

A Browser Based Mathematical Investigation with GXWeb and WolframAlpha

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Abstract: *The combination of Geometry Expressions with commercial CAS systems such as Mathematica or Maple makes a formidable tool kit for mathematical exploration. However, these systems are not appropriate for casual use, as they require installation, and are not free. In this paper, we demonstrate the use of the free browser based version of Geometry Expressions: GXWeb [1], along with the free browser based CAS tool WolframAlpha [2]. Both have taken great strides to enhance usability, while retaining the power of their underlying technologies. We root the discussion in an investigation of the Tschirnhausen cubic, the curve which appears as the catacaustic of parallel oblique beams of light reflecting in a parabolic mirror. We highlight techniques for moving information between GXWeb and WolframAlpha and for exploiting WolframAlpha's permissive approach to mixing mathematics with natural language.*

1. Introduction to the Tools

For those unfamiliar with GXWeb, a few words on the User Interface may be appropriate. The main GXWeb screen is a drawing area. The first of three drop down buttons lets you sketch geometric objects: points, lines circles, etc. The second drop down button lets you apply constraints: distances, angles etc.. The third drop down button handles constructions: bisectors, transforms, locus etc. Individual constraint and construction buttons are enabled based on the current selection. If nothing is selected, nothing is enabled, however a help icon on the button takes you directly to a help video which shows you how to make the appropriate selections to enable that button.

There are two drawers which can be opened out. On the right hand side the drawer lets you set numerical values of variables, and examine numeric measurements made from the drawing. At the bottom of the screen is a drawer which lets you examine symbolic measurements.

Short videos which may help the new user may be found at these web locations (or accessed directly from the Help button on GXWeb itself).

<https://geometryexpressions.com/gxweb/help/>

Wolfram Alpha be viewed as a free browser based version of Mathematica, but with a couple of radical user interface ideas: the application will do its best to interpret mathematics you type in, no matter what format you use, and it will give you a variety of related expressions to your input: different algebraic forms, derivatives, solutions, without being prompted, effectively giving you a smorgasbord of answers to choose from.

2. Reflections in a parabola

If we take a parabolic mirror, the reflection of any ray parallel to its axis passes through its focus (fig. 1). If we reflect a set of such parallel rays, they all pass through the same point and form an extremely concentrated region of light. Putting kindling here can cause a conflagration. If the parallel beams are not aligned with the axis, the concentration of reflected rays is less extreme and forms a curve. This curve is called the catacaustic, and mathematically is the envelope of the family of reflected rays. If the rays are perpendicular to the axis of the parabola, the curve is Tschirnhausen's Cubic [3]. In this paper, we will investigate what form the catacaustic takes when the rays are oblique.

We will start with the parabola $y = x^2$. To draw any function in GXWeb, you first sketch it, and the software gives you a guess as to what function you intended (with unknown parameters). If you sketch anything that looks vaguely like a parabola, its guess will be the generic quadratic $ax^2 + bx + c$. In any case, you are always able to enter a specific function. For the function $y = x^2$, we can identify the location of the focus by intersecting the reflection of an incident ray parallel to the y axis with the y axis (point B in figure 1).

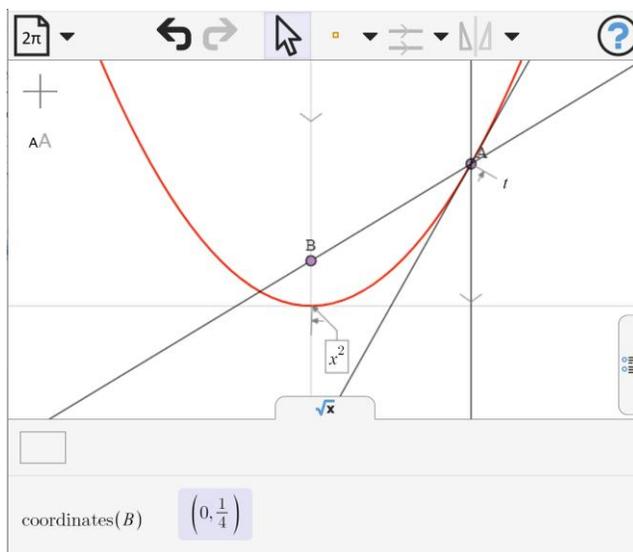


Figure 1: The reflection of any incident line parallel to the y-axis passes through point B.

As these coordinates do not depend on the location t of the point of incidence, we can infer that all reflected lines pass through the same point.

What happens if our incident rays are perpendicular to the y axis rather than parallel to it? In the model below (figure 2), an incident ray parallel to the x-axis is reflected in the tangent to the curve. The envelope curve defined by the family of reflected lines is the curve of light concentration, called the **catacaustic**, or simply the **caustic**. The envelope curve is defined to be the curve which is tangent to all the lines in the family. Alternatively, it can be thought of as the limit of the locus of the intersection between neighboring lines in the family as ‘neighboring’ tends to ‘identical’

The curve forms a loop, and GXWeb will output its equation which proves, on examination, to be a cubic. In fact this particular cubic has a name: the Tschirnhausen cubic, though it was also studied by l'Hopital and Catalan.

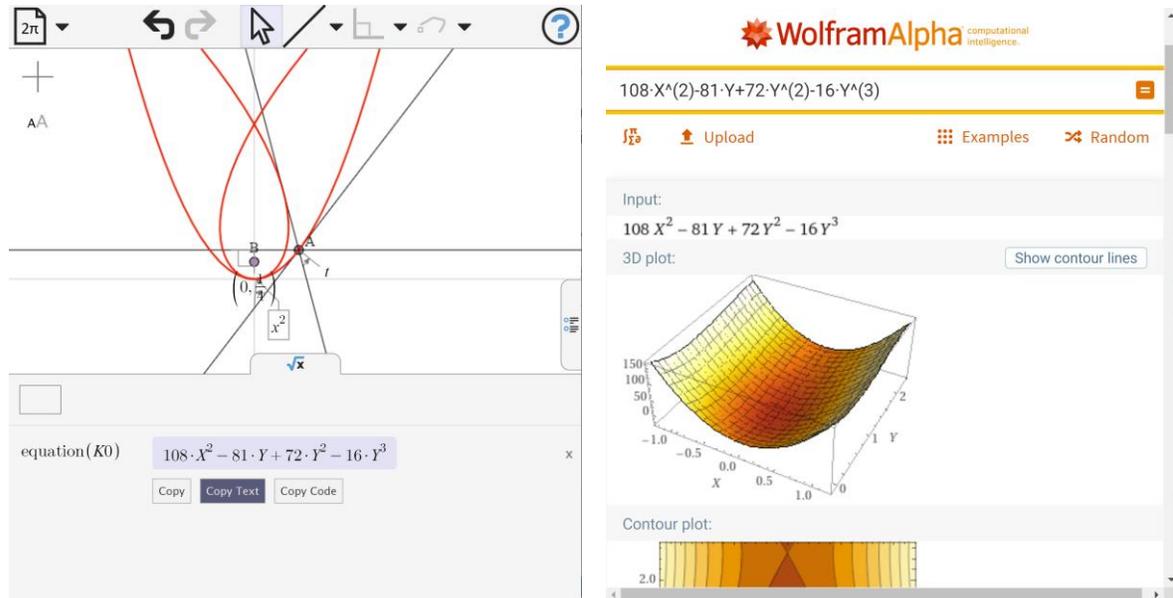


Figure 2: To copy an expression from GXWeb, first hover over the expression, then hover over the Copy button, then click on the Copy Text button.

