Exploring Locus Surfaces Involving Pseudo Antipodal Points

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Abstract

The discussions in this paper were inspired by a college entrance practice exam from China. It is extended to investigate the locus curve that involves a point on an ellipse and its pseudo antipodal point with respect to a fixed point. With the help of technological tools, we explore 2D locus for some regular closed curves. Later, we investigate how a locus curve can be extended to the corresponding 3D locus surfaces on surfaces like ellipsoid, cardioidal surface and etc. Secondly, we use the definition of a developable surface (including tangent developable surface) to construct the corresponding locus surface. It is well known that, in robotics, antipodal grasps can be achieved on curved objects. In addition, there are many applications already in engineering and architecture about the developable surfaces. We hope the discussions regarding the locus surfaces can inspire further interesting research in these areas.

1 Introduction

Technological tools have influenced our learning, teaching and research in mathematics in many different ways. In this paper, we start with a simple exam problem and with the help of technological tools, we are able to turn the problem into several challenging problems in 2D and 3D. The visualization benefited from exploration provides us crucial intuition of how we can analyze our solutions with a computer algebra system. Therefore, while implementing technological tools to allow exploration in our curriculum is definitely a must. How we can encourage learners to discover more dynamic content knowledge in mathematics and its applications remains a challenging task. In this paper, we use examples to demonstrate how the following pedagogical issues can be addressed.

1. How can our explorations be linked to real life applications?

2. Can our problems be made accessible to general audience with the help of technological tools?

3. Can graphical representations inspire more learners to do more explorations?

4. Can students make use of the existing Dynamic Geometry System (DGS) and Computer Algebra System (CAS) to verify their conjectures?

We consider the following problem that is being modified from ([4]).

An exam-based 2D Problem: We are given a fixed circle in black and a fixed point A in the interior of the circle $(x-a)^2+(y-b)^2=r^2$, where center $O=(a,b)$ (see Figures 1(a) or 1(b)). A line passes through A and intersects the circle at C and D respectively, and the point E is the **midpoint** of CD. Find the locus E. It is easy to see that the locus E is a circle (see Figure 1(a)), which we leave it to readers to explore.

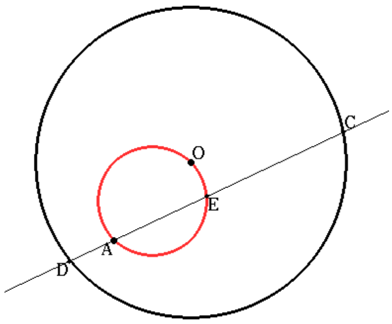


Figure 1(a). Locus and lines passing through a fixed point

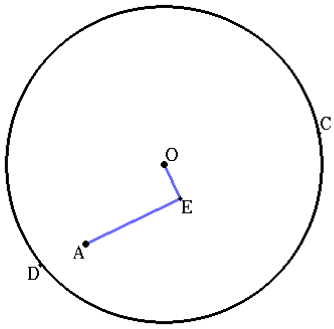


Figure 1(b). Locus, circle and perpendicular

We shall discuss the solution for a more general case in Section 2 below. They key relies on the Vieta's formulae to find the pseudo-antipodal points with respect to a fixed point. In Section 3, we demonstrate how we further apply the Vieta's formula on an ellipsoid to obtain the corresponding locus surface. In Section 4, we show that if the fixed point is at the origin $(0,0)$ in 2D, the pseudo-antipodal points with respect to the origin for a parametric equation $[r(t)\cos t, r(t)\sin t]$, where $r(t)$ represents a smooth closed curve and $t\in[0,2\pi]$, is simply $[r(t+\pi)\cos(t+\pi), r(t+\pi)\sin(t+\pi)]$. We use the rotation technique to find the locus surface for a cardioidal surface. In Section 5, we make use of the definition of a developable surface and use the base curve to be either an ellipse and cardioid to explore the corresponding locus surfaces. Finally, we discuss how developable surfaces have been discussed in many applications and hope our discussions on locus curves an surfaces can encourage more real-life applications.

2 Vieta's Formulae in Ellipse and its Locus Curve

We show that the Vieta's formulae can be applied when the implicit equation of a curve is of degree 2. We start with the case of ellipse below. First, we recall that for a real polynomial of degree n, say $p(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, having roots $r_1,r_2,\ldots,r_n$, then Vieta's formulas are:

$$
\begin{aligned}
r_1 + r_2 + \cdots + +r_{n-1} + r_n &= -\frac{a_{n-1}}{a_n}, \\
(r_1 r_2 + \cdots + r_1 r_n) + (r_2 r_3 + \cdots + r_2 r_n) + (r_{n-1} r_n) &= \frac{a_{n-2}}{a_n}, \\
&\vdots \\
r_1 r_2 \cdots r_{n-1} r_n &= (-1)^n \frac{a_0}{a_n}.
\end{aligned}
\tag{1}
$$

Since the method we discuss in this section is to find locus for the graph of a quadratic equation, the Vieta's formulae (1) we need is when $n = 2$.We demonstrate the process by using an ellipse, we leave other similar scenarios to readers to explore. Suppose we are given an ellipse

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{2}
$$

and the fixed point is at $A = (x_0, y_0)$. A line passes through A and intersects the ellipse at C and D respectively. If the point E lying on CD and satisfying $\overrightarrow{ED} = s\overrightarrow{CD}$. Then our objective is to find the locus of E. We shall call the point D as *the pseudo-antipodal point* of C with respect to the fixed point A.We describe how we find the pseudo-antipodal point D and the corresponding locus curve as follows:

1. First, we write the point $C = (x, y)$ on the ellipse in parametric form as $(a \cos u, b \sin u)$. Next we observe the followings:

   (a) We express the line equation AC as

   $$
   y = y_0 + \left(\frac{b \sin u - y_0}{a \cos u - x_0} \right) (x - x_0), \tag{3}
   $$

   (b) Substitute (3) in the implicit equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ to obtain a quadratic equation of x.

