Locus of Antipodal Projection When Fixed Point is Outside a Curve or Surface

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Abstract

We continue the investigations regarding the locus problems for certain curves and quadric surfaces, which are discussed in [6] and [7] in 2D and 3D cases. We explore the intersecting curves when the fixed point A is outside the surface and when A is at an infinity. The discoveries from explorations with the help of technological tools in this paper will assist learners to conduct further research in the area of projective geometry and beyond.

1 Introduction

In *[7]*, we considered the following:

Original problem: *We are given a fixed point A and a generic point C on a specified curve or surface Σ such that the line l passes through A and C and intersects a well-defined D on Σ, we want to determine the locus curve or locus surface $\triangle$ generated by the point E, lying on CD, which satisfies $\overrightarrow{ED} = s\overrightarrow{CD}$, where s is a real number parameter.*

We remark that the original locus problem leads to many interesting projects thanks to several parameters that need to be taken care of. The location of a fixed point A certainly determines the locus once the original surface Σ is chosen. The parameter s determines size of the locus surface $\triangle$. As we discussed in *[7]*, the 3D surface Σ we consider in this paper is either an ellipsoid or a hyperboloid with two sheets. In this paper, we discuss how the locus surface $\triangle$ will behave when point A is outside the specified surface Σ. Furthermore, since the location of the fixed point A will determine the locus surface, we shall distinguish the case when A is either at an infinity or not. We recall that we calculated the exact expression for the antipodal point D corresponding to the point C in *[7]*. In Section 2, we remind readers how we apply the Vieta formula to find the locus surface. In Section 3, we discuss the scenario, when A is outside a specified surface Σ but not at an infinity, how we can find the intersecting curves between Σ and its locus surface $\triangle$. It is interesting to note that once the fixed point A is chosen, the intersecting points or curve stays fixed regardless of the parameter s. In Section 4, we explore the locus surfaces when A is at an infinity. In such case, the point D_{inf} can be viewed as the "limit" of point D when A goes radially to infinity, or as the projection of C along the line l through the point A "fixed at infinity". The explorations lead to linear transformations involve further discussions in *[8]*.

2 Generic methodology to find locus surface

If Σ is the quadric surface $F(x, y, z) = 0$ we recall from *[7]* how we find the locus surface when the fixed point $A = (x_0, y_0, z_0)$ does not go to infinity. We represent a generic point on Σ as

$$C = \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \tag{1}$$

In order to calculate the coordinates of point $D = (x, y, z)$ (which is different from C), as the intersection between the quadric Σ and the line l passing through A and C, we make use of the parametric equation of line l as follows:

$$\begin{aligned} x - x_0 &= \lambda(\hat{x} - x_0), \\ y - y_0 &= \lambda(\hat{y} - y_0), \\ z - z_0 &= \lambda(\hat{z} - z_0). \end{aligned}$$

Hence, we obtain

$$\frac{y - y_0}{x - x_0} = \frac{\hat{y} - y_0}{\hat{x} - x_0}, \tag{2}$$

$$\frac{z - z_0}{x - x_0} = \frac{\hat{z} - z_0}{\hat{x} - x_0}. \tag{3}$$

Then we define two auxiliary functions, namely

$$k \doteq k(\hat{x}, \hat{y}) = \frac{\hat{y} - y_0}{\hat{x} - x_0} \tag{4}$$

$$m \doteq m(\hat{x}, \hat{y}) = \frac{\hat{z} - z_0}{\hat{x} - x_0} \tag{5}$$

Since both intersection points, C and D, satisfy the implicit equation of Σ, we can use (4) and (5) to get the x–coordinate of D, say x_1, by calculating the roots of the polynomial

$$p(x) = a_2 x^2 + a_1 x + a_0,$$

It follows from $p(\hat{x}) = 0$ and the Vieta's formulas that

$$x_1 = -\frac{a_1}{a_2} - \hat{x}.$$

It follows from (2) and (3) that

$$y_1 = y_0 + k(x_1 - x_0) \quad \text{and} \quad z_1 = z_0 + m(x_1 - x_0).$$

For a given s, the locus surface generated by point $E = sC + (1-s)D$ is defined as

$$\Delta(x_0, y_0, z_0) = \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1-s)x_1 \\ s\hat{y} + (1-s)y_1 \\ s\hat{z} + (1-s)z_1 \end{bmatrix}.$$

We remark that once the fixed point A is chosen, since A and C together determine the point E, the locus surface is thus fixed too.

3 Motivations

In this section, we shall explore how various factors such as, the original surface Σ, the fixed point A, and the scalar s will affect a locus surface $\Delta(\Sigma, A, s)$. For example, we may investigate the following scenarios:

1. How will the radius ρ of the fixed point $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$ affect a locus surface?

2. How will the angle (u_0, v_0) of the fixed point $A = (\rho \cos u_0 \sin v_0, \rho \sin u_0 \sin v_0, \rho \cos v_0)$ affect a locus surface?

3. How will the parameter $s > 1$ affect a locus surface?

We first use the following two dimensional ellipse to motivate our findings:

3.1 The ellipse case when the fixed point A is not at infinity

Consider the ellipse c

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{6}$$

and let $A = (x_0, y_0)$ be a fixed point "outside" c. In *[6]*, the locus curve $\gamma : [0, 2\pi] \to \mathbf{R}^2$ was determined in parametric form as

$$\gamma(t) = \begin{bmatrix} s\,a\cos(t) + (1-s)\frac{a^3 y_0^2 \cos(t) - 2a^2 b\, x_0 y_0 \sin(t) - a\, b^2 x_0^2 \cos(t) + 2a^2 b^2 x_0 - a^3 b^2 \cos(t)}{a^2 y_0^2 - 2a^2 b\, y_0 \sin(t) + b^2 x_0^2 - 2a\, b^2 x_0 \cos(t) + a^2 b^2} \\ \\ s\,b\sin(t) + (1-s)\frac{a^2 b\, y_0^2 \sin(t) + (2a\, b^2 x_0 \cos(t) - 2a^2 b^2) y_0 - b^3 x_0^2 \sin(t) + a^2 b^3 \sin(t)}{a^2 y_0^2 - 2a^2 b\, y_0 \sin(t) + b^2 x_0^2 - 2a\, b^2 x_0 \cos(t) + a^2 b^2} \end{bmatrix}$$

The Figure 1 shows the locus curves (orange) and the original ellipse (blue) for $a = 8,\ b = 6$, $A = (10, 10)$ and $s = 0.75, 1.5$, and 2.0 respectively.

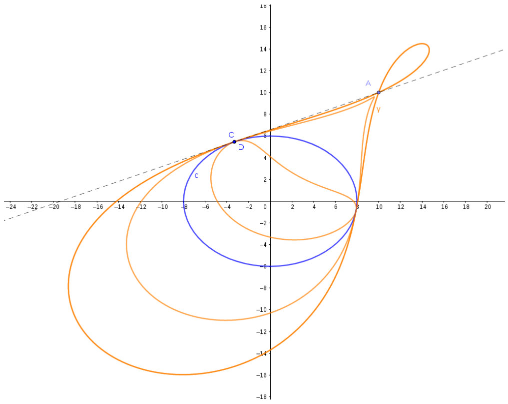

Figure 1. Locus curves when $s = 0.75, 1.5$ and 2.0 respectively.