As the curve has no constant term, it clearly passes through the origin. There is a double point on the y-axis. The y value of this point may be determined by solving the equation for Y, with 0 substituted for X. As this can be factored to a quadratic, it is quite manageable by hand. However, we can use this opportunity to apply WolframAlpha.

To copy an expression from GXWeb to WolframAlpha, you need to first hover over the expression, at which point a Copy button appears. Now if you hover over the **Copy** button, a **Copy Text** and **Copy Code** button appears. For WolframAlpha, you need to use **Copy Text**.

When you put a mathematical expression into WolframAlpha, it immediately gives you a large number of pieces of information about that expression. Sometimes you can use the information directly, other times you need to be more specific and request some particular derived quantity. We will see examples of this later. In this case, however, we see that a number of real roots of the equation are given, and though it does not give the root at (0,0), it does give one at $(0, \frac{9}{4})$. Creating a point at this location, we see it sitting on the double point of the curve.

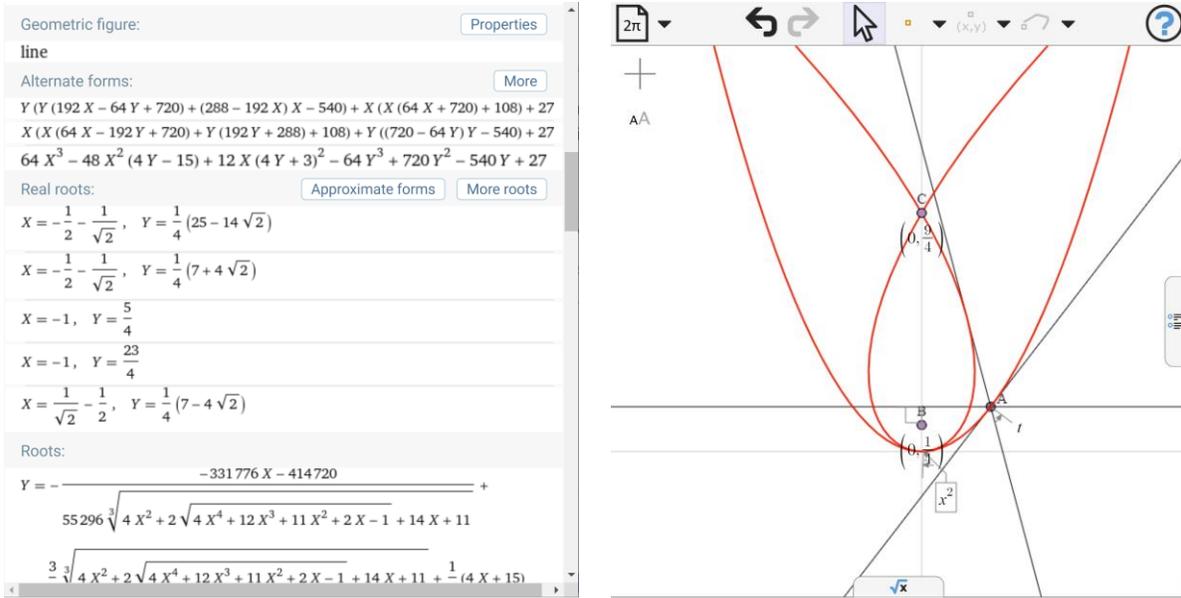


Figure 3: Wolfram Alpha immediately gives you a plethora of information about the input expression, including, in this case a number of roots. One of the roots can be seen to lie at the double point (crossover point) of the curve.

If the incident light is parallel to the y axis, we see that the catacaustic is composed of a single point. If it is parallel to the x-axis, the catacaustic is the Tschirnhausen cubic. What if the light comes in obliquely? In figure 4, we apply the above analysis to light coming in at 45 degrees to the axes.

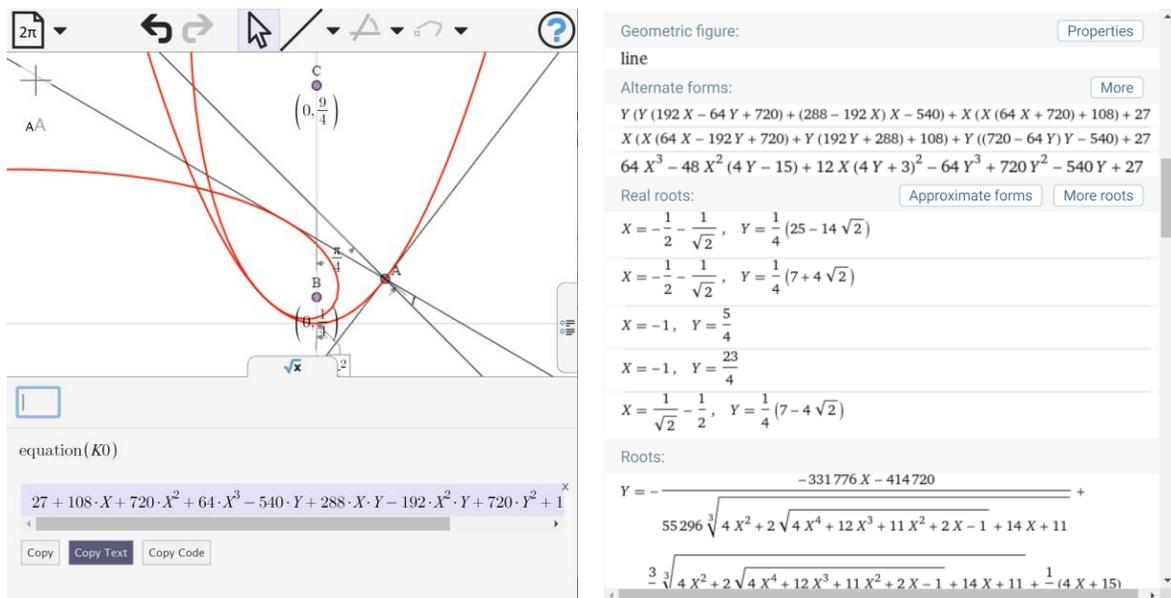


Figure 4: The equation of the caustic formed by light at 45 degrees to the axes. A number of real roots are shown in Wolfram Alpha.