   (c) We apply the Vieta's formulae to find the intersection point $D = (x_1, y_1)$ between the line AC and the ellipse, with the help of a CAS, accordingly as follows:

   $$
   x_1 = -\frac{a\left(\left(\left(b^2 - y_0^2\right) a^2 + b^2 x_0^2\right) \cos(u) - 2abx_0\left(-y_0 \sin(u) + b\right)\right)}{-2\cos(u)\, ab^2 x_0 - 2\, y_0 \sin(u)\, a^2 b + \left(b^2 + y_0^2\right) a^2 + b^2 x_0^2}, \tag{4}
   $$

   $$
   \begin{aligned}
   y_1 = y_0 + &\left(\frac{b \sin(u) - y_0}{a \cos(u) - x_0} \right) \times \\
   &\left(-\frac{a\left(\left(\left(b^2 - y_0^2\right) a^2 + b^2 x_0^2\right) \cos(u) - 2\, abx_0\left(-y_0 \sin(u) + b\right)\right)}{-2\cos(u)\, ab^2 x_0 - 2\, y_0 \sin(u)\, a^2 b + \left(b^2 + y_0^2\right) a^2 + b^2 x_0^2} - x_0 \right).
   \end{aligned}
   \tag{5}
   $$

2. We note the equation

   $$
   \overrightarrow{ED} = s\overrightarrow{CD} \tag{6}
   $$

   can be also written as

   $$
   \overrightarrow{OE} = s\overrightarrow{OC} + (1 - s)\overrightarrow{OD}, \tag{7}
   $$

we therefore set up the locus $E = (x_e, y_e)$ as follows:

$$x_e = s \cdot x + (1 - s) \cdot x_1, \tag{8}$$
$$y_e = s \cdot y + (1 - s) \cdot y_1. \tag{9}$$

We use Geometry Expressions [5] to capture the following screen shots to illustrate the locus E in red when the fixed point is at $A = (0.8215576, 0.3345135)$, $s = 0.2$ and 0.7 in Figures 2(a) and (b) respectively.

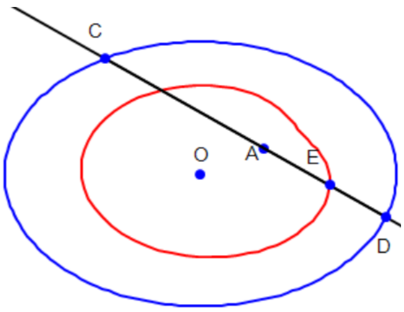


Figure 2(a). Locus of an ellipse when $s = 0.2$

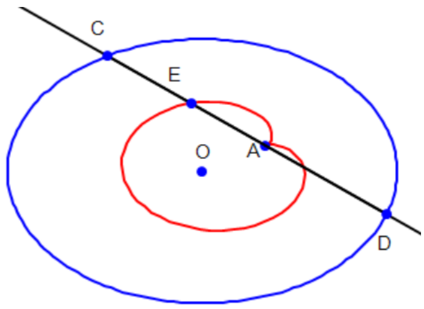


Figure 2(b). Locus of an ellipse when $s = 0.7$

Remarks:

1. With the help of technological tools, we are able to find the intersections between a line and an ellipse in a symbolic form such as equations (4) and (5).
2. We further can explore the locus curves by dragging the fixed point A and varying the parameter s with the symbolic Geometry Expressions [5].

When the given surfaces are quadrics, we can extend the techniques that we have done in the preceding 2D case to find the corresponding locus surfaces in 3D. We use the case of ellipsoid as demonstration in the next section.

3 Locus Surface for an Ellipsoid with Arbitrary Fixed point

We consider the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the fixed point is $A = (x_0, y_0, z_0)$. A line passes through A and intersects the ellipse at C and D respectively. If the point E lying on CD and satisfying $\overrightarrow{ED} = s\overrightarrow{CD}$. We want to find the locus of E.

