Limit and Continuity of a Function. Software Aspect

Vladimir Nodelman
nodelman@hit.ac.il
Department of Computer Sciences,
Holon Institute of Technology
52 Golomb St., Holon, 58102,
ISRAEL

Abstract: *The studies of fundamental concepts of limit and continuity of a function cause serious difficulties: students first time meet here definition with numerous quantifiers.*
Herewith, the existence of a quantifier of generality in the definitions of many concepts, being an objective obstacle, does not allow building adequate models of these notions due to the impossibility to provide an infinite number of checks.
This paper illustrates the methods of teaching and learning these concepts with intensive use of software. In general, the types of student's activities do not depend on the specifics of functions (whether they are real or complex), whereas, the supporting models are significantly different.
By means of author's non-profit software "VisuMatica" students not only use some ready models but construct proper models by themselves and explore with their help the studied contents.

1. The concept of Limit

The case of limit of real functions of real variables was discussed in [3][1].

1.1 Function of complex variable

The concept of *limit of a function of complex variable* is similar to the studied case of real variable. Domain and Range here can be illustrated by 2D-planes.

1.1.1 Definition *by Cauchy* (ε - δ definition)

D1 We call *limit of* $f(z)$ as z approaches an accumulation point z_0 is L, or that $\lim_{z \to z_0} f(z) = L$ if and only if for each positive number ε, there is a positive number δ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.

The graphical interpretation of $\lim_{z \to z_0} f(z) = L$ is as follows:

For each circle of radius ε centered at w_0 there exist correspondent radius δ of a circle with center L that images of all its interior points (not necessarily including z_0) that belong to Dom(f) lie inside the circle (L, ε).

[1] This report can be viewed as a sequel of [3].

VisuMatica in the "*Zone style*" tab of "*Mapping*" dialog includes elements for tuning the interface that help to grasp this logic (Fig.41).

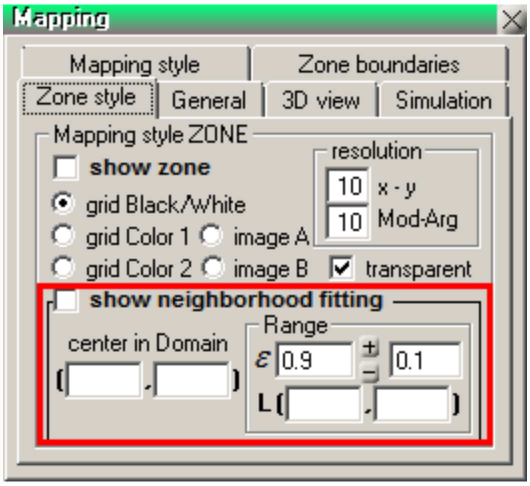

Figure 1

Checking the "*show neighborhood fitting*" checkbox starts the mechanism. From this moment the mouse pointer, while located in the Domain view (left half of Fig.2), symbolizes z_o. It is supplied by a pair of circles:

- A blue circle, in the Range view (right half of Fig.2) of radius ε is centered at $f(z_o)$ – the "suspected" w_o.
- A red disc, around mouse pointer of automatically calculated radius δ. Its red image lies inside the blue circle[2].

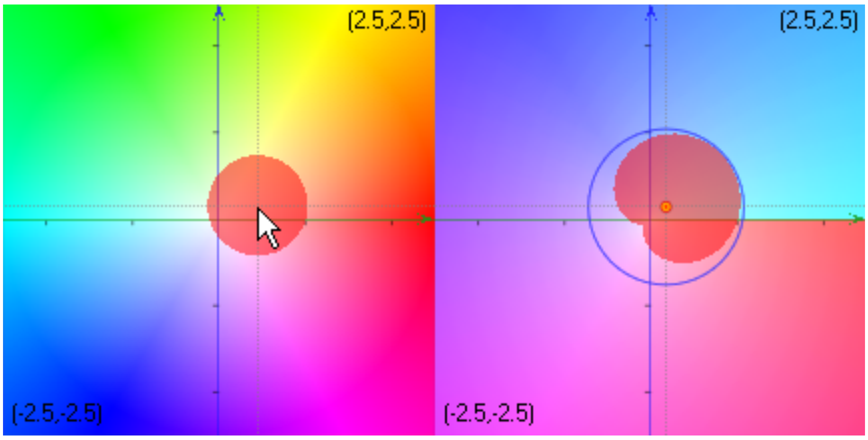

Figure 2

Fig.2 presents this pair in case of $w = z^2$ and $\varepsilon = 0.9$. This scene constantly follows after the moving mouse pointer so long as the "*show neighborhood fitting*" check box remains checked. Pressing the "+" / "-" buttons increases / decreases the value of ε by 0.1. Needless to say, these values (as well as the others) can be changed by straight retyping the contents of text boxes.