With the help of *[1]* we see that, for any $s > 1$, the corresponding locus curve intersects tangentially the ellipse at the same points, say P_1 and P_2. Consider now the following experiment: construct a point C on the ellipse and draw the corresponding tangent line, using for example the command `Tangent(Point,Conic)`. Turns out that the tangent lines to the ellipse at points P_1 and P_2 will contain the point A (this is what is expected from the geometric construction of the locus curve generated by point $E = sC + (1 - s)D$, because when $D = C$, we see $E = C$.

The intersecting points does not depend on the parameter s; the idea is that intersecting points are those that are equal to their "antipodal" points and are equal to the points of tangency of the lines from A to the ellipse. The Figure 2 below shows the construction when we drag C on P_1.

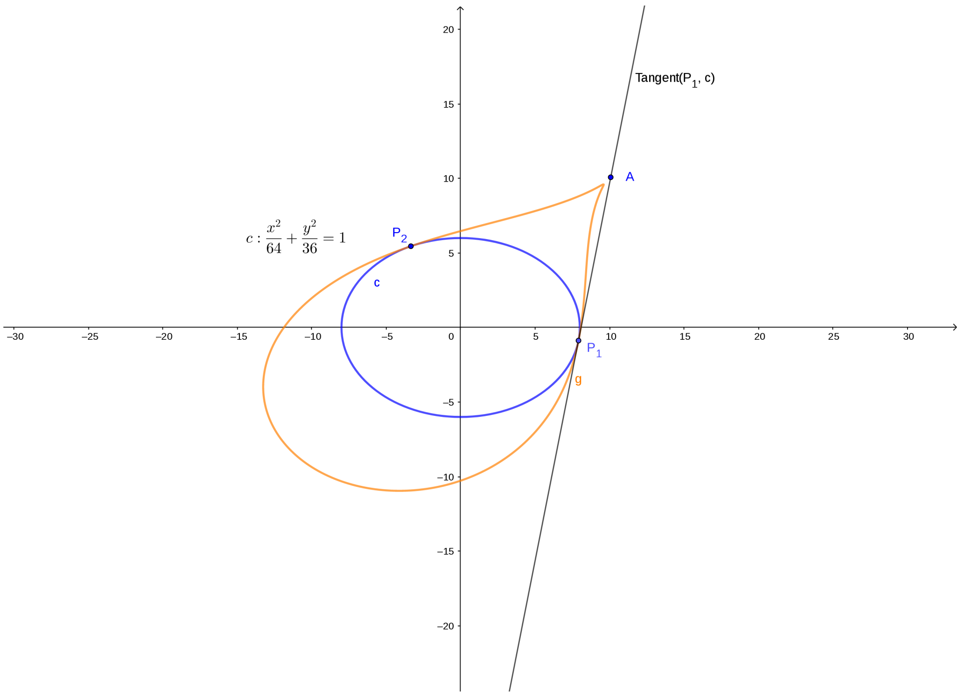

Figure 2. Tangent line for the ellipse at intersecting point P_1.

Therefore, when the fixed point A is given, the problem of finding the tangency points of the ellipse with any locus curve reduces to the simple problem to find the tangent lines passing through point A. We describe the procedure as follows:

1. We set the Eq1 as the equation the ellipse c :
$$F(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \tag{7}$$
2. We calculate the gradient of $F(x,y)$,
$$\nabla F(x,y)=\left(\frac{2x}{a^2},\frac{2y}{b^2}\right) \tag{8}$$
3. For generic point $P(x,y)$, we consider the vector $\overrightarrow{PA}=(x-x_0,y-y_0)$.
4. The condition of $\nabla F(x,y)\perp\ \overrightarrow{PA}$ yields to the following:
$$2x\frac{x-x_0}{a^2}+2y\frac{y-y_0}{b^2}=0. \tag{9}$$
5. We solve, using *[1]* (see [S1]), Eq1 and Eq2 for x and y to get the intersecting points,
$$P_1=\left(\frac{a^2y_0\sqrt{a^2y_0^2+b^2x_0^2-a^2b^2}+a^2b^2x_0}{a^2y_0^2+b^2x_0^2},-\frac{b^2x_0\sqrt{a^2y_0^2+b^2x_0^2-a^2b^2}-a^2b^2y_0}{a^2y_0^2+b^2x_0^2}\right)$$
and
$$P_2=\left(-\frac{a^2y_0\sqrt{a^2y_0^2+b^2x_0^2-a^2b^2}-a^2b^2x_0}{a^2y_0^2+b^2x_0^2},\frac{b^2x_0\sqrt{a^2y_0^2+b^2x_0^2-a^2b^2}+a^2b^2y_0}{a^2y_0^2+b^2x_0^2}\right)$$

3.2 The ellipsoid case when fixed point A is not at infinity

We want to find the tangent plane T at a point P on the ellipsoid $C=\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix}$ such that T is passing through the fixed point $A=(x_0,y_0,z_0)$. If the ellipsoid is the level surface of $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$. Then the gradient at a point of the ellipsoid is $\nabla F(x,y,z)=\left(\frac{2x}{a^2},\frac{2y}{b^2},\frac{2z}{c^2}\right)$, then we see the tangent plane as follows:
$$T(x,y,z)=\nabla F(x,y,z)\cdot(x-x_0,y-y_0,z-z_0)=0, \tag{10}$$
We thus solve $F(x,y,z)=0$ and $T(x,y,z)=0$ for the variables x, y and with the help of *[7]*, we obtain the followings:
$$x = -\frac{a^2\left(RootOf\left(\begin{array}{c}\left(a^2y0^2+b^2x0^2\right)\_Z^2+a^2c^4-2\,a^2c^2zz0\\ +a^2z^2z0^2-c^4x0^2+c^2x0^2z^2\\ +\left(-2\,a^2c^2y0+2\,a^2y0\,zz0\right)\_Z\end{array}\right)y0-c^2+zz0\right)}{c^2x0}, \tag{11}$$
$$y = \frac{RootOf\left(\begin{array}{c}\left(a^2y0^2+b^2x0^2\right)\_Z^2+a^2c^4-2\,a^2c^2zz0\\ +a^2z^2z0^2-c^4x0^2+c^2x0^2z^2\\ +\left(-2\,a^2c^2y0+2\,a^2y0\,zz0\right)\_Z\end{array}\right)b^2}{c^2} \tag{12}$$