Let $A = (x_0, y_0, z_0)$ and consider a point C on the ellipsoid

$$\Sigma = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}, \tag{10}$$

We write point C as

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} a\cos(u)\sin(v) \\ b\sin(u)\sin(v) \\ c\cos(v) \end{bmatrix} \tag{11}$$

In order to calculate the pseudo antipodal point $D = (x, y, z)$ of C with respective to A, we make use of the parametric equation of line l as follows:

$$\begin{aligned} x - x_0 &= \lambda(\hat{x} - x_0), \\ y - y_0 &= \lambda(\hat{y} - y_0), \\ z - z_0 &= \lambda(\hat{z} - z_0). \end{aligned}$$

Hence, we obtain

$$\frac{y - y_0}{x - x_0} = \frac{\hat{y} - y_0}{\hat{x} - x_0}, \tag{12}$$

$$\frac{z - z_0}{x - x_0} = \frac{\hat{z} - z_0}{\hat{x} - x_0}. \tag{13}$$

By substituting (11) into equations (12) and (13), we get some expressions for the left hand side in (12) and (13), allowing us to define two auxiliary functions, namely

$$k(u, v) = \frac{b\sin(u)\sin(v) - y_0}{a\cos(u)\sin(v) - x_0}, \tag{14}$$

$$m(u, v) = \frac{c\cos(v) - z_0}{a\cos(u)\sin(v) - x_0}. \tag{15}$$

Since both intersection points, C and D, satisfy the implicit equation of Σ, we use (14) and (15) to get the x–coordinate of D, say x_1, by calculating the roots of the polynomial

$$p(x) = a_2x^2 + a_1x + a_0,$$

where a_0, a_1 and a_2 can be found with help of a computer algebra system. It follows from $p(\hat{x}) = 0$ and the Vieta's formulas that

$$x_1 = -\frac{a_1}{a_2} - \hat{x}.$$

It follows from (12) and (13) that

$$y_1 = y_0 + k(x_1 - x_0) \quad \text{and} \quad z_1 = z_0 + m(x_1 - x_0).$$

For a given s, the locus surface generated by point $E = sC + (1 - s)D$ is defined as

$$\Delta(u, v) = \begin{bmatrix} x_e \\ y_e \\ z_e \end{bmatrix} = \begin{bmatrix} s\hat{x} + (1 - s)x_1 \\ s\hat{y} + (1 - s)y_1 \\ s\hat{z} + (1 - s)z_1 \end{bmatrix}.$$

The explicit form of the locus surface Δ can be found but we omit here. We use the following Example to demonstrate how we find the locus for a particular ellipsoid.

Example 1 *Consider the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

We explore the locus surface for the ellipsoid for the following scenarios using Netpad (see [6]). We plot the surface when $a = 5, b = 4, c = 3$, $s = 1.7$, *and the fixed point is at* $A = (2, -3, 4)$ *together with the locus trace when* v *is fixed at* 0.81 *and* u *varies between* 0 *and* 2π *in Figure 3 as follows.*

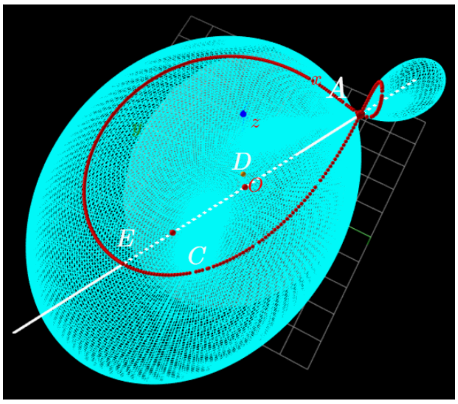


Figure 3. Locus of an ellipsoid when $a = 5, b = 4, c = 3, s = 1.7$ and $v = 0.81$

Remark: The web-based geometry software [6] indeed provides us a timely environment of visualizing the 3D locus surface for an ellipsoid.

4 Locus Curves in 2D and their Extensions in 3D

We observe that for a quadric surface, the Vieta's formulae can be successfully applied in finding the pseudo-antipodal point D of the point C with respective to a fixed point A is because the corresponding implicit equation is of degree 2 or when the implicit equation can be simplified to degree 2. We remark in the ellipse and ellipsoid cases that the Vieta's formulae are applicable even when the fixed point A is arbitrary at (x_0, y_0) or (x_0, y_0, z_0) respectively.

However, it is intuitive that Vieta's techniques may not be applied for any closed curve. For example, if the closed curve we consider is a cardioid $c(t) = (r(t)\cos(t), r(t)\sin(t))$, where $r(t) = 2 - 2\cos t$ and $t \in [0, 2\pi]$. Then the Vieta's formulae are applicable on its implicit equation, $p(x, y) = x^4 + 2y^2x^2 + y^4 + 4x^3 + 4xy^2 - 4y^2 = 0$, when the fixed point is at $A = (0, 0)$. On the other hand, the Vieta's formulae may not be applied at a fixed point other than $(0, 0)$ because the polynomial $p(x, y)$ may not be of degree 2 in x (after proper substitution of y, see [12]).

Unless otherwise specified, the parametric curves $\alpha(t)$ we consider in this paper are regular (if its derivative never vanishes) closed curve, twice continuously differentiable, and is written as $\alpha(t) = (r(t)\cos(t), r(t)\sin(t))$. It is interesting to see that the point $D \in (r(t+\pi)\cos(t+\pi), r(t+\pi)\sin(t+\pi))$ is the pseudo-antipodal point of $C \in (r(t)\cos(t), r(t)\sin(t))$ with respect to the fixed point $A = (0,0)$. Indeed, it can be verified that three points A, C and D are collinear, and more importantly, the locus curve $E = sC + (1-s)D$, in this case, turns out to be the same as the one obtained by using the Vieta's formulas (see [12]).