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[2] The mapping $w = f(z)$ is coded by colors: z in the D_f view and $f(z)$ in the R_f have the same color. The image of the mouse pointer in the Range view presented by a small red-yellow circle.

Till now the mechanism looks useless[3]. *We need a fixed* z_0 to start exploration. The common way to control the mouse location is an application of the *simulation* tool in the "*Mapping*" dialog. But it is not so comfortable to switch between tabs in order to change the value of ε. Much better is to set the mouse location by entering its coordinate's pair in the two text boxes of "*center in Domain*" (real and imaginary values of a complex z_0). Following the graphical interpretation of the definition of limit of function, it remains to change (decrease) the value of ε and ensure the existence of red $\delta-$ disk as well as its image belonging to the $\varepsilon-$ circle around the same point in the Range view – the suspicious limit.

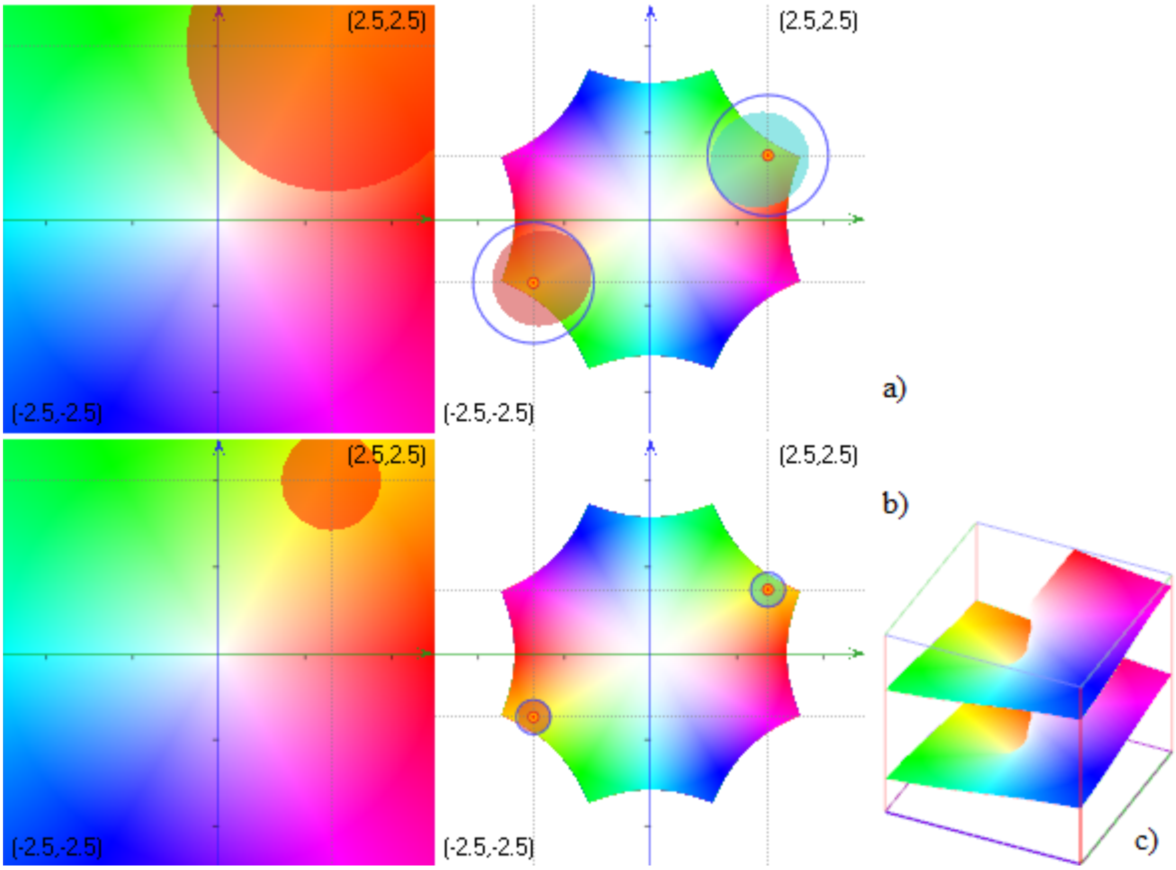

Figure 3

An interesting picture we get in case of multivalued functions. Fig.3 presents the scene in case of function $w = \sqrt{z}$. Here "*center in Domain*" is set to (1.3, 2) in both cases, $\varepsilon = 0.7$ (Fig.3 a), $\varepsilon = 0.2$ (Fig.3 b). We got two limits $\lim\limits_{z \to -1.7+1.6i} \sqrt{z}$. Their values and the reason and correctness of the issue itself of multiple limits may become a subject of a fruitful essay. Anyway, pay students' attention to the *VisuMatica*'s automatic coloring scheme of each image of the $\delta-$ circle, and the self-explanation potential of the "*Argument*" option of "*3D view*" (Fig.3 c).