Consequently, we have two branches for each of the respective variables x and y. We list them accordingly below:

$$
\begin{aligned}
x_1 &= \frac{a^2}{c^2 x_0 \left(a^2 {y_0}^2 + b^2 {x_0}^2\right)} \left(- \sqrt{-c^2 {x_0}^2 \begin{pmatrix} \left(\left(a^2 {y_0}^2 + b^2 {x_0}^2\right) c^2 + a^2 b^2 {z_0}^2\right) \\ \left(\cos\left(t\right)\right)^2 - 2\, a^2 c z_0 \cos\left(t\right) b^2 \\ + c^2 \left(\left(b^2 - {y_0}^2\right) a^2 - b^2 {x_0}^2\right) \end{pmatrix}} y_0 \right. \\
&\qquad \left. + b^2 c {x_0}^2 \left(-\cos\left(t\right) z_0 + c\right) \right) \\
x_2 &= \frac{a^2}{c^2 x_0 \left(a^2 {y_0}^2 + b^2 {x_0}^2\right)} \left(\sqrt{-c^2 {x_0}^2 \begin{pmatrix} \left(\left(a^2 {y_0}^2 + b^2 {x_0}^2\right) c^2 + a^2 b^2 {z_0}^2\right) \\ \left(\cos\left(t\right)\right)^2 - 2\, a^2 c z_0 \cos\left(t\right) b^2 \\ + c^2 \left(\left(b^2 - {y_0}^2\right) a^2 - b^2 {x_0}^2\right) \end{pmatrix}} y_0 \right. \\
&\qquad \left. + b^2 c {x_0}^2 \left(-\cos\left(t\right) z_0 + c\right) \right) \\
y_1 &= \frac{b^2}{\left(a^2 {y_0}^2 + b^2 {x_0}^2\right) c^2} \left(\sqrt{-c^2 {x_0}^2 \begin{pmatrix} \left(\left(a^2 {y_0}^2 + b^2 {x_0}^2\right) c^2 + a^2 b^2 {z_0}^2\right) \\ \left(\cos\left(t\right)\right)^2 - 2\, a^2 c z_0 \cos\left(t\right) b^2 \\ + c^2 \left(\left(b^2 - {y_0}^2\right) a^2 - b^2 {x_0}^2\right) \end{pmatrix}} \right. \\
&\qquad \left. + a^2 c y_0 \left(-\cos\left(t\right) z_0 + c\right) \right) \\
y_2 &= \frac{b^2}{\left(a^2 {y_0}^2 + b^2 {x_0}^2\right) c^2} \left(- \sqrt{-c^2 {x_0}^2 \begin{pmatrix} \left(\left(a^2 {y_0}^2 + b^2 {x_0}^2\right) c^2 + a^2 b^2 {z_0}^2\right) \\ \left(\cos\left(t\right)\right)^2 - 2\, a^2 c z_0 \cos\left(t\right) b^2 \\ + c^2 \left(\left(b^2 - {y_0}^2\right) a^2 - b^2 {x_0}^2\right) \end{pmatrix}} \right. \\
&\qquad \left. + a^2 c y_0 \left(-\cos\left(t\right) z_0 + c\right) \right)
\end{aligned}
$$

Accordingly, we have two branches of the space curves, which we describe them, respectively, as follows:

1. $r_1(t) = \left(x_1^*(t), y_1^*(t), z^*(t)\right)$, where $z^*(t) = c \cos t$, and $x_1^*(t), y_1^*(t)$ are shown below respectively. First, we let

$$
\delta = \sqrt{-c^2 {x_0}^2 \begin{pmatrix} \left(c^2 \left(a^2 {y_0}^2 + b^2 {x_0}^2\right) + a^2 b^2 {z_0}^2\right) \left(\cos\left(t\right)\right)^2 \\ -2\, b^2 c a^2 \cos\left(t\right) z_0 + c^2 \left(\left(b^2 - {y_0}^2\right) a^2 - b^2 {x_0}^2\right) \end{pmatrix}} \tag{13}
$$

$$
x_1^*(t) = \frac{-a^2}{c^2 x_0 \left(a^2 {y_0}^2 + b^2 {x_0}^2\right)} \left(\delta y_0 + b^2 c {x_0}^2 \left(-\cos\left(t\right) z_0 + c\right)\right), \tag{14}
$$

$$
y_1^*(t) = \frac{b^2}{c^2 \left(a^2 {y_0}^2 + b^2 {x_0}^2\right)} \left(\delta + a^2 c y_0 \left(-\cos\left(t\right) z_0 + c\right)\right). \tag{15}
$$

2. $r_2(t) = \left(x_2^*(t), y_2^*(t), z^*(t)\right)$, where $z^*(t) = c \cos t$, and $x_2^*(t), y_2^*(t)$ can be shown below respectively:

$$
x_2^*(t) = \frac{a^2}{c^2 x_0 \left(a^2 {y_0}^2 + b^2 {x_0}^2\right)} \left(\delta y_0 + b^2 c {x_0}^2 \left(-\cos\left(t\right) z_0 + c\right)\right), \tag{16}
$$

$$
y_2^*(t) = \frac{-b^2}{c^2 \left(a^2 {y_0}^2 + b^2 {x_0}^2\right)} \left(\delta + a^2 c y_0 \left(-\cos\left(t\right) z_0 + c\right)\right). \tag{17}
$$

We use $a = 5, b = 4, c = 3, x_0 = 7, y_0 = 8, z_0 = 9$ and plot the intersecting curve together with the ellipsoid and its locus surface when $s = 3$ in Figure 3. We remark that the intersecting curve in red will not vary when s varies once the fixed point $A = (x_0, y_0, z_0)$ is fixed. We use [S2] for exploration for this observation.

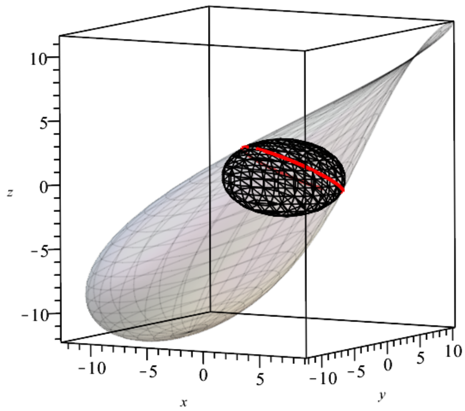

Figure 3. Intersecting curve between an ellipsoid and its locus surface.

3.3 Locus surface for a sphere

We let Σ be the sphere $x^2+y^2+z^2 = r^2$, and let the fixed point $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$ be on $S_1 : x^2 + y^2 + z^2 = \rho^2$ with $\rho \neq r$ and $\rho < \infty$. Because Σ is symmetric with respect to the origin and in view of preceding exploration with the locus for ellipsoid, it is natural to expect the shape of the locus for Σ stays unchanged and is coordinate free. Specifically, if we move the fixed points $A_1, A_2, ..., A_n \in S_1$ sequentially:

$$A_1 \to A_2 \to ... \to A_n \tag{18}$$

with $A_n = A$. Then Δ_i, the locus surface of Σ with respect to A_i, for $i = 1, 2, ...n$, moves sequentially

$$\Delta_1 \to \Delta_2 \to ... \to \Delta_n, \tag{19}$$

and we would expect that $\Delta_n = \Delta$. However, we shall explore that even if $\Delta_n = \Delta$, these two surfaces may be of different structures from differential geometry points of view. To begin with, we remark that it is an easy exercise to note that the antipodal points D of C for quadric surfaces by applying the Vieta's formula discussed in *[7]* is a **non-linear transformation** when $\rho < \infty$ and the fixed point A is not on the x, y or z axis. In the following, we describe how we can establish a sequence of non-linear transformations from a sphere to its locus surface Δ^* with the property of $\Delta = \Delta^*$.

Theorem 1 *For $s > 0$ given, let Σ be the sphere $x^2 + y^2 + z^2 = r^2$, $A_1 = (0, 0, \rho)$ and $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$. We denote Δ_1 to be the locus surface of Σ with respect to A_1 and Δ to be the locus surface of Σ with respect to A. If $R_y(v_0)$ represents the rotation by v_0 radians around y–axis, and $R_z(u_0)$ represents the rotation by u_0 radians around z–axis, then $R_z(u_0) \circ R_y(v_0)(\Delta_1) = \Delta$.*

Proof. Method 1. ($R_z(u_0) \circ R_y(v_0)(\Delta_1)$ is congruent to Δ.)