The root at $\left(-1, \frac{5}{4}\right)$ would appear to correspond to the crossover point of the caustic. A question of interest is, what parameter locations t on the parabola are responsible for the crossover point? To investigate this, we can look at the parametric equations of the curve. We can copy this into WolframAlpha, delete the y coordinate and equate the x coordinate to -1 . We can then ask Wolfram Alpha to solve. Here we can mix natural language with mathematics. We simply put the word 'solve' in front of our mathematical expression. As there is only one variable, WolframAlpha knows what we want to solve for and proceeds to give us the solutions

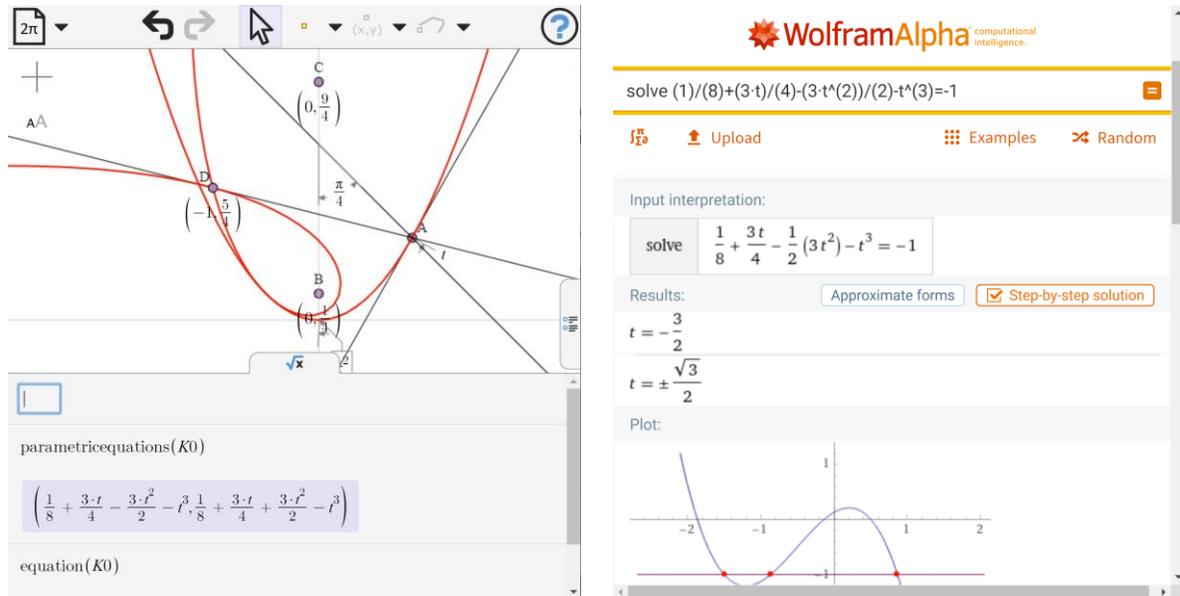


Figure 5: Parametric equations for the caustic curve and the parameter value corresponding to the double point.

A quick experiment shows that the responsible values of t are $\pm \frac{\sqrt{3}}{2}$

Repeating this process with other input angles reveals that the same value of t always yields the double point on the catacaustic, and that in fact these points lie on a circle passing through the focus and the two points $t = \pm \frac{\sqrt{3}}{2}$

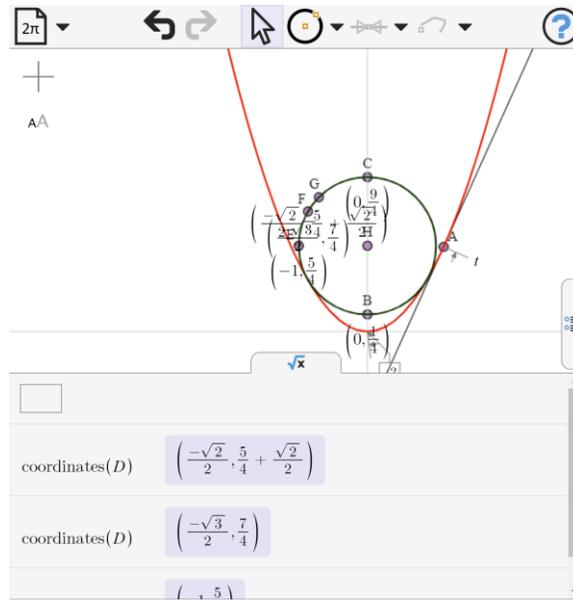


Figure 6: Double points for various incident light angles lie on a circle.

Here we have observed two things: first that the point on the caustic corresponding to a ray reflected at a particular point on the curve lies on a circle, secondly that the double point on the caustic is generated by the same points, regardless of the angle of incoming light. Are these two properties specific to the parabola, or general?

3. The locus of the intersection of the images of parallel rays under reflection in angled mirrors

The envelope of a family of lines may be viewed as the limit of the locus of the intersection point of two close neighbors. While the distance between neighbors remains finite, the reflection of two parallel beams of light can be modelled as follows.

In the diagram, points A and B represent the neighbors on the curve, while CA and CB are tangents to the curve. The reflections of parallel incident rays meet at point D. If angle ACB is θ , then the angle ADB is $2(\pi-\theta)$. If E is the circumcenter of ABC, then D may be seen (and proved) to be on the circumcircle of AEB. This circle is therefore the locus of the intersection point as the angle of the parallel incident rays varies.

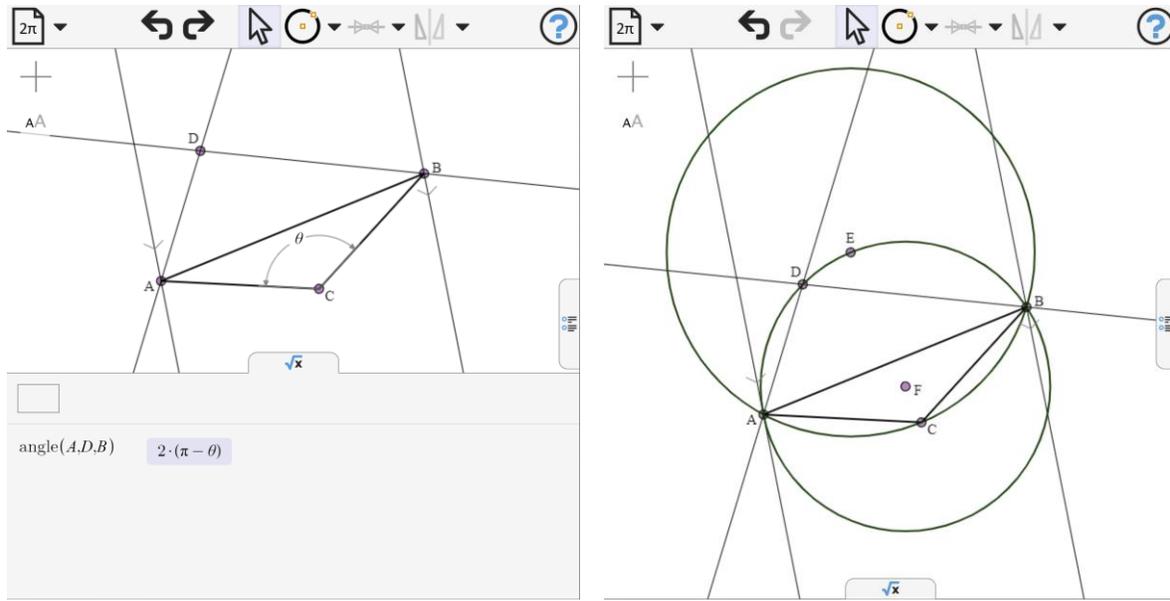


Figure 7: Point D is the intersection of the reflections of two parallel rays in lines AC and BC. It lies on the circumcircle of A, B and the circumcenter of ABC.

