Locus of Antipodal Projection When Fixed Point is Outside a Curve or Surface

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Objectives

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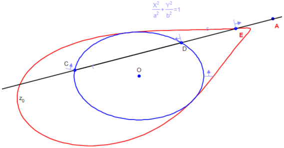
1. A self-contained problem that is easy to understand but leads to explorations touching various fields of mathematics.
2. Encourage students to come up with conjectures through geometric exploration.
3. Encourage students to make use of the existing Dynamic Geometry System (DGS) and Computer Algebra System (CAS) to verify their conjectures.
4. Inspire more students to be interested in mathematics by finding interesting problems with the help of technological tools.

Flash back: Locus of lines passing through a fixed point and a curve

Example

We are given an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and a fixed point A. A line passes through A intersects the circle at C and D respectively, and the point E is such that $\boxed{\overrightarrow{ED} = s\overrightarrow{CD}}$ or written as $\boxed{E = sC + (1-s)D}$. Find the locus E.



Locus surface is a convex linear combination of pseudo antipodal points C and D

Main Problem: We consider the ellipsoid $\Sigma : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the fixed point is $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$; $\rho \to \infty$. A line passes through A and intersects the ~~ellipse~~ at C and D respectively. If the point E lying on CD and satisfying $\overrightarrow{ED} = s\overrightarrow{CD}$ (or written as $E = sC + (1-s)D$.)

We want to find the locus surface

$$\Delta = \left\{ E \in \mathbb{R}^3 | C \in \Sigma \right\}.$$

ellipsoid

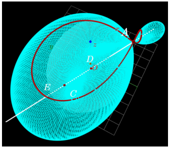
Example

Consider the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

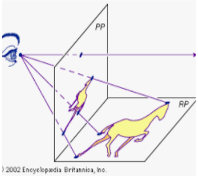
We explore the locus surface for the ellipsoid for the following scenarios

1. Explore with Maple (animation gifs)
2. We plot the surface when $a = 5, b = 4, c = 3, s = 1.7$, and the xed point is at $A = (2, -3, 4)$ together with the locus trace when v is
   xed at 0.81 and u varies between 0 and 2π in Figure 3 as follows.



Locus of an ellipsoid

Images of Projective Geometry (from Google)



projective geometry | Britannica
britannica.com

Projective geometry Ames Room Sli...
slidetodoc.com

Projective Geometry ...
semanticscholar.org

Applications of Projective Geometry
thescipub.com

Some Algebra

- In order to calculate the pseudo antipodal point $D = (x, y, z)$ of C with respective to A, we make use of the parametric equation of line CD as follows:
- We obtain

$$\frac{y - y_0}{x - x_0} = \frac{\hat{y} - y_0}{\hat{x} - x_0}, \tag{1}$$

$$\frac{z - z_0}{x - x_0} = \frac{\hat{z} - z_0}{\hat{x} - x_0}. \tag{2}$$

By substituting the point C into equations (1) and (2), we get some expressions for the left hand side in (1) and (2), allowing us to define two auxiliary functions, namely

$$k(u, v) = \frac{b \sin(u) \sin(v) - y_0}{a \cos(u) \sin(v) - x_0}, \tag{3}$$

$$m(u, v) = \frac{c \cos(v) - z_0}{a \cos(u) \sin(v) - x_0}. \tag{4}$$

Since both intersection points, C and D, satisfy the implicit equation of Σ, we use (3) and (4) to get the x–coordinate of D, say x_1, by calculating the roots of the polynomial

$$p(x) = a_2 x^2 + a_1 x + a_0,$$

where a_0, a_1 and a_2 can be found with help of a computer algebra system. It follows from $p(\hat{x}) = 0$ and the Vieta's formulas that

$$x_1 = -\frac{a_1}{a_2} - \hat{x}.$$

- It follows from (1) and (2) that

$$y_1 = y_0 + k(x_1 - x_0) \quad \text{and} \quad z_1 = z_0 + m(x_1 - x_0).$$

For a given s, the locus surface generated by point $E = sC + (1-s)D$ is defined as

$$\boxed{\Delta(u,v) = \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1-s)x_1 \\ s\hat{y} + (1-s)y_1 \\ s\hat{z} + (1-s)z_1 \end{bmatrix}.}$$

The explicit form of the locus surface Δ can be found but we omit here. We use the following Example to demonstrate how we find the locus for a particular ellipsoid.