Let us consider the transformations $T = R_z(u_0) \circ R_y(v_0)$ and $T^{-1} = R_y(-v_0) \circ R_z(-u_0)$. For arbitrary $E \in R_z(u_0) \circ R_y(v_0)(\Delta_1)$, there exists $E_1 \in \Delta_1$ such that $E = T(E_1)$ where

$$E_1 = sC_1 + (1-s)D_1 \quad \text{with} \quad C_1, D_1 \in \Sigma \cap \overleftrightarrow{A_1C_1}.$$

A direct calculation shows that $T(A_1) = A$ and, since T is a rigid transformation, for $C \doteq T(C_1)$ and $D \doteq T(D_1)$, we have that

$$E = sC + (1-s)D \quad \text{with} \quad C, D \in \Sigma \cap \overleftrightarrow{AC}.$$

This shows that $R_z(u_0) \circ R_y(v_0)(\Delta_1) \subset \Delta$. The contention of $\Delta \subset R_z(u_0) \circ R_y(v_0)(\Delta_1)$ follows from a similar argument by using T^{-1}.

Method 2. (Expressing $R_z(u_0) \circ R_y(v_0)(\Delta_1)$ via a non-linear transformation) We let Δ_1 be the locus surface for Σ when $u_0 = v_0 = 0$, and let $C_1 = \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \in \Sigma$ and D_1 be the antipodal point C_1. By the definition of D_1, we may write

$$D_1 = \begin{bmatrix} x_1^1 \\ y_1^1 \\ z_1^1 \end{bmatrix} = \frac{1}{\rho^2 + r^2 - 2\rho r \cos v} \begin{pmatrix} \rho^2 - r^2 & 0 & 0 \\ 0 & \rho^2 - r^2 & 0 \\ 0 & 0 & \frac{-\left(\rho^2+r^2\right)+2r^2\rho}{\cos v} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}. \tag{20}$$

Consequently, we define $T_1 : \Sigma \to \Delta_1$ by

$$\begin{aligned} T_1\left(\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}\right) &= s\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} + (1-s)\begin{bmatrix} x_1^1 \\ y_1^1 \\ z_1^1 \end{bmatrix}, \\ &= s\begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} + \frac{1-s}{\rho^2 + r^2 - 2\rho r \cos v} \begin{pmatrix} \rho^2 - r^2 & 0 & 0 \\ 0 & \rho^2 - r^2 & 0 \\ 0 & 0 & \frac{-\left(\rho^2+r^2\right)\cos v+2r^2\rho}{\cos v} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \\ &= \begin{pmatrix} s + \frac{\left(\rho^2-r^2\right)(1-s)}{\rho^2+r^2-2\rho r\cos v} & 0 & 0 \\ 0 & s + \frac{\left(\rho^2-r^2\right)(1-s)}{\rho^2+r^2-2\rho r\cos v} & 0 \\ 0 & 0 & s - \frac{(1-s)\left(\left(\rho^2+r^2\right)\cos v - 2r\rho\right)}{\left(\rho^2+r^2-2\rho r\cos v\right)\cos v} \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \\ &= \begin{pmatrix} sr\cos(u)\sin(v) - \dfrac{(1-s)\cos(u)\left(r^2-\rho^2\right)\sin(v)\, r}{-2\rho\cos(v)\, r + r^2 + \rho^2} \\ sr\sin(u)\sin(v) - \dfrac{(1-s)\, r\sin(u)\sin(v)\left(r^2-\rho^2\right)}{-2\rho\cos(v)\, r + r^2 + \rho^2} \\ sr\cos(v) - \dfrac{(1-s)\, r\left(\left(r^2+\rho^2\right)\cos(v) - 2r\rho\right)}{-2\rho\cos(v)\, r + r^2 + \rho^2} \end{pmatrix} \\ &= \Delta_1 \end{aligned} \tag{21}$$

We remark that matrix

$$M=\begin{pmatrix} s+\frac{\left(\rho^2-r^2\right)(1-s)}{\rho^2+r^2-2\rho r\cos v} & 0 & 0\\ 0 & s+\frac{\left(\rho^2-r^2\right)(1-s)}{\rho^2+r^2-2\rho r\cos v} & 0\\ 0 & 0 & s-\frac{(1-s)\left(\left(\rho^2+r^2\right)\cos v-2r\rho\right)}{(\rho^2+r^2-2\rho r\cos v)\cos v}\end{pmatrix}$$

is a non-linear transformation and Δ_1 is an image of a non-linear transformation. If we let $R_y\left(v_0\right)$ and $R_z(u_0)$ represent the rotation matrix around y and z axes respectively as follows:

$$R_y\left(v_0\right)=\begin{pmatrix}\cos v_0 & 0 & \sin v_0\\ 0 & 1 & 0\\ -\sin v_0 & 0 & \cos v_0\end{pmatrix}, R_z\left(u_0\right)=\begin{pmatrix}\cos u_0 & -\sin u_0 & 0\\ \sin u_0 & \cos u_0 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{22}$$

We leave it to a CAS to prove that $R_z\left(u_0\right)\circ R_y\left(v_0\right)\left(\Delta_1\right)$ produces the same surface as Δ.

Remarks:

1. In the preceding theorem, $R_z\left(u_0\right)\circ R_y\left(v_0\right)\left(\Delta_1\right)$ is congruent to Δ. However, these two locus surfaces have different characteristics in differential geometry sense. For example, the cross sections are different for $R_z\left(u_0\right)\circ R_y\left(v_0\right)\left(\Delta_1\right)$ and Δ respectively, when u or v is being kept as constant, and yet they have different characteristics in differential geometry sense. See the Example below.

2. In view of the preceding Theorem, if the fixed point A is on the x, y or z axis, the transformation from Σ to the corresponding locus surface is a non-linear transformation. The following result is trivial, which we omit the proof.

Corollary. If Σ is the sphere $x^2+y^2+z^2=r^2$, and we let $A=(\rho\sin v_0\cos u_0, \rho\sin v_0\sin u_0, \rho\cos v_0)$ be on $S_1: x^2+y^2+z^2=\rho^2$ with $\rho\neq r$, and A is not on the x, y or z axis. Then there exists a transformation T^* on Σ such that $T^*(\Sigma)$ and Δ are same surface but with different cross sections, where Δ is the locus surface of Σ with respect to the fixed point A.

We use the following Example to demonstrate the effect of the preceding theorem.

Example 2 *Consider the sphere S_0 of $x^2+y^2+z^2=25$, and the fixed point of*

$$A=\left(\left(7\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right),\left(7\sin\frac{\pi}{4}\right)\left(\sin\frac{\pi}{4}\right),7\cos\frac{\pi}{4}\right). \tag{23}$$

We shall show that to obtain the locus surface Δ for A when $s=3$. We may pursue in the following ways. First, we pick $u_0=v_0=0$ for the fixed point

$$A_1=\left(\left(7\sin 0\right)\left(\cos 0\right),\left(7\sin 0\right)\left(\sin 0\right),7\cos 0\right)=(0,0,7), \tag{24}$$

and let the locus surface Δ_1 be the one with respect to A_1. We first depict the surface Δ for A when $s=3$ together with S_0 in Figure 4 below.