The limit of this circumcircle as points A and B approach each other along the curve is then the locus of a the point on the catacaustic defined by the merged point as the direction of the incident ray varies.

In the case of the parabola, we can create a model where the points A and B lie on the curve at parametric locations (x-coordinates) t-h and t+h. The radius of the locus circle can then be read off from GXWeb and exported to Wolfram Alpha where we may take the limit.

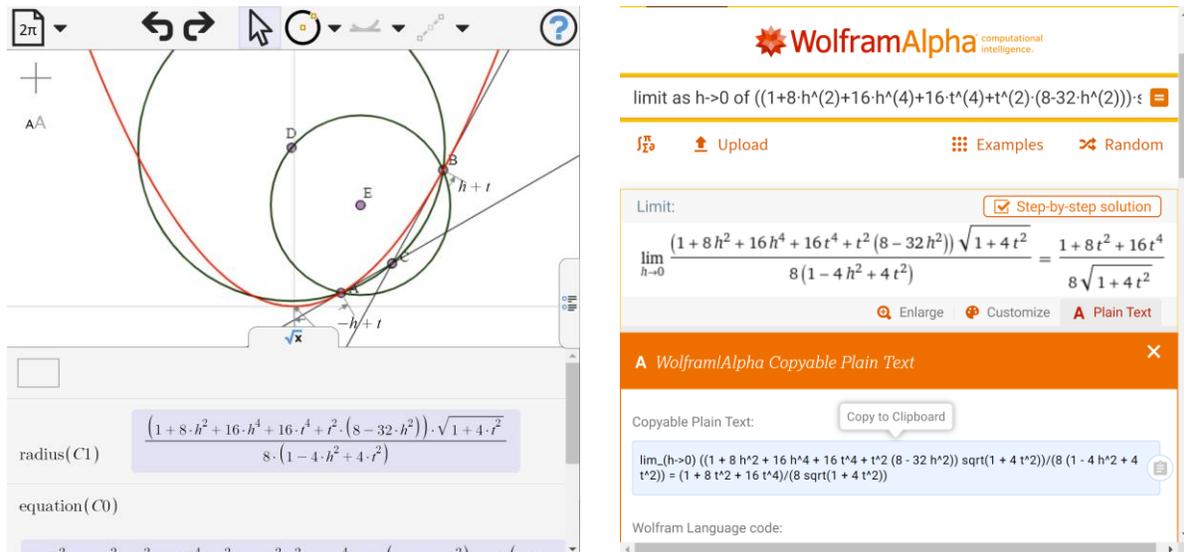


Figure 8: Using Wolfram Alpha to find the limit of the radius of the locus of Figure 7.

Notice that we can use natural language to ask for the limit, simply typing **limit as h->0** in front of our pasted expression. We could have put the **limit of** up front, followed by the expression

followed by **as h tends to 0**, or we could have expressed the question in any of a number of different ways. Wolfram Alpha is designed to infer the intent of the question. This doesn't mean that it always infers correctly. However it does display its interpretation first for you to check. The benefit of this approach is that you don't need to remember or look up a set of commands, you just experiment with natural language inquiry.

To copy back from WolframAlpha into GXWeb, you need to first display Copyable Plain Text and copy this (as shown above). In this case the copyable plain text contains both the problem and the solution. We would like to create a GXWeb model with the circle in place, through point A, tangent to the parabola tangent at this point and with the correct radius, so we need to paste in the radius on its own, but not the limit problem. Two ways of achieving this are illustrated in figure 9. In the first the plain text is pasted back into WolframAlpha, and everything deleted except the solution. This can then be cleanly copied from WolframAlpha's Copyable Plain Text region. The second strategy is to copy the expression into a text editing application (in this case Notepad) and then select the piece that you want to paste into GXWeb.

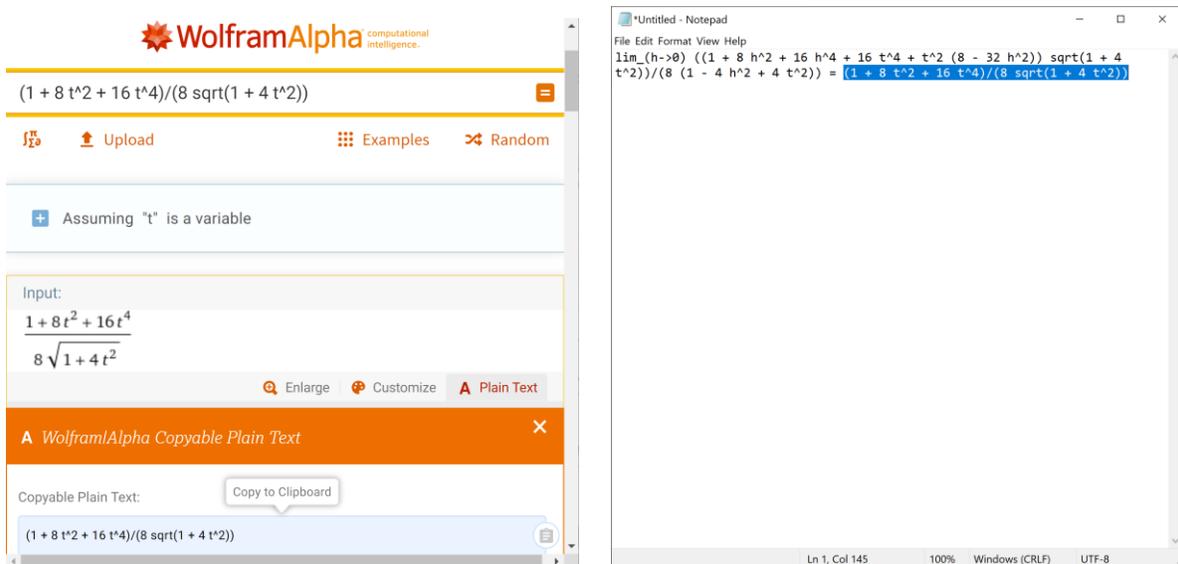


Figure 9: Wolfram Alpha responses may contain the question as well as the answer. To copy only part of such a response, either paste it back into Wolfram Alpha and remove the unwanted portion, or paste into some text processing program and copy only the required piece.