Our task now is to see how the locus problem in 2D can be extended to the corresponding locus surface in 3D. First, we rephrase the following terminologies:

1. We let $c_c(u)$ denote the 2D curve for the point C.
2. We let $c_d(u)$ denote the 2D curve for the point D, which is a pseudo antipodal point of C with respect to the fixed point A.
3. We let $c_e(s,u)$ denote the locus curve for the locus E, with respect to s, satisfying

$$\overrightarrow{OE} = s\overrightarrow{OC} + (1-s)\overrightarrow{OD} \tag{16}$$

We assume there exists a transformation that will extend the 2D curves of c_c and c_d to the corresponding 3D surfaces of $S_c(u,v)$ and $S_d(u,v)$ respectively. Furthermore if we denote $S_e^t(s,u,v)$ to be the 3D surface, which is resulted by the same transformation by lifting from the 2D locus curve $c_e(s,u)$. We are interested in exploring when the following equality holds

$$\begin{aligned} S_e^t(s,u,v) &= S_e(s,u,v) && (17)\\ &= sS_c(u,v) + (1-s)\,S_d(u,v), && (18) \end{aligned}$$

and the fixed point A, C, D and E are colinear.

Remark: We note here that the points C, D and E are well defined in 3D. However, the fixed point A in 3D needs to be specifically chosen or adjusted from its 2D fixed point. We shall see how the fixed points A can be adjusted in later examples.

In view of an isometry is a distance-preserving transformation between metric spaces, which maps elements to the same or another metric space such that the distance between the image elements in the new metric space is equal to the distance between the elements in the original metric space. We therefore consider the following

Definition. If (17) holds and the adjusted fixed point A in 3D together with the points C, D and E are all colinear, then we call such transformation to be a *locus isometry.*

The following is a trivial case locus isometry.

Theorem 2 *The surfaces of revolution are of locus isometry.*

Proof: If the surfaces $S_c(u,v)$ and $S_d(u,v)$ are obtained by rotating the curves c_c and c_d respectively about a proper axis, L, which contains the fixed point A. Then it is easy to see that the surface of revolution is locus isometry. In other words,

$$S_e^t(s,u,v) = sS_c(u,v) + (1-s)\,S_d(u,v).$$

We demonstrate how we can obtain a 3D locus surface by rotating the 2D cardioid curve $c(u) = (r(u)\cos(u), r(u)\sin(u))$ around the x-axis, where $r(u) = 2 - 2\cos u$:

1. We rotate the curve $c(u)$ around x–axis to get the 3D cardioidal surface,
$$S_c(u,v) = \begin{bmatrix} (2-2\cos u)\cos u \\ (2-2\cos u)\sin u\cos v \\ (2-2\cos u)\sin u\sin v \end{bmatrix},$$
2. We apply the same rotation on $c(u+\pi) = (r(u+\pi)\cos(u+\pi), r(u+\pi)\sin(u+\pi))$ to obtain the surface $S_d(u,v)$. We remark that if C is a point $S_c(u,v)$ and $D \in S_d(u,v)$, then D is the pseudo-antipodal point of C with respect to the fixed point $A = (0,0,0)$.
3. We rotate the 2D locus curve $c_e(s,u)$ around the $x-axis$ to obtain the surface of $S_e^t(s,u,v)$.
4. It is easy to show that
$$S_e^t(s,u,v) = sS_c(u,v) + (1-s)\,S_d(u,v). \tag{19}$$
We use a technological tool to visualize the following locus surface.

Example 3 *The parametric equation for the cardioidal surface Σ is*
$$\begin{bmatrix} (2-2\cos u)\cos u \\ (2-2\cos u)\sin u\cos v \\ (2-2\cos u)\sin u\sin v \end{bmatrix}.$$
A line passes through $A = (0,0,0)$ and intersects the at C and D respectively. If the point E lying on CD and satisfying $\overrightarrow{ED} = s\overrightarrow{CD}$. Then explore the locus of E.

We refer readers to [7] for exploring the locus E. We capture the screen shot of the locus surface when $s = 0.7$ and the trace of $u = 0.88$ in yellow in Figure 4. (The green curve is the trace for the point C or D.)

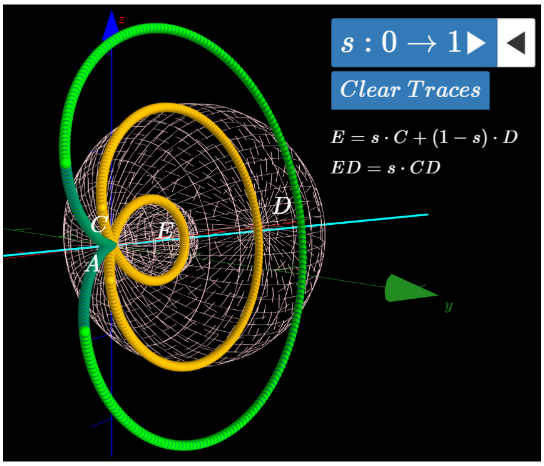


Figure 4. Locus surface when $s = 0.7$,
trace of $v = 4.88$ (yellow).

We explore some more cases when locus isometry is valid by lifting a 2D locus curve to the corresponding 3D locus surface in the next section.