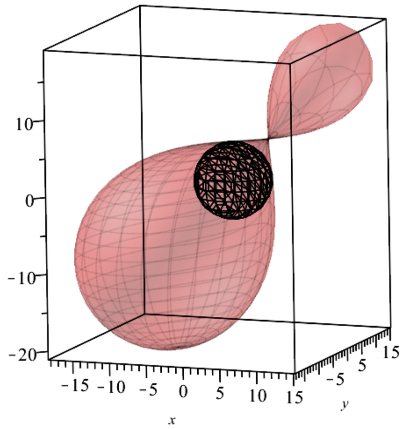

Figure 4. Locus surface Δ together with S_0.

We notice that Δ *is tangent to* S_0 *at an intersecting curve as we have discussed in the preceding section. Furthermore, we depict the locus* Δ *in Figure 5(a). Next we compute* $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$*, with* $u_0 = v_0 = \frac{\pi}{4}$*, and the plot can be seen as in Figure 5(b).*

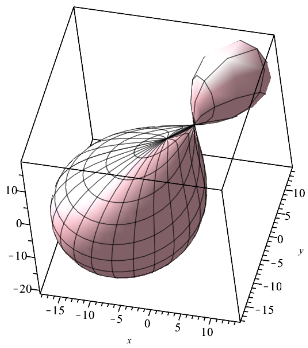

Figure 5(a). The locus for Δ*.*

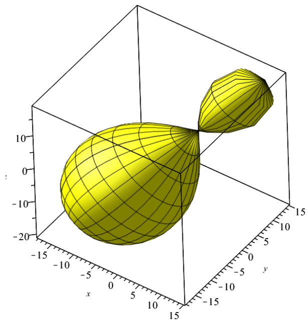

Figure 5(b). The locus surface $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$*.*

The traces for Δ and $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$ when $u = \frac{\pi}{2}$ can be seen in Figures 5(c) and 5(d) respectively.

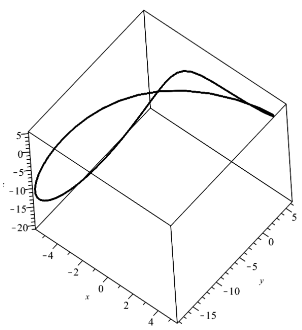

Figure 5(c) The traces for Δ when $u = \frac{\pi}{2}$.

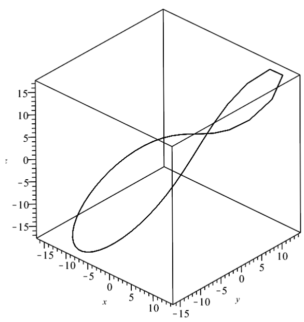

Figure 5(d). The traces for $R_z(u_0) \circ R_y(v_0) \circ \Delta_1$ when $u = \frac{\pi}{2}$.

It is a good exercise, which we leave it to readers to explore with a CAS or DGS that the the trace for Δ when $u=\frac{\pi}{2}$ (Figure 5(c)) does not lie on the same plane but the trace for $R_z(u_0)\circ R_y(v_0)\circ\Delta_1$ when $u=\frac{\pi}{2}$ does lie on the same plane (Figure 5(d)). See [S3] for explorations.

In the next section, we show that when the fixed point A is at infinity, the mapping which sends a quadric surface to its locus surface is a linear transformation.

4 When the fixed point is at an infinity

We first describe two approaches how we may obtain the locus surfaces when the fixed point A, written in its spherical coordinate, $(\rho\cos u_0\sin v_0, \rho\sin u_0\sin v_0, \rho\cos v_0)$, is at an infinity. We remark that the following Method 2 is essentially identical to the Method 1 after letting $\rho\to\infty$.

Method 1. We let the radius of spherical coordinate for $A=(\rho\cos u_0\sin v_0, \rho\sin u_0\sin v_0, \rho\cos v_0)$, ρ, go to infinity

We describe the locus surface in the following steps:

1. If $A=(\rho\cos u_0\sin v_0, \rho\sin u_0\sin v_0, \rho\cos v_0)$. Let us note that 4 and 5 become,

$$k=\frac{\hat{y}-\rho\sin u_0\sin v_0}{\hat{x}-\rho\cos u_0\sin v_0} \tag{25}$$

$$m=\frac{\hat{z}-\rho\cos v_0}{\hat{x}-\rho\cos u_0\sin v_0} \tag{26}$$

2. We follow the usual procedure to find the intersection between the line AC and the quadric surface at $D=(x_1,y_1,z_1)$ respectively by adopting the Vieta's formula.

3. Next we let $\rho\to\infty$ to obtain the corresponding intersection point $D_{\text{inf}}=(x_{1\,\text{inf}}, y_{1\,\text{inf}}, z_{1\,\text{inf}})$

4. The corresponding locus surface, is defined as $E_{\text{inf}}=(x_{e\,\text{inf}}, y_{e\,\text{inf}}, z_{e\,\text{inf}})$ where

$$\begin{aligned} x_{e\,\text{inf}} &= s\hat{x}+(1-s)(x_{1\,\text{inf}}) \\ y_{e\,\text{inf}} &= s\hat{y}+(1-s)(y_{1\,\text{inf}}) \\ z_{e\,\text{inf}} &= s\hat{z}+(1-s)(z_{1\,\text{inf}}). \end{aligned}$$

Method 2. We take k (4) and m (5) to be fixed angles after letting $\rho\to\infty$.

1. We fix the angles $u_0\in(0,2\pi)-\{\frac{\pi}{2},\frac{3\pi}{2}\}$ and $v_0\in(0,\pi)$, and let the point A going to infinity in the direction $(\sin v_0\cos u_0, \sin v_0\sin u_0, \cos v_0)$. Taking the limit of 25 and 26 when $\rho\to\infty$ we get,

$$k_0\doteq k(u_0,v_0)=\frac{\sin v_0\sin u_0}{\sin v_0\cos u_0}=\tan u_0, \tag{27}$$

and

$$m_0\doteq m_(u_0,v_0)=\frac{\cos v_0}{\sin v_0\cos u_0}=\cot v_0\sec u_0.. \tag{28}$$

2. By using the followings and substitute into the implicit equation of the quadric, $F(x,y,z) = 0$,

$$\begin{aligned} y &= \hat{y} + k_0(x - \hat{x}), \\ z &= \hat{z} + m_0(x - \hat{x}), \end{aligned}$$

we follow the Vieta's formula to find the x–coordinate of the the antipodal point D'_{inf}, say x'_{1inf}, by calculating the roots of the polynomial

$$p(x) = a_2x^2 + a_1x + a_0.$$

3. For a given s, the locus surface generated by point $E'_{\text{inf}} = sC + (1-s)D'_{\text{inf}}$ is defined as

$$\Delta'_{\text{inf}}(s, u_0, v_0) = \begin{bmatrix} x'_{e\,\text{inf}} \\ y'_{e\,\text{inf}} \\ z'_{e\,\text{inf}} \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1-s)x'_{1\,\text{inf}} \\ s\hat{y} + (1-s)y'_{1\,\text{inf}} \\ s\hat{z} + (1-s)z'_{1\,\text{inf}} \end{bmatrix}.$$

Calculations in Exploration [S4] shows that $D_{\text{inf}} = D'_{\text{inf}}$, and therefore $E_{\text{inf}} = E'_{\text{inf}}$, so the locus surfaces $\Delta_{\text{inf}}(s, u_0, v_0)$ and $\Delta'_{\text{inf}}(s, u_0, v_0)$ produced by Method 1 and Method 2, respectively, are identical.