Once pasted in we observe the locus circle. As the reflection of every vertical line passes through the focus, we would expect this circle to pass through the focus, and this can be verified by inspecting the coordinates of its intersection with the y -axis.

In the above, we determined that the locations $t = \pm \frac{\sqrt{3}}{2}$ were special in that they defined the double point in the catacaustic, regardless of the direction of the light. We observe that this point is special with respect to the locus circle: this is the point at which the locus circle touches both sides of the parabola (fig. 10).

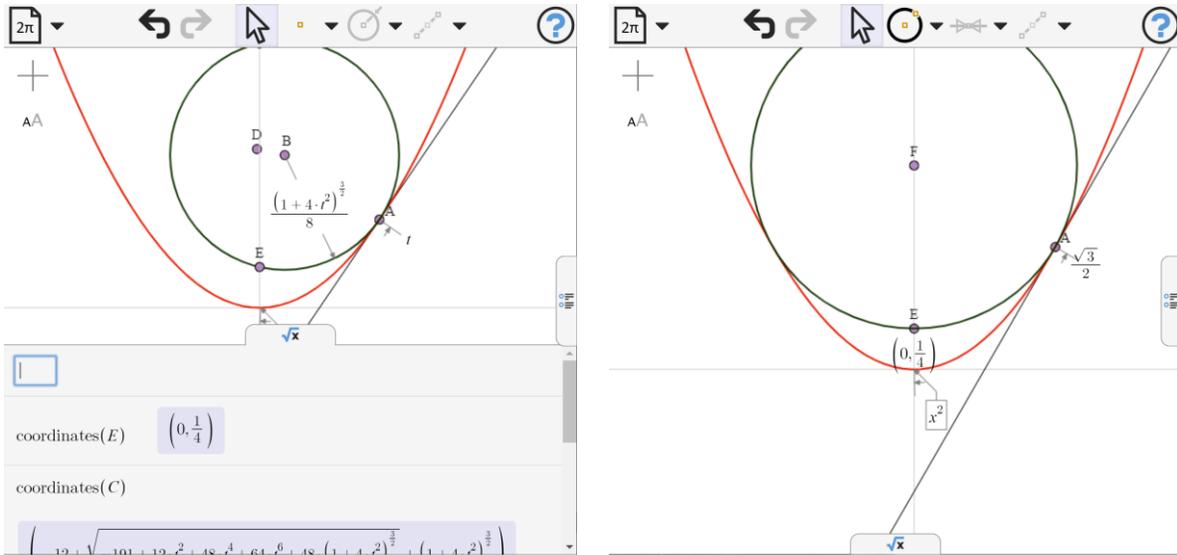


Figure 10: The circle shown is the locus of the point on the caustic due to reflection at t , for any angle of incident light. As vertical incident light passes through the focus, so does this circle.

The circle touches both sides of the parabola when $t = \frac{\sqrt{3}}{2}$

4. Oblique light Catacaustic

Looking at the caustic curve, we observe that for incident light angle θ to the horizontal, the axis of the caustic curve (from the focus of the parabola to its double point) is at angle θ to the vertical, and the distance between the focus and the double point is $2\cos(\theta)$.

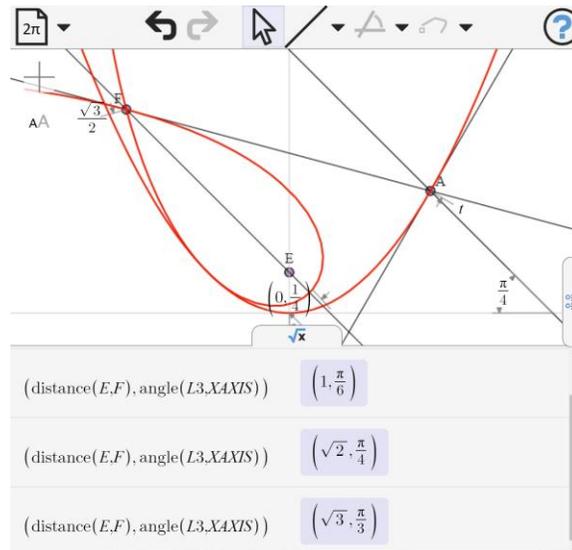


Figure 11: Distance between the focus E and the double point F , along with the angle that the caustic's axis of symmetry makes with the x axis, for 3 different angles of incident light.

To the naked eye, it appears that the caustic is simple a rotated, scaled version of the Tschirnhausen cubic. This can be tested for a specific angle, by examining the equation of the caustic generated

by that angle, and comparing it with the curve generated by the caustic formed by a family of horizontal rays, rotated about the parabola focus and scaled.

This can be tested most readily by examining the parametric equations of, first the caustic at general angle θ , and then the caustic at angle 0, rotated by θ about the parabola focus and then dilated about the focus by $\cos(\theta)$.

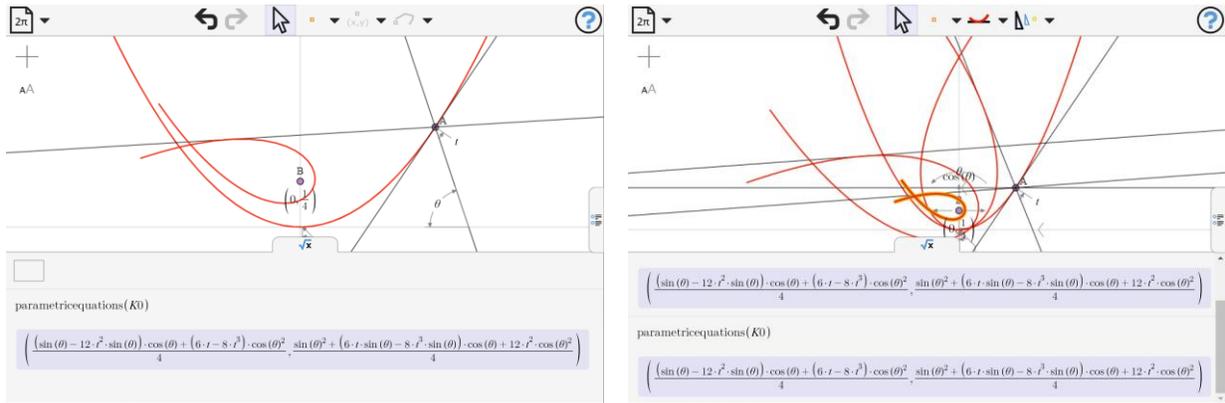


Figure 12: The caustic for incident light at angle θ to the horizontal is identical to that for horizontal incident light, rotated about the focus by θ then dilated about the focus by $\cos(\theta)$.

When θ becomes $\frac{\pi}{2}$ the scale factor becomes 0 and the entire catacaustic collapses to a single point: the parabola focus.

A quick counterexample suffices to show that this property is not general. Changing our base curve to $y = x^4$ and displaying the caustic for horizontal incident rays and for oblique rays, clearly the curves are not of the same shape.

