s(r^2+s^2)y + a^2(r^2+s^2)^2 = 0$

which is a conic, see Fig. 7. Note that the point O is the focus of the conic, as we can compute

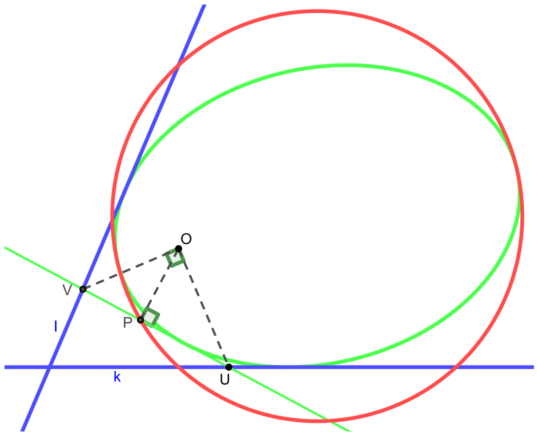

Figure 8: The locus of points P is the director circle of the conic

from the equation above. □

To display the envelope of the family of lines p we can use GeoGebra command `Envelope`, Fig. 7.

Now it is easy to arrive at the circle. It is well-known that feet of perpendiculars from the focus of a conic to all its tangents form the director circle of the conic, see Fig. 8.

In the case when the lines k and l are mutually orthogonal, the envelope is a parabola, with the focus at O. Then the feet of perpendiculars from the focal point O to tangents of the parabola

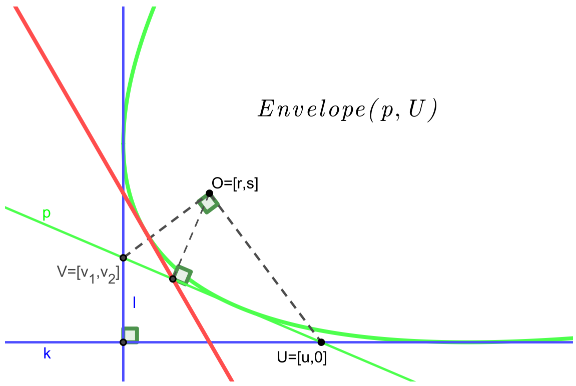

Figure 9: If $k \perp l$ then the envelope of the family of lines p is a parabola

form the directrix line, see Fig. 9.

Similarly, the director circle appears in the case of hyperbola.

4 Conclusions

In the paper locus of points related to chords of conic sections and its properties are described. The locus is explored using both dynamic geometry and computer algebra software. It is

shown how to compute the locus equation in the case when we obtain the zero elimination ideal. Finally, connection of the locus to the envelope of a parametric family of lines and its relation to the Frgier point is given.
There are some questions for a future work. The first one is relating to generalization of the locus construction for an arbitrary angle. The second one relates to the 3D version of the construction.

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