An interactive visual introduction to curvature flows

Extended abstract without illustrations

Matthias Kawski
kawski@asu.edu
School of Mathematical and Statistical Sciences
Arizona State University
Tempe, AZ 85287-1804
United States of America

Abstract

After a short motivation we introduce several different curvature flows: A naive flow on the curvature under the heat equation, the curve-shortening flow, the mean curvature flow for imbedded surfaces, and the Ricci flow on surfaces of revolution and for abstract 2-manifolds. The main focus is on interactive visualizations using animations of curves, surfaces, and in the case of the Ricci also using flow of a metric field similar to Tissot's indicatrix. Our code is written in the computer algebra system MAPLE and thus it is very easy to vary the examples and experiment with different initial shapes.

1 Introduction

Curvature flows are a beautiful mathematical topic that is very tangible and accessible to school students in the form of science experiments with soap bubbles and films [7]. Calculus students are familiar with problems of minimizing area given a perimeter constraint. The classical version that does not limit itself to polygons is the isoperimetric problem or Dido's problem which has had a huge impact on mathematics [3]. Rather than just looking at static minimal (networks of) curves and curves it is natural to try to continuously deform arbitrary such into minimally. Great models for the way nature does this are different notions of curvature flows. The main idea is that curvature should spread out, just like hot and cold spots on a bar or plate should diffuse and eventually yield a constant curvature. Curvature flows have been highly publicized in the past 25 years, especially Hamilton's Ricci flow that Perelman utilized to prove the Poincaré conjecture [10]. However the Ricci flow is a highly abstract flow lives on abstract manifolds, and is is very inaccessible for non-experts.. Our objective is to present several similar curvature flows that have many similar properties, yet are much more accessible and tangible.

2 A simple model at the level of multi-variable calculus

A very simple example, accessible to multi-variable calculus students, simply lets the curvature of a fixed length curve evolve according to the heat equation. Note that according to the fundamental theorem of the differential geometry of curves (with never zero speed) every smooth planar curve is. up to translation and rotation. completely determined by its curvature as a function of arc length. We illustrate how to recover the curve from its curvature, and then animate many examples, especially closed curves that straighten out to become circles. In the literature, length constrained curvature flows, is discussed primarily for a modification of the curve-shortening flow (see the next sections), e.g., [9].

Just for fun we also present similar animation with the curvature evolving according to the wave equation, reminiscing the motions undergone by giant soap bubbles mentioned in the discussion.

3 Curve Shortening Flow

Extensively studied has been the curve shortening flow which is the one dimensional version of the mean curvature flow discussed in the next section. The flow is again generated by a simple analog of the heat equation. Physically, every point on the curve moves in the direction normal to the curve (in the convex direction) at a speed controlled by the curvature. It is well known that the flow exists globally and smooth closed curves eventually become convex and then exponentially become circular and shrink into a point at infinite time.

We compare this well studied flow with the naive flow from the first section. Visually particularly appealing animations that start from curves that initially are made up from (finitely) many pairs of spirals *closed by circular end caps*. One observes that the flow cannot cause self-intersections, which can be proven by very intuitive arguments.

4 Mean curvature flow for surfaces

The mean curvature flow for surfaces is defined analogously to that of curves. This flow, especially its stable equilibria, are most familiar from every-day life. In particular, this flow is often used to model the motion of soap bubbles, and in general surfaces that are under tension. In this case the stable equilibria are minimal surfaces, that is, surfaces which locally minimize area.

In the context of soap bubbles, it is a popular science project to dip frames made of copper or other wire into soap water and study the rapidly stabilizing soap films, which consist of often many minimal surfaces that meet each other at angles of 120 degrees.

While smooth closed (uniformly) convex surfaces also shrink into a point, they do so in finite time. However, flows initialized by nonconvex surfaces may develop singularities with the curvature becoming unbounded in finite time, see, e.g., [4]. One of the best known examples is a surface in the shape of a smooth dumbbell whose neck's diameter will shrink much faster than the *radii* of the *sphere like* ends, than *pinch off* in finite time and become two disconnected surfaces. Positive results on the flow through singularities are presented in [1].

For this flow we will only briefly demonstrate some examples of surfaces that are graphs of functions, and some which are compact. At the end (see the last section) we will also briefly compare this flow with the Ricci flow when using Tissot's indicatrix to visualize the flow.

5 Ricci flow on imbedded surfaces of revolution

Hamilton's Ricci flow was introduced in the 1980s [6], and quickly become key tool instrumental in Perelman's celebrated proof [10] of the Poincaré conjecture.