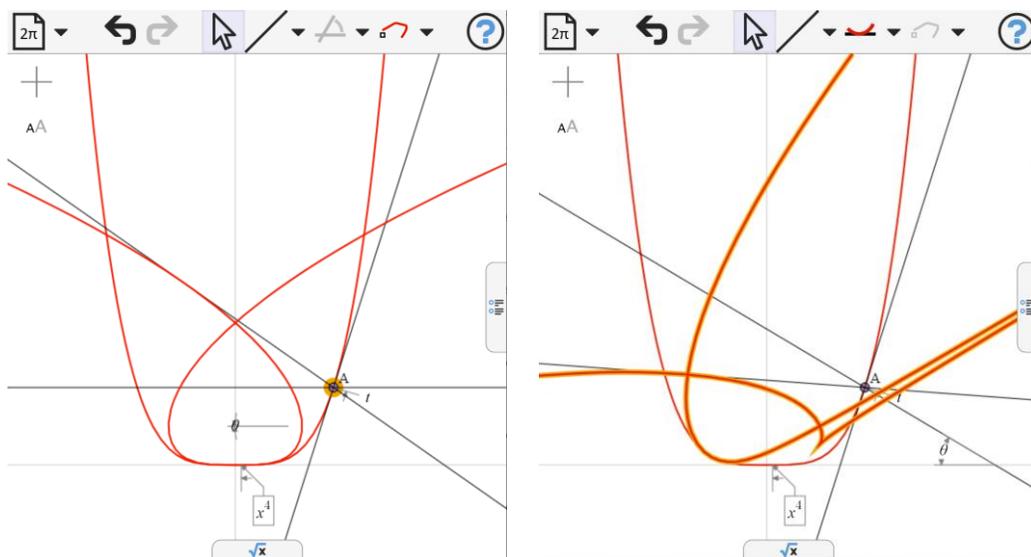


Figure 13: The caustic for the curve $y = x^4$, first for horizontal, then for oblique incident light.

5. Bezier Curves

We will examine the representation of the caustic as a Bezier curve and thus derive a feeling for the geometry behind its curious spiral symmetry. First we remind the reader of some relationships between the Bezier Curve and its control polygon which we will use in the discussion.

A cubic Bezier Curve may be defined as a locus in terms of a control polygon ABCD (Figure 14). Points E F and G are defined to be at proportion t along segments AB, BC, CD respectively. Points H and I are the same proportion t along segments EF and FG respectively. Point J is defined to be the same proportion t along HI. The Bezier Curve is the locus of J as t varies between 0 and 1.

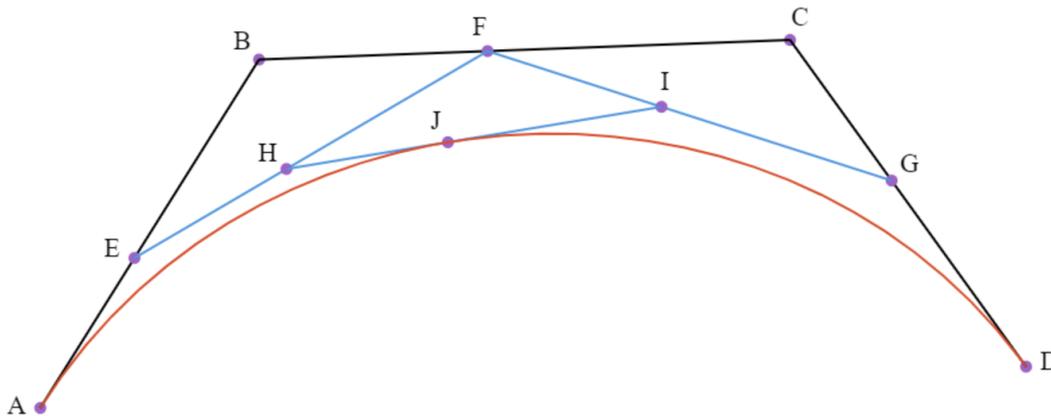


Figure 14: A Cubic Bezier Curve and its control polygon

The curve is a parametric cubic, and 6th order as an implicit curve.

Important properties, which we will use in the following are that AB is tangent to the curve at A and CD is tangent to the curve at D, while HI is tangent to the curve at I. Hence the curve is the envelope of the family of lines defined by HI as t varies.

A quadratic Bezier Curve may be defines as a locus in terms of its control polygon ABC (figure 15). Di and E are defined to be proportion t along segments AB and BC respectively, while F is proportion t along DE. The quadratic Bezier Curve with control polygon ABC is the locus of F as t varies.

Important properties are that AB and BC are tangent to the curve at A and C, while DE is tangent to the curve at E. The curve is a parametric quadratic, a portion of a parabola.

Given two points U,V on a parabola, the control polygon for the portion of the parabola between these points is U,V,W, where W is the intersection of the tangents to the parabola at U and V.

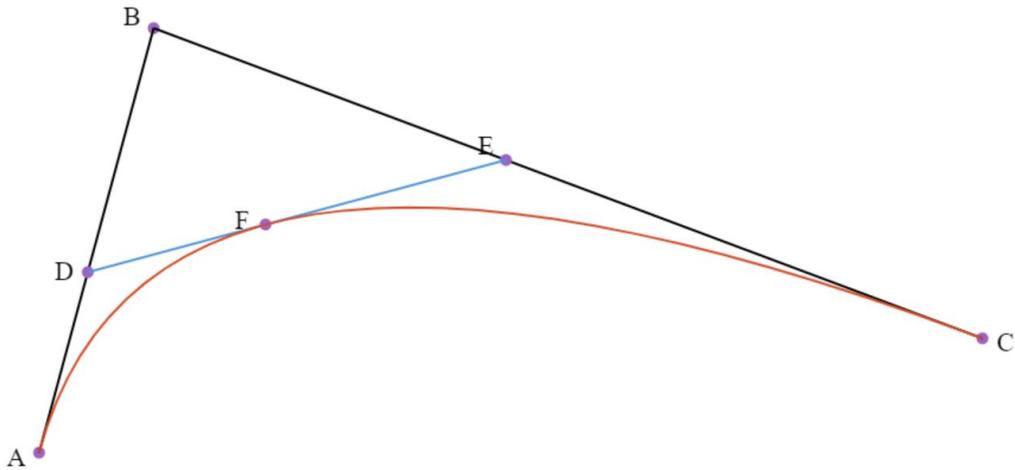


Figure 15: A Quadratic Bezier Curve and its control polygon

6. Tschirnhausen's Cubic as a Bezier Curve

Any portion of a parabola may be expressed as a quadratic Bezier curve whose control points are the endpoints of the portion and the intersection of its tangents at those points. The portion of the parabola $y = x^2$ between $x = -\frac{\sqrt{3}}{2}$ and $x = \frac{\sqrt{3}}{2}$ can be expressed as the Bezier curve with control points $A = \left(-\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$, $B = \left(0, -\frac{3}{4}\right)$, $C = \left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$ as shown in figure 16.