- *Can function of a single real variable have more then one value of limit as its argument approaches to some definite value?*

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[3] Until studies of the concept of *continuity*.

Exploration tasks (types) with VisuMatica

1. Estimate the value of the $\lim_{z \to z_0} f(z)$ for $f(z) = \dfrac{z^3}{z - i}$, $z_0 = 2\text{-}i$.
2. Check if $\lim_{z \to z_0} f(z) = L$.

   a) $f(z) = 2z$, $a = -1 + 2i$, $L = -2 + 4i$.

   b) $f(z) = \dfrac{\sin 2z}{\sin z}$, $z_0 = 0$, $L = 2$.

   - Can there be a limit of function at a point that does not belong to its domain?
3. Prove that $\lim_{z \to z_0} f(z) \neq L$.

   $f(z) = 2z$, $a = 1 - 2i$, $L = 3.5 + 3i$.

The study in the cases of tasks of types 2 and 3, where given values of both z_0 and L, starts from entering them as coordinates of "*center in Domain*" and "*L*" in the proper text boxes (Fig.1). However, the visual outcomes (immediate, or resulted by the subsequent activity of decreasing the value of ε) will be *radically different.*

- It is important to draw students' attention to the white point (2,0) (Fig.4 a) inside a blue circle while modeling the task 2 b). -*Where is it from?*
- A small change of the value of z_0 say to (0, 0.0001) leads to replacement of the white point with the familiar small red-yellow circle (Fig.4 b). Figuring out the reason of this difference can help in the interpretation of the "effect" of white point.

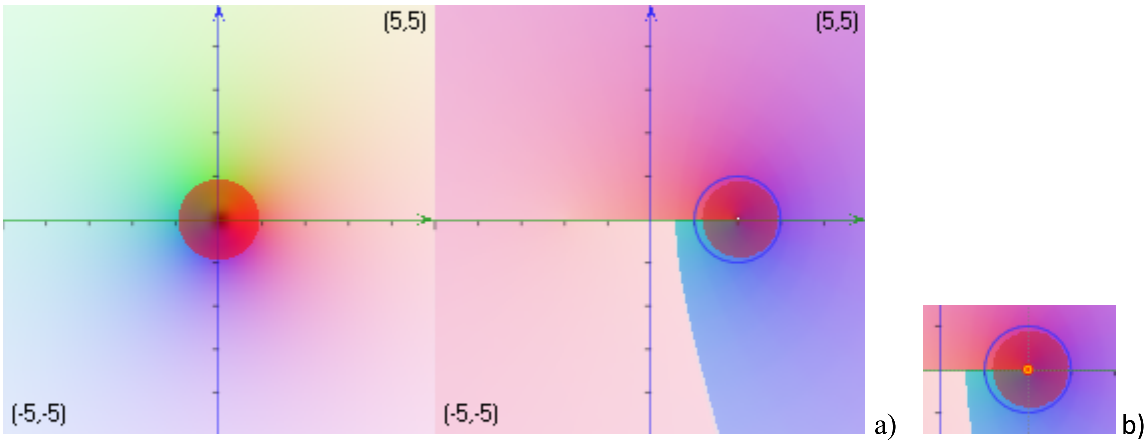


Figure 4

Fig.5 a) presents the disappointing display in case of task 3.