The basic idea of the Ricci flow is very similar to the above extrinsic examples of curvature flows. However, it is an intrinsic flow that lives in the space of abstract Riemannian manifolds, not on surfaces imbedded in 3-space, and thus it is very hard to visualize. Intrinsic means that it only depends on the manifold and its metric, not how the manifold is imbedded in some ambient or surrounding space. The recent thesis [12] provides a nice very readable survey on intrinsic geometric flows on manifolds of revolution For a meticulous introduction to key properties of the Ricci flow in two dimensions see, e.g., [8]

The most familiar example of a two-dimensional manifold that cannot be imbedded in 3-space (without self-intersections) is a Klein bottle. Mathematically we think of the Klein bottle just as a rectangle with opposing edges identified, but unlike the torus or donut, one edges with reversed direction. (There are many ways to equip either of these with a Riemannian metric).

Even if the flow starts with an initial surface that is imbedded in 3-space, in general, the Ricci flow instantaneously deforms the surface into a manifold that cannot be imbedded into 3-space. However, in the special case of surfaces of revolution the Ricci flow will stay within in this class. For a survey and study of the flow in this class of surfaces, with many sequences of pictures see the article [11], and also [5]. We will present very similar animations.

6 Visualizing the 2D Ricci-flow via Tissot's indicatrix

Many years ago we experimented visualizing *time-varying* differential equations in the plane by animations of *dancing* vector fields. The solution curves grow in such a way that at every time the leading point moves in the direction of the vector field at this point at this time. Instead of plotting single vector fields, to visualize a metric in a coordinate patch we may basically draw two arrows at every point (corresponding to the eigenvectors scaled by the eigenvalues of a 2×2 matrix representing the metric at that point). A natural geometric alternative is to draw the ellipse with these two scaled eigenvectors as principal axes.

This is basically Tissot's indicatrix which was used extensively in the mid 19-th century in cartography in geography. It nicely communicates geometric properties of various projections such as preserving angles or areas, and showing where the biggest distortions happen. See, e.g., [2] for a discussion of the history and mathematics of Tissot's indicatrix. We will provide animations of how the metrics evolve under the Ricci flow in a coordinate patch, and try to demonstrate how one can see curvature and bending in this special kind of visual presentation.

References

[1] Sigurd Angenent, *On the formation of singularities in the curve shortening flow*, J. Differential Geom. **33** (1991), no. 3, 601–633. MR 1100205

[2] Papadopoulos. Athanase, *A note on nicolas-auguste tissot: At the origin of quasiconformal mappings*, (to appear in Vol. VII of the Handbook of Teichml̈ler Theory). 2020. ffhal-02429961, 2020.

[3] Catherine Bandle, *Dido's problem and its impact on modern mathematics*, Notices Amer. Math. Soc. **64** (2017), no. 9, 980–984. MR 3699773

[4] Tobias Holck Colding, William P. Minicozzi, II, and Erik Kjæ r Pedersen, *Mean curvature flow*, Bull. Amer. Math. Soc. (N.S.) **52** (2015), no. 2, 297–333. MR 3312634

[5] Vincent E. Coll, Jr., Jeff Dodd, and David L. Johnson, *Ricci flow on surfaces of revolution: an extrinsic view*, Geom. Dedicata **207** (2020), 81–94. MR 4117562

[6] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geometry **17** (1982), no. 2, 255–306. MR 664497

[7] Cyril Isenberg, *The science of soap films and soap bubbles*, Tieto Ltd., Clevedon, 1978, With a foreword by George Porter. MR 620322

[8] James Isenberg, Rafe Mazzeo, and Natasa Sesum, *Ricci flow in two dimensions*, Surveys in geometric analysis and relativity, Adv. Lect. Math. (ALM), vol. 20, Int. Press, Somerville, MA, 2011, pp. 259–280. MR 2906929

[9] James McCoy, Glen Wheeler, and Yuhan Wu, *Evolution of closed curves by length-constrained curve diffusion*, Proc. Amer. Math. Soc. **147** (2019), no. 8, 3493–3506. MR 3981127

[10] Grigori Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109, 2003.

[11] J. Hyam Rubinstein and Robert Sinclair, *Visualizing Ricci flow of manifolds of revolution*, Experiment. Math. **14** (2005), no. 3, 285–298. MR 2172707

[12] Jefferson C. Taft, *Intrinsic geometric flows on manifolds of revolution*, ProQuest LLC, Ann Arbor, MI, 2010, Thesis (Ph.D.)–The University of Arizona. MR 2794762