Point D is positioned proportion t along AB and point E proportion t along BC. Point F is proportion t along DE. The locus of F as t varies is the desired parabola. This can readily be checked by asking GXWeb for the equation of the locus. Further, DE is tangent to the parabola.

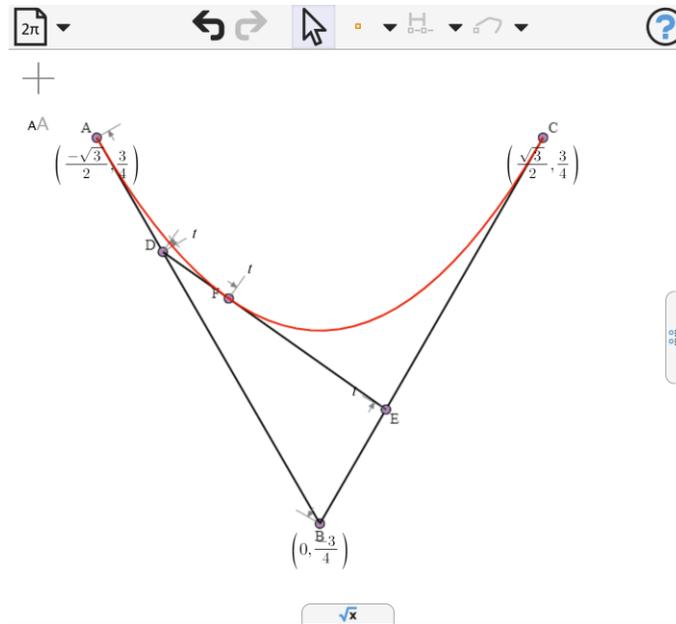


Figure 16: Control polygon for $y = x^2$ between $x = -\frac{\sqrt{3}}{2}$ and $x = \frac{\sqrt{3}}{2}$ viewed as a quadratic Bezier Curve

We create a point $G = (0, \frac{9}{4})$, and form a triangle with side GH passing through A and GI passing through C, and HI the horizontal line through B. We put J at proportion t along GH, K at proportion t along HI and L at proportion t along IG.

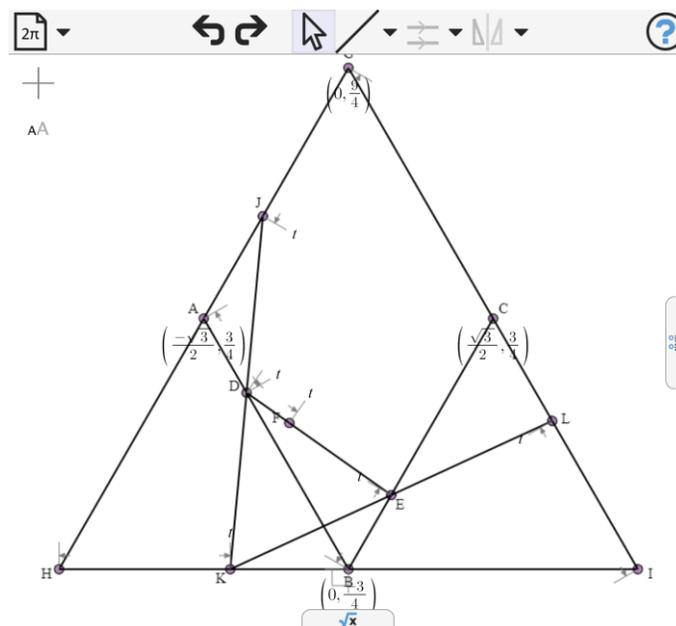


Figure 17: A control polygon for the caustic curve's loop as a cubic Bezier Curve

We note in passing that D is the midpoint of JK and E is the midpoint of KL.

We create a line through F parallel to HI and reflect it in DE. As F is on the parabola, and DE its tangent, this represents reflecting the line in the parabola at that point.

Let M be the intersection of the reflected line with JK and let N be its intersection with KL.

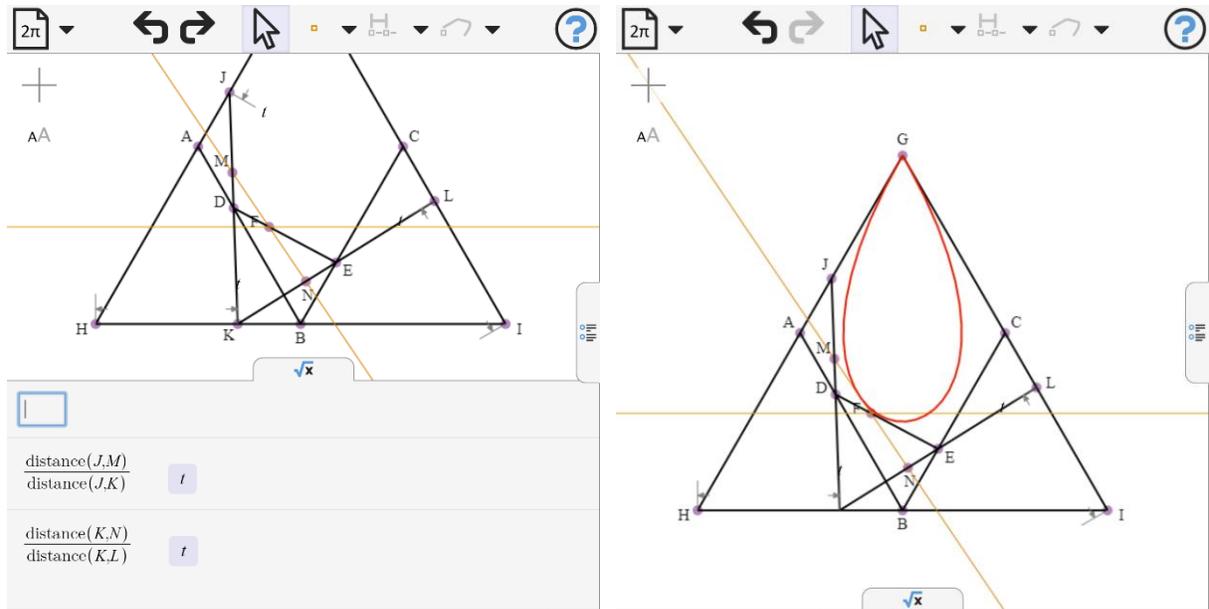


Figure 18: The reflection of horizontal ray striking point F on the parabola is shown to be identical to the tangent to the Bezier curve.

GXWeb can display the ratios $\frac{JM}{JK}$ and $\frac{KN}{KL}$. Both evaluate to t . Hence the caustic, which is the envelope of MN is the cubic Bezier curve defined by the control points GHIG

A similar analysis for oblique incident ray is shown in figure 19. Points A,B,C are defined as above. Parallel incident rays at an angle θ to horizontal impinge on A and C and are reflected in AC and BC. The reflected lines meet at point D. An equilateral triangle DEF is drawn such that DE passes through A, DF passes through C and EF passes through B.