We explore the locus surfaces for ellipsoids and hyperbolic with two sheets when A is at an infinity in the following subsections.

4.1 Ellipsoid

Let Σ be the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Following the methodology set out in the previous section, we calculate the roots of the polynomial

$$p(x) = a_2x^2 + a_1x + a_0,$$

where

$$a_2 = \frac{a^2b^2m^2 + a^2c^2k^2 + b^2c^2}{a^2b^2c^2} \tag{29}$$

$$a_1 = \frac{2\big(z_0b^2m + y_0c^2k - x_0(b^2m^2 + c^2k^2)\big)}{b^2c^2} \tag{30}$$

$$a_0 = \frac{x_0^2\big(b^2m^2 + c^2k^2\big) - 2x_0\big(z_0b^2m + y_0c^2k\big) + y_0^2c^2 + z_0^2b^2 - b^2c^2}{b^2c^2}. \tag{31}$$

The explicit expressions for D_{inf}, E_{inf} and $\Delta_{\text{inf}}(s, u_0, v_0)$ are calculated in Exploration [S5]. See [S6] for dynamic explorations.

We depict the locus surface (blue) when $s = 2, a = 5, b = 4, c = 3$, with $u_0 = \frac{\pi}{3}, v_0 = \frac{\pi}{4}$ and $\rho \to \infty$ together with the original ellipsoid (yellow) in Figure 6.

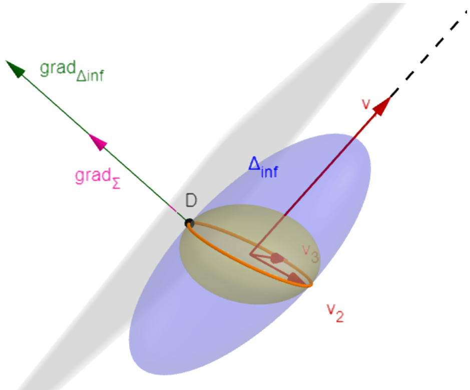

Figure 6. Locus ellipsoid when $s = 2$.

4.2 Hyperboloid with two sheets

Let Σ be the hyperboloid with two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$. Following the methodology set out in the previous section, we calculate the roots of the polynomial

$$p(x) = a_2 x^2 + a_1 x + a_0,$$

where

$$a_2 = \frac{a^2c^2k^2 - a^2b^2m^2 + b^2c^2}{a^2b^2c^2} \tag{32}$$

$$a_1 = \frac{2y_0c^2k - 2x_0\left(c^2k^2 - b^2m^2\right) - 2z_0b^2m}{b^2c^2} \tag{33}$$

$$a_0 = \frac{2x_0z_0b^2m - 2x_0y_0c^2k - x_0^2\left(b^2m^2 - c^2k^2\right) + y_0^2c^2 - z_0^2b^2 + b^2c^2}{b^2c^2}. \tag{34}$$

The explicit expressions for D_{inf}, E_{inf} and $\Delta_{\text{inf}}(s, u_0, v_0)$ are calculated in Exploration [S7]. See [S8] for dynamic explorations.

We depict the locus surface (blue) when $s = 0.8, a = 5, b = 4, c = 3$, with $u_0 = \frac{\pi}{6}, v_0 = \frac{\pi}{3}$ and $\rho \to \infty$ together with the original hyperboloid (yellow) in Figure 7.

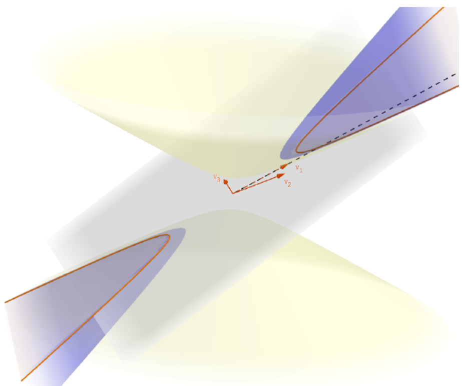

Figure 7. Locus hyperboloid when $s = 0.8$

5 Special Cases

We show that both methods coincide when the point $A = (x_0, y_0 z_0)$ is at infinity on $x - axis, y - axis$ or $z - axis$ respectively. In other words, when $x_0 \to \pm\infty, y_0 \to \pm\infty$ or $z_0 \to \pm\infty$ respectively, both methods produce the same locus surfaces.

1. If the fixed point $A = (x_0, y_0 z_0)$ is on the $x - axis$ and we let $x_0 \to \pm\infty$, $D = (x_1, y_1, z_1) = (-\hat{x}, \hat{y}, \hat{z}))$ is simply a reflection of C along the $x - axis$. Consequently, the locus surfaces is
$$\begin{bmatrix} s\hat{x} + (1-s)(-\hat{x}) \\ s\hat{y} + (1-s)\hat{y} \\ s\hat{z} + (1-s)\hat{z} \end{bmatrix} = \begin{bmatrix} (2s-1)\hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}. \tag{35}$$

2. If the fixed point $A = (x_0, y_0 z_0)$ is on the $y - axis$ and we let $y_0 \to \pm\infty$, $D = (x_1, y_1, z_1) = (\hat{x}, -\hat{y}, \hat{z})$ is simply a reflection of C along the $y - axis$. Consequently, the locus surfaces is
$$\begin{bmatrix} s\hat{x} + (1-s)\hat{x} \\ s\hat{y} + (1-s)(-\hat{y}) \\ s\hat{z} + (1-s)\hat{z} \end{bmatrix} = \begin{bmatrix} \hat{x} \\ (2s-1)\hat{y} \\ \hat{z} \end{bmatrix}. \tag{36}$$

3. If the fixed point $A = (x_0, y_0 z_0)$ is on the $z - axis$ and we let $z_0 \to \infty$, $D = (x_1, y_1, z_1) = (\hat{x}, \hat{y}, -\hat{z})$ is simply a reflection of C along the $z - axis$. Consequently, the locus surfaces is
$$\begin{bmatrix} s\hat{x} + (1-s)\hat{x} \\ s\hat{y} + (1-s)\hat{y} \\ s\hat{z} + (1-s)(-\hat{z}) \end{bmatrix} = \begin{bmatrix} \hat{x} \\ \hat{y} \\ (2s-1)\hat{z} \end{bmatrix}. \tag{37}$$

5.1 Remarks for the ellipsoid case

When the point $A = (x_0, y_0, z_0)$ is at infinity on $x - axis$, say $x_0 \to +\infty$, the above calculations show the followings:

1 The locus surface $\Delta_{\text{inf}}(s, u_0 = 0, v_0 = \pi/2)$ is the image of Σ under the linear transformation given by the matrix
$$L = \begin{bmatrix} 2s-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
Let us note that this locus surface is an ellipsoid, say:
$$\Delta_{x\,\text{inf}} \doteq \Delta_{\text{inf}}(s, u_0 = 0, v_0 = \pi/2) = \left\{(x, z, z) \in \mathbb{R}^3 \,:\, \frac{x^2}{((2s-1)a)^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}$$