5 Ruled Spaces and Locus Isometry

We recall that a **ruled surface** if through every point of S there is a straight line that lies on S. It is the union of a one parametric family of lines. The lines of this family are the generators of the ruled surface. A ruled surface can also be described by a parametric representation of the form

$$\widetilde{x}(u,v) = c(u) + v \cdot r(u). \tag{20}$$

The curve $c(u)$ is the **directrix** or the **base curve** of the representation. For fixed $u = u_0$, the space curve $\widetilde{x}(u_0, v)$ is a generator. The vectors $r(u)$ describe the **directions of the generators**. We shall describe ruled surfaces when the base curves are restricted to those $c_c(u)$ in 2D, where its locus curve $c_e(u)$ can be found. There are many applications in computer-aided geometric design (CAGD). For examples, there are interesting simplified structures that are related to ruled-surfaces.

5.1 Developable Surfaces and Locus Isometry

In the next two subsections, we explore locus isometry for two special developable spaces. The results are intuitive based on the definition of developable surfaces. We shall use graphical visualization to lead readers to appreciate the connection between the locus isometry and special surfaces. A **developable surface** is a ruled surface with zero Gaussian curvature. That is, it is a surface that can be flattened onto a plane without distortion (i.e. it can be bent without stretching or compression). Many developable surfaces can be visualized as the surface formed by moving a straight line in space. For example, a cone is formed by keeping one end-point of a line fixed whilst moving the other end-point in a circle. [11]. For developable surfaces they form one family of its lines of curvature. It can be shown that any developable surface is a cone, a cylinder or a surface formed by all tangents of a space curve.

Remarks:

1. Spheres are not developable surfaces under any metric as they cannot be unrolled onto a plane.

2. The helicoid $x(u,v) = (v\cos u, v\sin u, cu) = u(0,0,c) + v(\cos u, \sin u, 0) = c(u) + v \cdot r(u)$ is a ruled surface by definition, but it is not a developable surface, because their Gaussian curvature does not vanish at each point and every point of helicoid is hyperbolic.

Next we describe how we can obtain a locus isometry for a ruled surface if we define it in the following way. We let $c_c(u)$ be the 2D curve for the point $C = (x_c(u), y_c(u))$, $c_d(u)$ be the 2D curve for the point $D = (x_d(u), y_d(u))$, which is a pseudo antipodal point of C with respect to the fixed point $A = (x_0, y_0)$. Finally, we let $c_e(s,u)$ denote the locus curve for the locus E, with respect to s, satisfying

$$\overrightarrow{OE} = s\overrightarrow{OC} + (1-s)\overrightarrow{OD}. \tag{21}$$

In other words,

$$c_e(s,u) = \begin{pmatrix} x_e(s,u) \\ y_e(s,u) \end{pmatrix} = s \begin{pmatrix} x_c(u) \\ y_c(u) \end{pmatrix} + (1-s) \begin{pmatrix} x_d(u) \\ y_d(u) \end{pmatrix} \tag{22}$$

First we define three respective ruled surfaces based on $c_c(u), c_d(u)$ and $c_e(s,u)$ respectively as follows, where f and g are differentiable functions.

$$S_c(u,v) = \begin{pmatrix} x_c(u) \\ y_c(u) \\ f(u) \end{pmatrix} + v \cdot \begin{pmatrix} x_c(u) \\ y_c(u) \\ g(u) \end{pmatrix},$$
$$S_d(u,v) = \begin{pmatrix} x_d(u) \\ y_d(u) \\ f(u) \end{pmatrix} + v \cdot \begin{pmatrix} x_d(u) \\ y_d(u) \\ g(u) \end{pmatrix},$$
$$S_e^t(s,u,v) = \begin{pmatrix} x_e(u) \\ y_e(u) \\ f(u) \end{pmatrix} + v \cdot \begin{pmatrix} x_e(u) \\ y_e(u) \\ g(u) \end{pmatrix}. \tag{23}$$

Theorem 4 *The ruled surface $S_e^t(s,u,v)$ defined by (23) is of locus isometry if the fixed point $A = (x_0, y_0)$ is adjusted to be $A^* = (x_0, y_0, f(u) + v \cdot g(u))$.*

Proof: First we prove that

$$S_e^t(s,u,v) = sS_c(u,v) + (1-s)\, S_d(u,v). \tag{24}$$

We observe that

$$\begin{aligned} & sS_c(u,v) + (1-s)\, S_d(u,v) \\ &= s\left(\begin{pmatrix} x_c(u) \\ y_c(u) \\ f(u) \end{pmatrix} + v \cdot \begin{pmatrix} x_c(u) \\ y_c(u) \\ g(u) \end{pmatrix}\right) + (1-s)\left(\begin{pmatrix} x_d(u) \\ y_d(u) \\ f(u) \end{pmatrix} + v \cdot \begin{pmatrix} x_d(u) \\ y_d(u) \\ g(u) \end{pmatrix}\right) \\ &= s\begin{pmatrix} x_c(u) \\ y_c(u) \\ f(u) \end{pmatrix} + (1-s)\begin{pmatrix} x_d(u) \\ y_d(u) \\ f(u) \end{pmatrix} + v \cdot \left(s \cdot \begin{pmatrix} x_c(u) \\ y_c(u) \\ g(u) \end{pmatrix} + (1-s) \cdot \begin{pmatrix} x_d(u) \\ y_d(u) \\ g(u) \end{pmatrix}\right) \\ &= \begin{pmatrix} x_e(u) \\ y_e(u) \\ f(u) \end{pmatrix} + v \cdot \begin{pmatrix} x_e(u) \\ y_e(u) \\ g(u) \end{pmatrix} = S_e^t(s,u,v). \end{aligned}$$

Finally we adjust the point A to be $A^* = (x_0, y_0, f(u) + v \cdot r(u))$, and hence the proof is complete.