Intersecting points or curve are the fixed points of the transformation

Example

Consider the ellipse c

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{5}$$

and let $A = (x_0, y_0)$ be a fixed point "outside" c. The locus curve $\gamma : [0, 2\pi] \rightarrow R^2$ was determined in parametric form. The Figure 1 shows the locus curves (orange) and the original ellipse (blue) for $a = 8,\ b = 6,\ A = (10, 10)$ and $s = 0.75, 1.5,$ and 2.0 respectively.

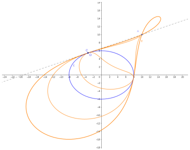
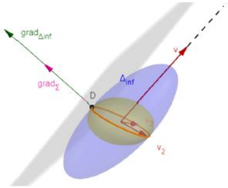
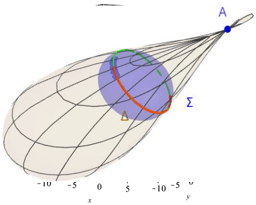


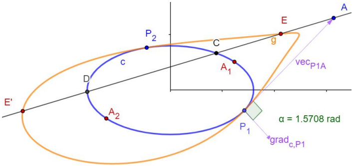
Figure 1. Locus

We remark that the intersecting curve in red will not vary when s varies once the fixed point $A = (x_0, y_0, z_0)$ is fixed. We use [S2] for exploration for this observation.





Intersecting curve is invariant when s changes-animated gif



Intersecting curve for an ellipsoid case

We want to find the tangent plane T at a point P on the ellipsoid $C = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$ such that T is passing through the fixed point $A = (x_0, y_0, z_0)$. If the ellipsoid is the level surface of $\boxed{F(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0}$. Then the gradient at a point of the ellipsoid is $\nabla F(x,y,z) = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)$, then we see the tangent plane as follows:

$$\boxed{T(x,y,z) = \nabla F(x,y,z) \cdot (x - x_0, y - y_0, z - z_0) = 0,} \tag{6}$$

We thus solve $F(x,y,z) = 0$ and $T(x,y,z) = 0$ for the variables x, y and with the help of [2], We use $a = 5, b = 4, c = 3, x_0 = 7, y_0 = 8, z_0 = 9$ and plot the intersecting curve together with the ellipsoid and its locus surface when $s = 3$ in Figure 2..

Locus of a sphere is an invariant ellipsoid when A is at an infinity

1. We let Σ be the sphere $x^2 + y^2 + z^2 = r^2$, and let the fixed point $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$ be on $S_1 : x^2 + y^2 + z^2 = \rho^2$ with $\rho \neq r$ and $\rho < \infty$. Because Σ is symmetric with respect to the origin and in view of preceding exploration with the locus for ellipsoid, it is natural to expect the shape of the locus for Σ stays unchanged and is coordinate free. Specifically, if we move the fixed points $A_1, A_2, ..., A_n \in S_1$ sequentially:

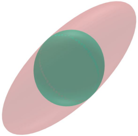
$$A_1 \to A_2 \to ... \to A_n \tag{7}$$

3 min

with $A_n = A$. Then Δ_i, the locus surface of Σ with respect to A_i, for $i = 1, 2, ...n$, moves sequentially

Not true if we start with an ellipsoid

$$\Delta_1 \to \Delta_2 \to ... \to \Delta_n, \tag{8}$$



and we would expect that $\Delta_n = \Delta$.

Locus when the fixed point A is at an infinity

Let Σ be the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The explicit expressions for D_{inf}, E_{inf} and $\Delta_{\text{inf}}(s, u_0, v_0)$ are calculated in Exploration [S5]. See [S6] for dynamic explorations. We depict the locus surface (blue) when $s = 2$, $a = 5$, $b = 4$, $c = 3$, with $u_0 = \frac{\pi}{3}$, $v_0 = \frac{\pi}{4}$ and $\rho \longrightarrow \infty$ together with the original ellipsoid (yellow) in Figure 6.

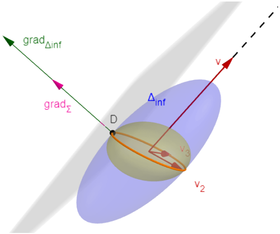


Figure 6. Locus ellipsoid when $s = 2$

Meaning of eigenspace corresponds to the eigenvalue of -1

Theorem

Let Σ *be an ellipsoid in the standard form,* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, *and let* $A_{\text{inf}}(u_0, v_0)$ *be the fixed point at infinity in the direction of* $(\cos u_0 \sin v_0, \sin u_0 \sin v_0, \cos v_0)$, $C \in \Sigma$ *and* D_{inf} *be the "antipodal" point of* C *corresponding to* $A_{\text{inf}}(u_0, v_0)$, *and let* $E_{\text{inf}} = sC + (1-s)D_{\text{inf}}$. *Then for a given* $s > 0$, *there exists a matrix* $L_D^e = \left[l_{ij}^e\right]_{3\times 3}$ *such that*