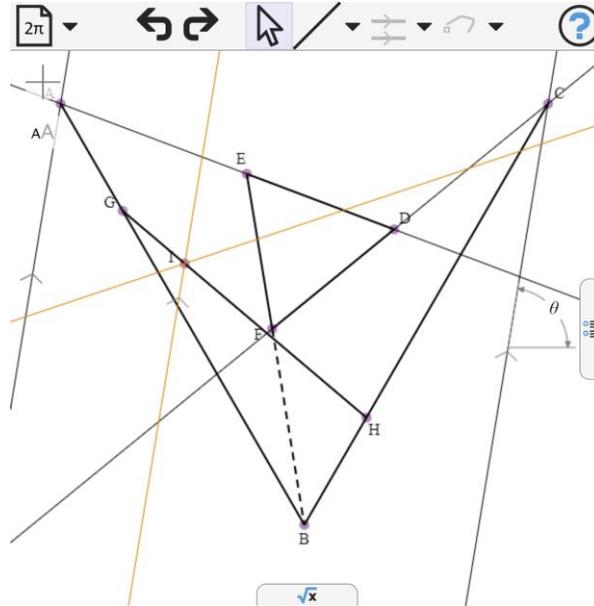


Figure 19: Geometric derivation of the control polygon for the caustic for light angle θ to horizontal along with the reflection of the ray in a point of $y = x^2$.

The caustic is the cubic Bezier curve defined by the control points DEFD. To prove this, we add G proportion t along AB and H proportion t along BC, then I proportion t along GH. We draw a line through I and constrain it to be parallel to the ray incident to A. We then reflect this line in GH. This reflected line is tangent to the desired caustic curve.

We now create the traditional construction for the cubic Bezier curve with control polygon DEFD. Point J is placed at proportion t along DE, point K at proportion t along EF and point L at proportion t along FD. Point M is positioned at proportion t along JK and point N at proportion t along KL. The line MN is tangent to the Bezier curve generated by this configuration. We need to show that MN is the same as the reflected line through I. This can be done by verifying that the symbolic distance between each point and the line is 0.

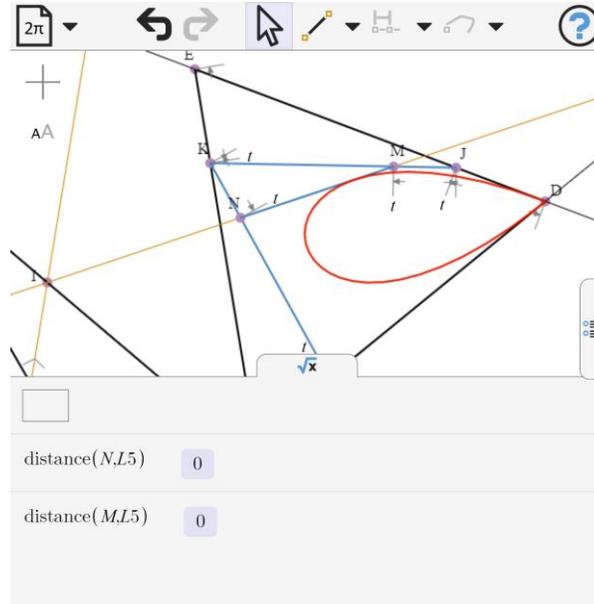


Figure 20: The cubic Bezier curve construction on the control polygon of figure 19. The line MN, which is by construction tangent to the curve, is shown to lie on the reflected ray.

A further observation is that the centroid of ABC is the point $(0, \frac{1}{4})$ which is the focus of the parabola. The same point is also the centroid of DEF.

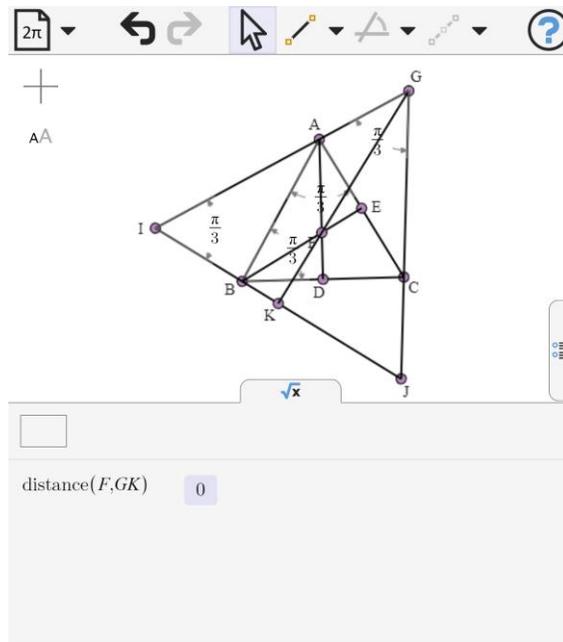


Figure 21: An equilateral triangle and an inscribed equilateral triangle have the same centroid.

A simple model (fig. 21) can be used to verify this. In figure 21, triangle ABC is constrained to be equilateral, and its centroid F determined by intersecting two medians. Triangle GIJ is constrained to be equilateral, with GI passing through A and IJ passing through B and JG passing through C.

GK is a median of GIJ. The distance between F and the median is (symbolically) 0. This (and a similar result for another median) shows that the centroid of GIJ is the same as the centroid of ABC.

7. Further Investigation

A question for further thought: which other curves have a catacaustic which is the same shape regardless of the angle of the incident rays? The circle, certainly has caustics which are rotated but not scaled. Are there any other curves with this property?

8. Conclusion

We have illustrated the use of free browser based symbolic geometry (GXWeb) and CAS (Wolfram Alpha) in an exploration of the mathematics of the catacaustic of the parabola. The conclusion of the investigation is that the catacaustic has the same shape (scaled and rotated) for any set of parallel incident rays. This is the Tschirnhausen Cubic, though reduced to a single point when the rays are parallel to the axis of the parabola.

We have further shown how to view Tschirnhausen's cubic as a Bezier curve. The connection between the construction of a parabola as a quadratic Bezier curve and the construction of the Tschirnhausen Cubic as a cubic Bezier curve sheds a little light on the conundrum of why the parabola has this curious property of yielding a similar caustic from every angle.

In the course of the investigation, we have illustrated how expressions can be copied and pasted from GXWeb into WolframAlpha and back again. We have shown how the free-form command entry in Wolfram Alpha along with its technique of generating a plethora of information related to an input facilitates casual use, as does the simplicity of the GXWeb graphical interface.

In conclusion, these browser based applications work well together and constitute a powerful toolkit for mathematical modelling

9. References

- [1] GXWeb www.geometryexpressions.com/gxweb
- [2] Wolfram Alpha www.wolframalpha.com
- [3] Dennis, L. J., & Lawrence, J. (1972). A catalog of special plane curves. Dover Publications,