We describe how we can incorporate a closed curve and its antipodal curve as base curves when constructing the respective ruled spaces. Specifically, in the next example we incorporate a cardioid curve, its antipodal curve into respective developable cone surfaces.

Example 5 *We let $C \in c_c(u) = [x(u), y(u)] = [(2 - 2\cos u)\cos u, (2 - 2\cos u)\sin u]$. Then we have shown that when the point D is a point on $c_d(u) = [x_1(u), y_1(u)] = [x(u+\pi), y(u+\pi)]$, the point D is a pseudo antipodal point of C with respect to the fixed point at $A = (0,0)$. Here we would like to discuss the following corresponding two developable surfaces based on a cone, whose respective base curves are plain cardioidal curves c_c and c_d respectively.*

1. First we define two surfaces that are induced by the plane curves $c_c(u) = [x(u), y(u)]$ and $c_d(u) = [x_1(u), y_1(u)] = [x(u+\pi), y(u+\pi)]$ respectively as follows:

$$\begin{bmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{bmatrix} = \begin{bmatrix} x(u) \\ y(u) \\ 1 \end{bmatrix} + v \begin{bmatrix} x(u) \\ y(u) \\ 1 \end{bmatrix}, \tag{25}$$

$$\begin{bmatrix} X_1(u,v) \\ Y_1(u,v) \\ Z_1(u,v) \end{bmatrix} = \begin{bmatrix} x_1(u) \\ y_1(u) \\ 1 \end{bmatrix} + v \begin{bmatrix} x_1(u) \\ y_1(u) \\ 1 \end{bmatrix}. \tag{26}$$

Since both surfaces $S_c(u,v) = [X(u,v), Y(u,v), Z(u,v)]$ and $S_d(u,v) = [X_1(u,v), Y_1(u,v), Z_1(u,v)]$ are constructed by lifting respective 2D cardioid curves to 3D surfaces by setting $Z(u,v) = Z_1(u,v) = 1+v$, we conjecture first both $S_c(u,v)$ and $S_d(u,v)$ are developable spaces. After further investigation, we see that if

$$P(u,v) = q(u) + v \cdot t(u) \tag{27}$$

is a ruled surface. Then the Gaussian curvature $K = 0$ if and only if the tangent plane at $P(u,v)$ does not depend on v. Indeed, it can be shown that respective the normal vectors of the tangents at both $\begin{bmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{bmatrix}$ and $\begin{bmatrix} X_1(u,v) \\ Y_1(u,v) \\ Z_1(u,v) \end{bmatrix}$ do not depend on v. Therefore both $\begin{bmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{bmatrix}$ and $\begin{bmatrix} X_1(u,v) \\ Y_1(u,v) \\ Z_1(u,v) \end{bmatrix}$ are developable surfaces.

2. However, the locus surface E satisfying $E = sC + (1-s)D$ is topological transformation between surfaces $S_c(u,v)$ and $S_d(u,v)$, whether or not the locus surface E is also developable depends on the coefficient s.

3. Furthermore, we observe that the locus surface $S_e^t(s,u,v) = [X_e(s,u,v), Y_e(s,u,v), Z_e(s,u,v)]$ can be constructed from its 2D base curve $[x_e(u), y_e(u)]$ directly as follows:

$$\begin{bmatrix} X_e(s,u,v) \\ Y_e(s,u,v) \\ Z_e(s,u,v) \end{bmatrix} = \begin{bmatrix} x_e(u) \\ y_e(u) \\ 1 \end{bmatrix} + v \begin{bmatrix} x_e(u) \\ y_e(u) \\ 1 \end{bmatrix} \tag{28}$$

4. Alternatively, the locus surface $[X_e(s,u,v), Y_e(s,u,v), Z_e(s,u,v)]$ can be constructed once both 3D surfaces of $S_c(u,v)$ and $S_d(u,v)$ are defined. In other words, we have $S_e(s,u,v) = sS_c(u,v) + (1-s)S_d(u,v)$. It is clear that the locus surface is locus isometry. In other words, the locus surface stays the same regardless of which way we use to construct such locus surface.

5. When exploring the space curves $S_e(s_0,u,v_0), S_d(u,v_0)$ and $S_c(u,v_0)$ with fixed $s = s_0$ and $v = v_0$, in order for the points A, C, D and E to be colinear, we need to adjust the fixed point to be

$$A = (0,0,1+v) \tag{29}$$

in this case.