2 If $s \in \mathbb{R}^+ \setminus \{1\}$ then Σ and $\Delta_{x\,\text{inf}}$ intersect tangentially just at an elliptical curve, say:
$$\gamma_{x=0} = \left\{(x, y, z) \in \mathbb{R}^3 \,:\, x = 0,\ \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\right\}.$$

Proof. Clearly, $\gamma_{x=0} \subseteq \Sigma \cap \Delta_{x\inf}$. Now, $(\bar{x}, \bar{y}, \bar{z}) \in \Sigma \cap \Delta_{x\inf}$ implies that

$$1 = \frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{b^2} + \frac{\bar{z}^2}{c^2} = \frac{\bar{x}^2}{((2s-1)a)^2} + \frac{\bar{y}^2}{b^2} + \frac{\bar{z}^2}{c^2}$$

so, $(2s-1)^2\,\bar{x}^2 = \bar{x}^2$. Since $s \neq 1$, we conclude that $\bar{x} = 0$, that is, $(\bar{x}, \bar{y}, \bar{z}) \in \gamma_{x=0}$, and therefore $\Sigma \cap \Delta_{x\inf} \subseteq \gamma_{x=0}$.

Remark 3 *1 Let us denote by*

$$\underline{\Sigma} = \left\{ (x, y, z) \in \mathbb{R}^3 \;:\; \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

the solid ellipsoid which boundary is Σ. Then we see $\Delta_{x\inf} \subsetneq \Sigma$ for $0 < s < 1$.

Proof.

Given $(x, y, z) \in \Delta_{x\inf}$, choose $(\hat{x}, \hat{y}, \hat{z}) \in \Sigma$ such that $x = (2s-1)\hat{x}$, $y = \hat{y}$ and $z = \hat{z}$. Then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{(2s-1)^2\hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} + \frac{\hat{z}^2}{c^2} < \frac{\hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} + \frac{\hat{z}^2}{c^2} \leq 1.$$

2 Let us denote by

$$\underline{\Delta_{x,\inf}} = \left\{ (x, y, z) \in \mathbb{R}^3 \;:\; \frac{x^2}{((2s-1)a)^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

the solid ellipsoid which boundary is $\Delta_{x,\inf}$. Then, $s > 1$ implies $\Sigma \subsetneq \underline{\Delta_{x,\inf}}$.

Proof.

The proof is similar to (1). Similar statements hold when $x_0 \to -\infty, y_0 \to \pm\infty$ or $z_0 \to \pm\infty$ respectively.

In view of 1, we prove the result for A being at an infinity as follows:

Theorem 4 *For $s > 0$ given, let Σ be the sphere $x^2 + y^2 + z^2 = r^2$, A_1 be at the infinity on the z axis, and $A = (\rho \sin v_0 \cos u_0, \rho \sin v_0 \sin u_0, \rho \cos v_0)$ when $\rho \to \infty$. We denote Δ_1 to be the locus surface of Σ with respect to A_1 and Δ to be the locus surface of Σ with respect to A. If $R_y(v_0)$ represents the rotation by v_0 radians around y–axis, and $R_z(u_0)$ represents the rotation by u_0 radians around z–axis, then $R_z(u_0) \circ R_y(v_0)(\Delta_1) = \Delta$.*

Proof. We let $\rho \to \infty$ for $A_1 = (0, 0, \rho)$, and compute the matrices in method 2 in 1. It turns out, as expected, that the antipodal point D_1 is $\begin{pmatrix} r\cos u \sin v \\ r \sin u \sin v \\ -r\cos v \end{pmatrix}$ and the locus Δ_1 is

$$\begin{pmatrix} sr\cos u \sin v + (1-s)r\cos u \sin v \\ sr \sin u \sin v + (1-s)r \sin u \sin v \\ sr\cos v - (1-s)r\cos v \end{pmatrix}. \tag{38}$$

We proceed and compute $R_z(u_0) \circ R_y(v_0)(\Delta_1)$ and let a CAS to verify that $R_z(u_0) \circ R_y(v_0)(\Delta_1) = \Delta$.

5.2 Remarks for the hyperboloid with two sheets

When the point $A = (x_0, y_0, z_0)$ is at infinity on $x - axis$, say $x_0 \to +\infty$, the above calculations show the followings:

1 The locus surface $\Delta_{\text{inf}}(s, u_0 = 0, v_0 = \pi/2)$ is the image of Σ under the linear transformation given by the matrix:

$$L = \begin{bmatrix} 2s-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let us note that this locus surface is an hyperboloid with two sheets, say:

$$\Delta_{x\,\text{inf}} \doteq \Delta_{\text{inf}}(s, u_0 = 0, v_0 = \pi/2) = \left\{ (x, z, z) \in \mathbb{R}^3 \ : \ \frac{x^2}{((2s-1)a)^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \right\}$$

2 If $s \in \mathbb{R}^+ \setminus \{1\}$ then Σ and $\Delta_{x\,\text{inf}}$ intersect tangentially just at an hyperbolic curve, say: $\gamma_{x=0} = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ x = 0, \ \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \right\}$.

Proof. Clearly, $\gamma_{x=0} \subseteq \Sigma \cap \Delta_{x\,\text{inf}}$. Now, $(\bar{x}, \bar{y}, \bar{z}) \in \Sigma \cap \Delta_{x\,\text{inf}}$ implies that,

$$-1 = \frac{\bar{x}^2}{a^2} + \frac{\bar{y}^2}{b^2} - \frac{\bar{z}^2}{c^2} = \frac{\bar{x}^2}{((2s-1)a)^2} + \frac{\bar{y}^2}{b^2} - \frac{\bar{z}^2}{c^2}$$

so, $(2s-1)^2\,\bar{x}^2 = \bar{x}^2$. Since $s \neq 1$, we conclude that $\bar{x} = 0$, that is, $(\bar{x}, \bar{y}, \bar{z}) \in \gamma_{x=0}$, and therefore $\Sigma \cap \Delta_{x\,\text{inf}} \subseteq \gamma_{x=0}$.

3 Let us denote by

$$\underline{\Sigma} = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} \leq -1 \right\}$$

the solid hyperboloid with two sheets whose boundary is Σ. Then we have $\Delta_{x\,\text{inf}} \subsetneq \underline{\Sigma}$ for $0 < s < 1$.

Proof. Given $(x, y, z) \in \Delta_{x\,\text{inf}}$, choose $(\hat{x}, \hat{y}, \hat{z}) \in \Sigma$ such that $x = (2s-1)\hat{x}, \ y = \hat{y}$ and $z = \hat{z}$. Then,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{(2s-1)^2\hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} - \frac{\hat{z}^2}{c^2} < \frac{\hat{x}^2}{a^2} + \frac{\hat{y}^2}{b^2} - \frac{\hat{z}^2}{c^2} \leq -1.$$

4 Let us denote by

$$\underline{\Delta_{x,\text{inf}}} = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ \frac{x^2}{((2s-1)a)^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} \leq -1 \right\}$$

the solid hyperboloid with two sheets whose boundary is $\Delta_{x,\text{inf}}$. Then we have $\Sigma \subsetneq \underline{\Delta_{x,\text{inf}}}$ for $s > 1$.