- *Where from came the small red point (2, -4)?*
- *What is the meaning of the red sign ⊗, of its position and size?*

Clean out both text boxes of "*center in Domain*" (keep *L* coordinates (3.5, 3) unchanged) and start moving the mouse pointer… Occasionally you will transform the red sign ⊗ to a regular blue circle with some red spot inside.

- *Where to locate the mouse pointer to ensure such transformation of the red sign ⊗?*

Discussion of this problem should lead students to its interpretation in terms of looking for preimages of points inside the red crossed circle.

Fortunately, *VisuMatica* allows getting preimage of a set of points in the function's range, if the set defined by inequality. Just do as follows:

1. Check the "*inequalities in Range*" check box of the "*Mapping style*" tab of "*Mapping*" dialog.

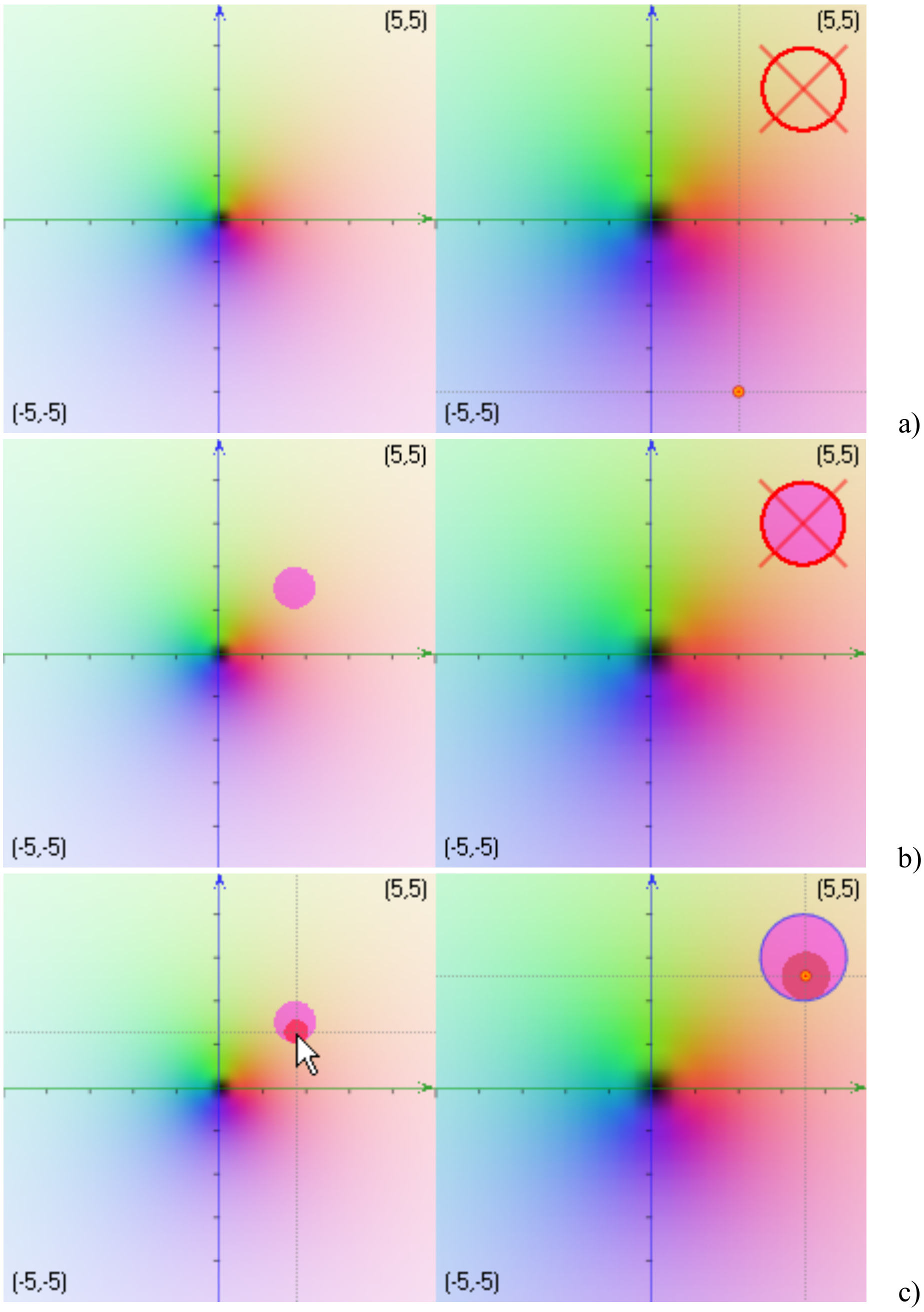


Figure 5

2. Type the proper inequality "|z-(3.5+3*i)|<1" in the main input text box, select its color (say, magenta), and press the "*Add*" button.
   The magenta disc – graph of the solution of inequality - fills in the crossed circle and its preimage (in case of $f(z) = 2z$ it forms a small disc) appears in the Domain view (Fig.5 b).
   Remains to check correctness of our idea, and its result… Move mouse pointer inside the small magenta disk in D_f. Voila! It comes, - the expected blue circle instead of the crossed one (Fig.5 c). Moreover, one can recognize a red disc centered at the mouse pointer with maximal radius that allows all its point to lie inside the magenta disk in Domain, and its image – a red disc inside the blue circle in the R_f.

... Where lays z_o?

Certainly, inside the magenta disk of D_f.

... Is it the center of the disk?

Magenta area is rarely a disk. *By the way, when*?

Redefine $f(z)$ to cause other form(s) of the magenta region.