6. We depict the screen shots below, including the space curves $c_c\left(u, \frac{\pi}{4}\right), c_d\left(u, \frac{\pi}{4}\right), c_e\left(\frac{1}{3}, u, \frac{\pi}{4}\right)$ when we set $s = \frac{1}{3}, v = \frac{\pi}{4}$ together with the locus surface $\begin{bmatrix} X_e(\frac{1}{3}, u, v) \\ Y_e(\frac{1}{3}, u, v) \\ Z_e(\frac{1}{3}, u, v) \end{bmatrix}$ (See Figure 5(a)). When we project the space curves to $z = 0$, we get back to the 2D cardioid case, see Figure 5(b). With the help of a DGS or CAS, we can visualize from Figure 5(b) that the surface can be flattened to a plane curve, which helps us to *conjecture* that the locus surface $\begin{bmatrix} X_e(\frac{1}{3}, u, v) \\ Y_e(\frac{1}{3}, u, v) \\ Z_e(\frac{1}{3}, u, v) \end{bmatrix}$ is a developable surface. In addition, we notice that A, C, D and E are all colinear satisfying

$$\overrightarrow{ED} = \frac{1}{3}\overrightarrow{CD}. \tag{30}$$

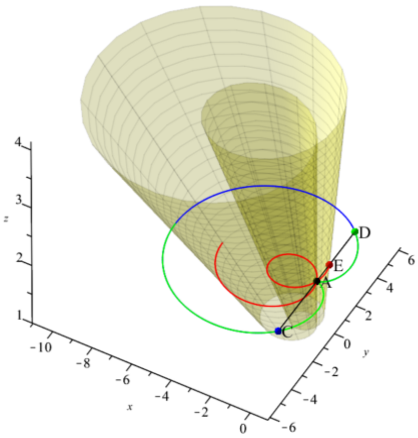


Figure 5(a). A developable surface when base curve is a cardioid.

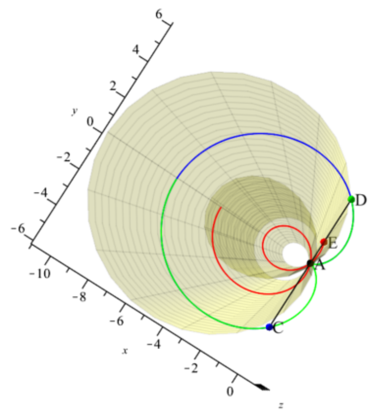
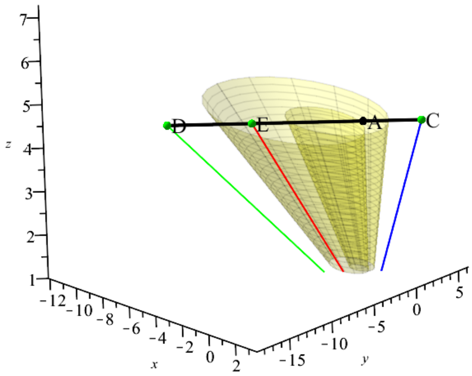


Figure 5(b). The projection of $z = 0$ becomes a 2D cardioid

We depict the screen shot in Figure 6, including the space curves $c_c(1, v), c_d(1, v), c_e\left(\frac{1}{3}, 1, v\right)$ when we set $s = \frac{1}{3}, u = 1$, together with the locus surface $\begin{bmatrix} X_e(\frac{1}{3}, u, v) \\ Y_e(\frac{1}{3}, u, v) \\ Z_e(\frac{1}{3}, u, v) \end{bmatrix}$. We can see that the generators $c_c(1, v), c_d(1, v), c_e\left(\frac{1}{3}, 1, v\right)$ for their respective surfaces are lines as expected. More importantly, we see that A, C, D and E are all colinear and

$$\overrightarrow{ED} = \frac{1}{3}\overrightarrow{CD}. \tag{31}$$



curves when $s = 1/2$ and $u = 1$

5.2 Tangent Developable Surfaces and Isometry

We refer to [9] that a tangent developable is a particular kind of developable surface obtained from a curve in Euclidean space as the surface swept out by the tangent lines to the curve. Such a surface is also the envelope of the tangent planes to the curve. Let

$$\gamma(u) = [x(u), y(u), z(u)], \tag{32}$$

where u is real, be a twice-differentiable function with nowhere-vanishing derivative. We see γ represents a space curve, we shall concentrate on the tangent developable surface that is written in the form of

$$\begin{bmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \end{bmatrix} = \gamma(u) + v \cdot \gamma'(u), \tag{33}$$

where $u, v \in \mathbb{R}$. The original curve $\gamma(u)$ forms a boundary of the tangent developable, and is called its directrix or edge of regression. The following observation is clear, which we omit the proof.

Theorem 6 *The tangent developable transformation is locus isometry.*

Example 7 *We start with the plane curve an ellipse of* $[x(u), y(u)] = [3\cos u, 2\sin u]$, *where* $u \in [0, 2\pi]$. *As we have discussed in Section 2 that if the fixed point A is* $(4, 3)$, *the pseudo antipodal curve* c_d *can be found as* $[x_1(u), y_1(u)]$ *using 4 and 5 respectively. We define* $x_e(s,u) = s \cdot x(u) + (1-s)x_1(u)$ *and* $y_e(s) = s \cdot y(u) + (1-s)y_1(u)$. *If we choose* $z(u) = z_1(u) = \sin u \cos u$ *for the edge of regression*