$$L_D^e\, C = D_{\text{inf}}.$$

The L_D^e *has the eigenvalue* $\mu_1 = -1$ *of multiplicity* 1 *with associated eigenvector* $\overrightarrow{v_1} = \left[\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1\right]^t$, *we see*

$$\boxed{L_D^e\left(r\,\overrightarrow{v_1}\right) = -r\,\overrightarrow{v_1}.}$$

So if C_1 is a point on the ellipsoid in the direction of $\overrightarrow{v_1}$, $L_D^e\left(C_1\right) = -C_1$.

Meaning of eigenspace corresponds to the eigenvalue of 1

In addition, L_D^e has the eigenvalue $\mu_2 = 1$ of multiplicity 2 with the following associated eigenvectors $\overrightarrow{v_2} = \left[\frac{-a^2}{c^2} \sec u_0 \cot v_0, 0, 1\right]^t$, and $\overrightarrow{v_3} = \left[-\frac{a^2}{b^2} \tan u_0, 1, 0\right]^t$.

$$\boxed{L_D^e\left(r\overrightarrow{v_2} + s\overrightarrow{v_3}\right) = rL_D^e\left(\overrightarrow{v_2}\right) + sL_D^e\left(\overrightarrow{v_3}\right) = r\overrightarrow{v_2} + s\overrightarrow{v_3}.}$$

In other words, **the vectors lying on the intersecting elliptical disk, spanned by $\overrightarrow{v_2}$ and $\overrightarrow{v_3}$ are fixed under** L_D^e.

2D Graphical Representation

3D Graphical Representation

Principal Axis Theorem

1. $\Sigma : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $C \in \Sigma$. In matrix quadratic form: $[x\ y\ z\ 1] Q_\Sigma [x\ y\ z\ 1]^t = 0$.
2. We want to find the directions and lengths of the axes for the locus $\boxed{\Delta = L_E^e(C)}$.
3. We consider the implicit equation form as follows:

$$\boxed{(X)\, Q_\Delta^e\, (X)^T = 0,}$$

where $X = [x\ y\ z\ 1]$ and

$$Q_\Delta^e = \begin{bmatrix} l_{11}^e & l_{21}^e & l_{31}^e & 0 \\ l_{12}^e & l_{22}^e & l_{32}^e & 0 \\ l_{13}^e & l_{23}^e & l_{33}^e & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b^2c^2 & 0 & 0 & 0 \\ 0 & a^2c^2 & 0 & 0 \\ 0 & 0 & a^2b^2 & 0 \\ 0 & 0 & 0 & -a^2b^2c^2 \end{bmatrix} \begin{bmatrix} l_{11}^e & l_{12}^e & l_{13}^e & 0 \\ l_{21}^e & l_{22}^e & l_{23}^e & 0 \\ l_{31}^e & l_{32}^e & l_{33}^e & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1}.$$

CAS, DGS and numerical software

Locus surface when the scaling factor s is large

Example

Consider the our locus surface for the ellipsoid Σ

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

with $s = 20$, $a = 5$, $b = 4$, $c = 3$ and the fixed point is at the infinity when $u_0 = \frac{\pi}{3}$, $v_0 = \frac{\pi}{4}$. The numerical approximation of Q_Δ^e is

$$\begin{bmatrix} 135.4334986093818 & -23.18377445571889 & -47.5916743923231 & 0 \\ -23.18377445571889 & 162.2570698929334 & -128.7987469762161 & 0 \\ -47.5916743923231 & -128.7987469762161 & 135.6018089315383 & 0 \\ 0 & 0 & 0 & -3600 \end{bmatrix}$$

The equation $XQ_\Delta^e X^T = 0$ of the locus ellipsoid becomes
$135.4334986x^2 - 46.36754892xy - 95.18334882xz + 162.2570699y^2 - 257.597494yz + 135.6018089z^2 - 3600 = 0.$

1. Eigenvalues are $\lambda_1 = 0.1987812414007027$, $\lambda_2 = 153.0821898482854$, $\lambda_3 = 280.0114063441675$, and $\lambda_4 = -3600$ respectively.
2. The corresponding unit eigenvectors are written as column vectors respectively below

$$\begin{bmatrix} 0.3537757151666001 & -0.9290517690585807 & -0.1081922075173726 & 0 \\ 0.6124527321391638 & 0.3175226342422298 & -0.7239344083818289 & 0 \\ 0.7069260175249132 & 0.1898477999688088 & 0.6813320912692791 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} w1 & w2 & w3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