Great! Let's assume that mouse points exactly to z_o, and check if the limit is the center of the blue circle. We have to decrease its radius ε and make sure that the red disk around the mouse pointer and its image in the Range view still exist. Proceed as follows:

- Click the mouse at the suspected location of z_o. Leave it there.
- Press "F7"/ "F8" keyboard buttons to increase/decrease the value of ε.[4]

Make conclusion.

The task of finding the exact value of z_0 by known value of the limit of a function can be complicated and unsolvable. Even in case of $L \in R_f$ it is the task of solving equation $f(z) = L$ with complex variable.
The "*simulation*" mechanism helps out us in finding z_0 (Fig.6).

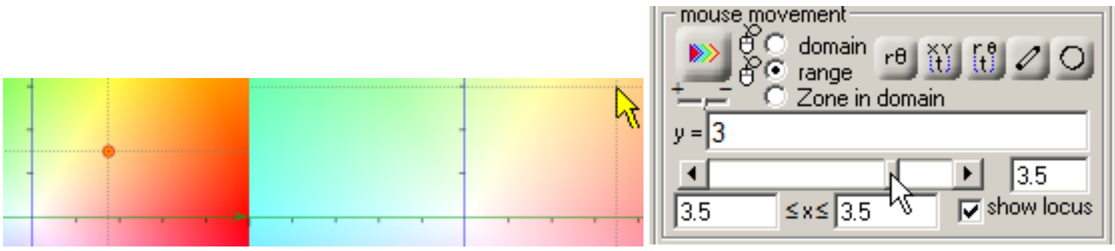


Figure 6

Select the "*Simulation*" tab of the "*Mapping*" dialog box. Choose the "*range*" radio button. Enter the $L_{im} = 3$ value as y. Set both limits of the x interval to $L_{re} = 3.5$. Try to reposition slider of the horizontal scroll bar.
The small red circle(s) centered at z_0 - candidate(s).

The "*center in Domain*" z_o, L, and ε can be defined not only numerically but also by expressions in the proper text boxes, allowing more flexibility in controlling their values and modeling in whole.

1.1.2 Definition *by Heine*

D2 We call L *the limit of function* $f(z)$ as $z \to a$ if for any sequence $\{z_n\}$ converging to z_0 with terms $z_n \neq z_0$ for all $n \in \mathbf{N}$, the sequence $\{f(z_n)\}$ converges to L as $n \to \infty$, i.e. $\lim_{n\to\infty} z_n = z_0 \Rightarrow \lim_{n\to\infty} f(z_n) = L$

This is a copy of the definition of the limit of a function of real variable applied to a complex one. Not surprising, the educational methods and tools [3] here will be similar. Function "*RandomlyTo*(…)" will help us also in this case. *VisuMatica* at each moment shows the mapping of one concrete complex function, so there is no need to specify it as an argument of

[4] "F7"/ "F8" keyboard buttons duplicate the action of "+"/"-" buttons.