Proof. The proof is similar to (3). Similar statements hold when $x_0 \to -\infty, y_0 \to \pm\infty$ or $z_0 \to \pm\infty$ respectively.

6 Main results

This section is to say the locus problem is a linear transformation between the specified surface Σ and the locus surface Δ when A is at an infinity. However, due to the length requirement for the paper, we simply state the following results and their respective proofs can be found in [8].

Theorem 5 *Let Σ be a quadric surface, and let $A_{\text{inf}}(u_0, v_0)$ be the fixed point at infinity in the direction of $(\cos u_0 \sin v_0, \sin u_0 \sin v_0, \cos v_0)$, $C \in \Sigma$ and D_{inf} be the "antipodal" point of C corresponding to $A_{\text{inf}}(u_0, v_0)$ as described in previous sections. Then there exists an affine transformation $\mathcal{A}_D : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\mathcal{A}_D(C) = D_{inf}$.*

Corollary. Given $s > 0$, consider same hypothesis as in Theorem 5 and let $E_{\text{inf}} = sC + (1-s)D_{\text{inf}}$. Then the affine transformation

$$\mathcal{A}_E = sI + (1-s)\mathcal{A}_D$$

is such that $\mathcal{A}_E(C) = E_{inf}$, where I is the identity mapping from R^3 to R^3.

Proposition 6 *In Theorem 5, if Σ is the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, then there exists a matrix $L^e_D = \left[l^e_{ij}\right]_{3\times 3}$ such that $L^e_D C = D_{\text{inf}}$.*

Corollary. Given $s > 0$, consider same hypothesis as in Proposition 6 and let $E_{\text{inf}} = sC + (1-s)D_{\text{inf}}$. Then the matrix

$$L^e_E = sI + (1-s)L^e_D$$

is such that $L^e_E C = E_{\text{inf}}$, and therefore, the locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ is the image of Σ under the linear transformation given by the matrix $L^e_E = [l^e_{ij}]_{3\times 3}$.

Proposition 7 *For $s \in \mathbb{R} \setminus \{1\}$, the ellipsoid Σ and locus ellipsoid $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect themselves tangentially at an elliptical curve.*

Exploration [S6] contains an animation to exemplify the previous result.

Proposition 8 *For $s \in \mathbb{R}^+ \setminus \{1\}$, if the hyperboloid with two sheets Σ and corresponding locus surface $\Delta_{\text{inf}}(s, u_0, v_0)$ intersect themselves, they do it tangentially at an hyperbolical curve.*

Finally, we can verify that the gradient of Σ and $\Delta_{\text{inf}}(s, u_0, v_0)$ are colinear when evaluated at any point on γ when Σ is a hyperboloids with two sheets, see Figure 8 and exploration [S8].

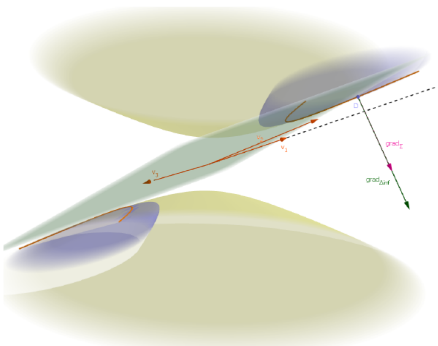

Figure 8. Intersection of the hyperboloid $\sum$ and its corresponding locus.

In [8], we shall further discuss how the eigenvectors of a linear transformation on a quadric surface, when the fixed point A is at an infinity, will affect the shapes of locus surfaces when s gets larger and larger and when $s \to \infty$ respectively.

7 Conclusions

In this paper, we have explored the locus problems when the fixed point A is outside a specified curve or surface. When A is not at an infinity, although the projection $T : \Sigma \to \Delta$ is not a linear map, the result is interesting because the intersecting points in 2D or intersecting curve in 3D remains fixed regardless of the parameter s. When the fixed point A is at an infinity, the projection $T : \Sigma \to \Delta$ becomes a linear transformation. We shall further discuss this linear transformation in [8]. It is delighted to see a simple college entrance exam problem originated from China [6] has led to many interesting discoveries in projective geometry, differential geometry (see [6]), and possibly other areas. The explorations, discussed all papers that are elated to this locus problem, are very accessible to undergraduate or graduate students, we believe that the concepts involved can be comprehended to future math teachers. Only when the math contents are enriched for our math teachers, can we increase our success in teaching math for our future generations.

It is clear that technological tools provide us with many crucial intuitions before we attempt more rigorous analytical solutions, and lead to many unexpected discoveries. Here we have gained geometric intuitions while using a DGS. In the meantime, we use a CAS for verifying that our analytical solutions are consistent with our initial intuitions. Incorporating a DGS and CAS into exploring a problem definitely has made mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in many parts of the world. However, encouraging a greater interest in mathematics for students, and in particular, providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil and paper, is an important step for cultivating creativity and innovation.

8 Supplementary Electronic Materials

[S1] GeoGebra worksheet for ellipse case in Section 3.1.

[S2] GeoGebra worksheet for ellipsoid case in Section 3.2.

[S3] Maple worksheet for Sections 3.2, 3.3 and Example 2.

[S4] wxMaxima worksheet for methods 1 and 2 in Section 4.1.

[S5] wxMaxima for ellipsoid case in Sections 4.1 and 6.

[S6] GeoGebra worksheet for ellipsoid case in Sections 4.1 and 6.

[S7] wxMaxima worksheet for hyperboloid case in Sections 4.2 and 6.

[S8] GeoGebra worksheet for hyperboloid case in Sections 4.2 and 6.

References

[1] GeoGebra (release 6.0.562 / October 2019), see https://www.geogebra.org/.

[2] Maple, A product of Maplesoft, see http://Maplesoft.com/.

[3] Maxima (release 5.43.0 / May 2019), see http://maxima.sourceforge.net/.

[4] Image of a sphere is an ellipsoid: https://sites.math.northwestern.edu/~clark/354/2002/ellipse.pdf.

[5] Mathworld: https://mathworld.wolfram.com/QuadraticSurface.html

[6] Yang, W.-C., *Locus Resulted From Lines Passing Through A Fixed Point And A Closed Curve*, the Electronic Journal of Mathematics and Technology, Volume 14, Number 1, 2020. ISSN 1933-2823, published by Mathematics and Technology, LLC.

[7] Yang, W.-C. & Morante, A., *3D Locus Problems of Lines Passing Through A Fixed Point*, the Electronic Journal of Mathematics and Technology, Volume 15, Number 1, 2021. ISSN 1933-2823, published by Mathematics and Technology, LLC.

[8] Yang, W.-C. & Morante, A., *Locus Surfaces and Linear Transformations when Fixed Point is at an Infinity*, to appear at the Electronic Journal of Mathematics and Technology.

[9] Yang, W.-C., *Exploring Locus Surfaces Involving Pseudo Antipodal Point*, the Electronic Proceedings of the 25th Asian Technology Conference in Mathematics, Published by Mathematics and Technology, LLC, ISSN 1940-4204 (online version), see https://atcm.mathandtech.org/EP2020/abstracts.html#21829.