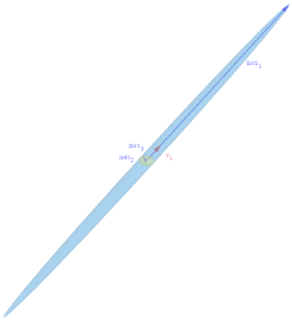
Apply the Principal Axis Theorem

1. Using the rotated system of coordinates $\tilde{x}$–$\tilde{y}$–$\tilde{z}$ determined by eigenvectors w_1, w_2, w_3, we got the equation of the locus ellipsoid in standard form $\lambda_1\tilde{x}^2 + \lambda_2\tilde{y}^2 + \lambda_3\tilde{z}^2 = -\lambda_4$, or

$$\frac{\tilde{x}^2}{\left(\sqrt{\frac{-\lambda_4}{\lambda_1}}\right)^2} + \frac{\tilde{y}^2}{\left(\sqrt{\frac{-\lambda_4}{\lambda_2}}\right)^2} + \frac{\tilde{z}^2}{\left(\sqrt{\frac{-\lambda_4}{\lambda_3}}\right)^2} = 1.$$

Note that $\boxed{\sqrt{\frac{-\lambda_4}{\lambda_1}}} = 134.5747405338178 > \boxed{\sqrt{\frac{-\lambda_4}{\lambda_2}}} = 4.849410151798864 > \boxed{\sqrt{\frac{-\lambda_4}{\lambda_3}}} = 3.585612795294109$

2. It follows from the direct calculations using Maxima that the angle between w_1 and eigenvector $v_1 = [\cos u_0 \tan v_0, \sin u_0 \tan v_0, 1]^t$ for L_E^e. is equal to 0.0002975755987288942 radians. It is expected that the eigenvector for the locus ellipsoid w_1 is close to v_1 when s is increased.



GGB-2 min

Locus ellipsoid when $s = 20, a = 5, b = 4, c = 3, u_0 = \frac{\pi}{3}$ and $v_0 = \frac{\pi}{4}$

Locus surfaces in the limit as the parameter s goes to infinity

It is expected that it is too complicated for a CAS to compute the eigenvectors for Q^e_Δdirectly if we let $s \longrightarrow \infty$.Instead, we take $s \longrightarrow \infty$for each entry of the matrix $Q^e_\Delta = \begin{bmatrix} A & B/2 & C/2 & 0 \\ B/2 & D & E/2 & 0 \\ C/2 & E/2 & F & 0 \\ 0 & 0 & 0 & J \end{bmatrix}$. We can see that the eigenvector $X = \begin{bmatrix} \boxed{\begin{matrix} \cos u_0 \tan v_0 \\ \sin u_0 \tan v_0 \\ 1 \end{matrix}} \\ 0 \end{bmatrix}$ corresponds to the $\boxed{\text{eigenvalue } 0}$ for Q^e_Δ.

1 min

Since we have $\boxed{(Q^e_\Delta)' X = 0}$,this transformation $\boxed{\text{sends the vector } X \text{ to}}$ $\boxed{\text{the origin}}$, and consequently the locus surface becomes an open ended elliptical cylinder.

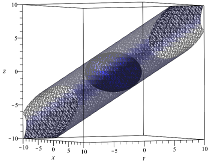


Figure 5(b). The ellipsoid and its locus are tangent to each other

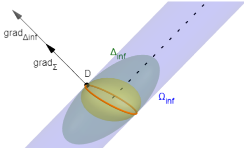


Figure 5(c). Ellipsoid and when locus surfaces $s \longrightarrow \infty$

1.5 min

Our geometric construction sends hyperboloids with two sheets to another hyperboloids with two sheets

Since $L_D^e : C \longrightarrow D$ is linear

Theorem

If $s \in \mathbb{R}^+ \backslash \{1/2\}$, the locus surface $\Delta_{\inf}(s, u_0, v_0)$ is also a hyperboloid of two sheets.