$$\gamma(u) = [x(u), y(u), z(u)], \tag{34}$$

and

$$\gamma_1(u) = [x_1(u), y_1(u), z_1(u)]. \tag{35}$$

respectively. We obtain two respective surfaces $S_c(u,v) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix} + v \begin{bmatrix} x'(u) \\ y'(u) \\ z'(u) \end{bmatrix}$, *and* $S_d(u,v) = \begin{bmatrix} x_1(u) \\ y_1(u) \\ z_1(u) \end{bmatrix} + v \begin{bmatrix} x_1'(u) \\ y_1'(u) \\ z_1'(u) \end{bmatrix}$. *It follows from the preceding Theorem that we can write the locus surface* $S_e^t(s,u,v)$ *as follows:*

$$S_e^t(s,u,v) = sS_c(u,v) + (1-s)\,S_d(u,v). \tag{36}$$

We discuss more about $S_c(u,v), S_d(u,v)$ *and* $S_e^t(s,u,v)$ *as follows:*

1. It can be shown that both surfaces $S_c(u,v)$ and $S_d(u,v)$ are tangent developable surfaces.
2. We depict $S_c(u,v)$ together with the space curves $S_c(u,0)$ (in red, which is also the edge of regression $\gamma(u)$ in Figure 7). We note that the space curve $S_c(u,0)$ forms a boundary of the tangent developable.

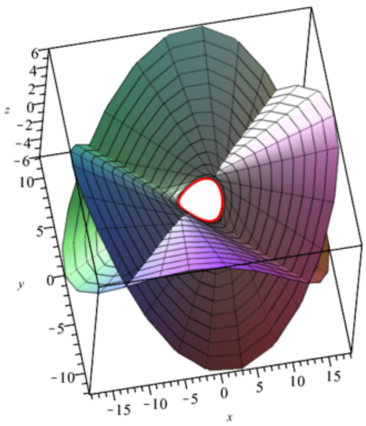


Figure 7. A developable surface and its generator in red

3. We remark that the developable spaces $S_c(u,v), S_d(u,v)$ and $S_e^t(s,u,v)$ are lifted from 2D. The choice of the fixed point A needs to be adjusted to

$$A = (x_0, y_0, z(u)) \tag{37}$$

when exploring space curves $S_e^t(s_0,u,v_0), S_d(u,v_0)$ and $S_c(u,v_0)$, where s_0 and v_0 are fixed constants. For example, when $s_0 = \frac{3}{5}$ and $v = 0$, we depict the space curves of $S_e^t(\frac{3}{5},u,0)$ (in red), $S_d(u,0)$ (in green) and $S_c(u,0)$ (in blue) in Figure 8below. More importantly, we notice that A, C, D and E are all colinear and satisfy

$$\overrightarrow{ED} = \frac{3}{5}\overrightarrow{CD}. \tag{38}$$

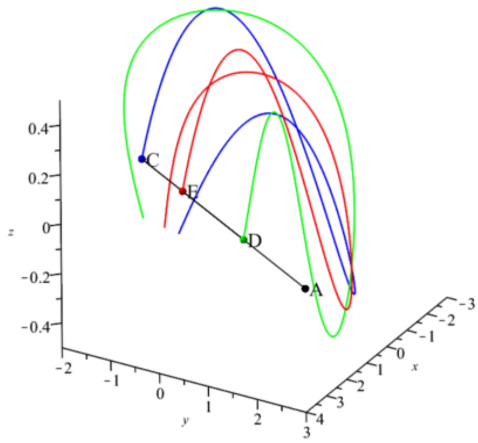


Figure 8. Space curves when $s = \frac{3}{5}$ and $v = 0$

Remark: Since $S_e^t(s,u,v)$ is a topological transformation between two tangent developable spaces $S_c(u,v)$ and $S_d(u,v)$, which may not a tangent developable space again for certain parameter s. Therefore, it will be interesting to investigate for what value of s, $S_e^t(s,u,v)$ stays as a tangent developable surface.

5.3 Applications

In robotics it is well known that antipodal grasps can be achieved on curved objects. Developable surfaces have many applications in engineering, architecture and etc.. For example, understanding developable surfaces in mechanisms will enable engineers to understand hyper-compact mechanisms in several important application areas. The developable surfaces are also critical in the understanding of architectural geometry because then architects can understand the limitation and use of possible geometries to include in their work. We hope our findings involve pseudo-antipodal points, which lead to the locus surfaces, can assist and promote further research areas in applicable fields. We demonstrate some interesting architectures images that are related to either ruled or developable surfaces.

1. An architecture and ruled surface (see [1]).

Figure 9. An architecture and ruled surface

2. An architecture and developable surface (see [2]).

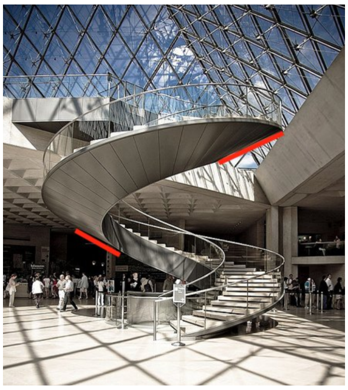


Figure 10. An architecture and developable surface

6 Conclusion

In this paper, we see technologies can assist a learner to venture from an elementary college entrance practice problem to several challenging explorations in 2D and 3D. It is indisputable that technological tools indeed provide us with many crucial intuitions before we can prove rigorous analytical solutions with a computer algebra system. Evolving technological tools definitely have made mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

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