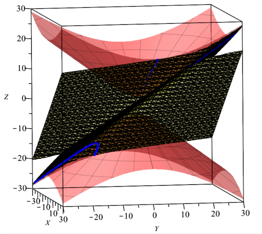
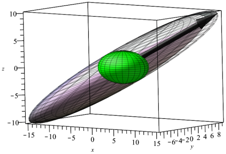


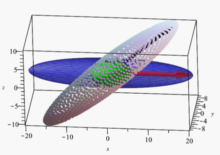
Figure. Hyperboloid with two sheets and its locus

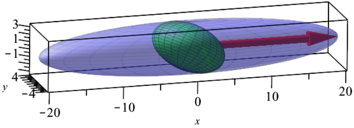
1 min

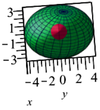
Future works: An excellent project showing why linear algebra is important

1. The green ellipsoid Σ_1 is given in a standard form), we apply the linear transformation on ellipsoid Σ_1 to get the pink ellipsoid locus Δ_1. Note the green is tangent to pink.
2. When we rotate the ellipsoid locus Δ_1 to the blue one in standard form, Δ_2, we want to find the corresponding ellipsoid Σ_2 that is inside Δ_2 and tangent to Δ_2.

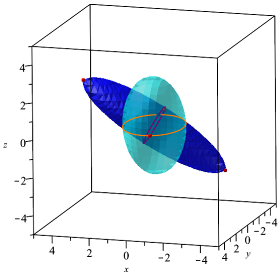




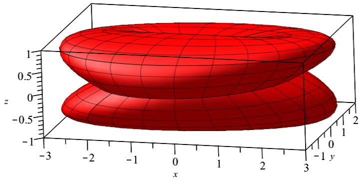




SVD Decompositions for the locus surfaces of ellipsoids



Topology: If an ellipsoid Σ is represented by $\begin{bmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{bmatrix}$ and we randomly select $\alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{3}, \alpha_3 = \frac{\pi}{6}, \beta_1 = \frac{\pi}{2}, \beta_2 = \frac{\pi}{3}$ and $\beta_3 = \frac{\pi}{4}$, then the surface $\Sigma_1 = \begin{bmatrix} f(u+\alpha_1, v+\beta_1) \\ g(u+\alpha_2, v+\beta_2) \\ h(u+\alpha_3, v+\beta_3) \end{bmatrix}$ can be seen below, which is a continuous deformation from the original ellipsoid. Therefore, this surface is topologically equivalent to the ellipsoid.



Conclusions

1. Technological Tools can *assist researchers and students for exploring new ideas*.
2. Provide us crucial intuitions before proving analytical results with a computer algebra system.
3. Evolving technological tools can link mathematics to cross-disciplinary subjects effectively.
4. People (including kids) ask why they need to learn mathematics when they have CAS and DGS.
   4.1 Connect proper contents (using technological tools for exploration) to real-life project, it will encourage greater interests in mathematics for learners.
   4.2 Challenge problems beyond the reach of pencil-and-paper with the help of CAS and DGS
   4.3 Everyone can expand their horizon of mathematics knowledge to a next level when exploring with technologies.
   4.4 Learn to explore mathematics with technologies and discover mathematics.

- Geometry Expressions, see http://www.geometryexpressions.com/.
- GeoGebra (release 6.0.562 / October 2019), see https://www.geogebra.org/.
- D. Hilbert S. Cohn-Vossen,'Geometry and the Imagination', ISBN 978-0828400879, Chelsea Publishing Company, 1st edition, January 1, 1952.
- Maple, A product of Maplesoft, see http://Maplesoft.com/, version Maple 2021.1.
- Maxima (release 5.43.0 / May 2019), see http://maxima.sourceforge.net/.
- Principal axis theorem: https://en.wikipedia.org/wiki/Principal_axis_theorem.
- Singular-Value-Decomposition: https://commons.wikimedia.org/wiki/File:Singular-Value-Decomposition.svg

- Quadric surface: https://mathworld.wolfram.com/QuadraticSurface.html.
- Yang, W.-C. & Morante, A., 3D Locus Problems of Lines Passing Through A Fixed Point, the Electronic Journal of Mathematics and Technology, ISSN 1933-2823, Volume 15, Number 1, 2021. Published by Mathematics and Technology, LLC.
- Yang, W.-C., Exploring Locus Surfaces Involving Pseudo Antipodal Point, the Electronic Proceedings of the 25th Asian Technology Conference in Mathematics, Published by Mathematics and Technology, LLC, ISSN 1940-4204 (online version), see https://atcm.mathandtech.org/EP2020/abstracts.html#21829.
- Yang, W.-C. & Morante, A., **Locus Surfaces and Linear Transformations when Fixed Point is at an Infinity**, to appear at the eJMT February 2022 issue.