"*RandomlyTo*(…)". Thus, two arguments (z_0 and *rate*) or even one of them (z_0) will sufficiently describe the task. In the last case, the *rate* has a default value 0.05. The first argument - z_0 – is presented by an expression, that includes names of geometric points (P, Q1 etc.), complex parameters (c - defined as "c:=2+3*i" -, z_1 etc.) and complex expressions (0, 0.5+i, a+i*b etc.). The common model **M1** (Fig.7) for studies of the definition by Heine includes point P and a series, defined as "$s(n)$=*RandomlyTo*(P, 0.05)"

Figure 7

It remains only to add the function $f(z)$ being examined for the presence of a limit at P and to start the exploration.

Example 1. Let us start looking for the $\lim\limits_{z\to 0.8+i} z^2$. We add function w=z^2 to our model and receive the following show (Fig.8).

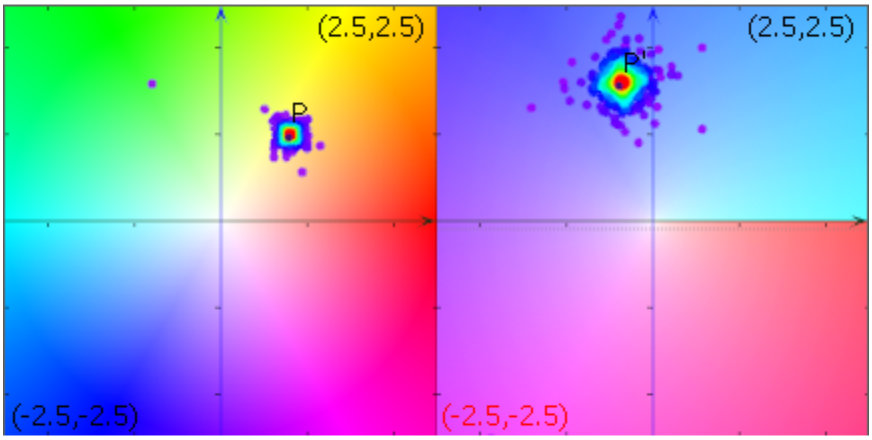


Figure 8

Using the same technique as in the real case (repeatedly pressing the F5 key to generate different random sequences with the same *rate*, changing the value of the *rate* parameter and pressing the F5 key again), "make sure" that the desired limit exists: the red dots are centered in a one depinite place in the Range view.

We draw the students' attention to the fact that red points are concentrated around P' - the image of point P. Thus, it turns out that P' is the desired limit. Select point P (click on it in the Domain View). P and P' become red-colored and as result – at the bottom of *VisuMatica* (Fig.9) the application's StatusBar expresses the situation: the image of P(0.8, 1.00) is P'(-0.36, 1.60). In terms of our study f(0.8+i) = -0.36+1.60i or $\lim\limits_{z\to 0.8+i} z^2 = -0.36+1.6i$.[5]

[5] Point P can slightly shift to mouse position when been pressed, so it is easier to use the simulation mechanism to fix the mouse position.

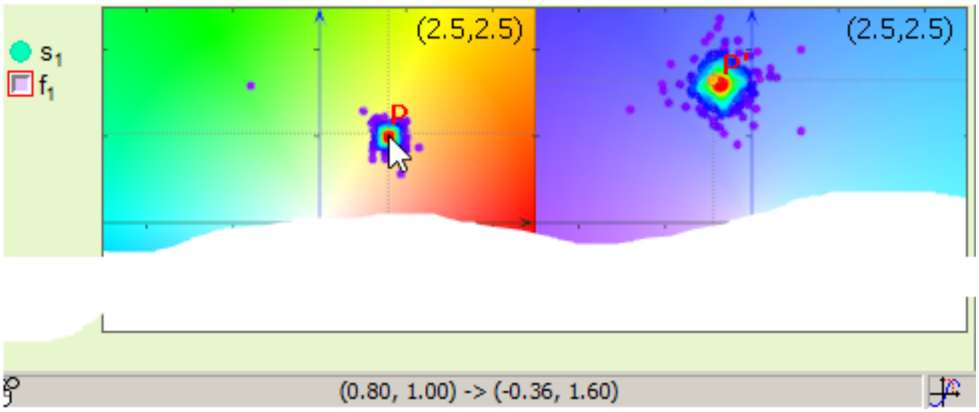


Figure 9

<u>*Example* 2.</u> Consider $\lim\limits_{z \to P} \frac{z^2}{z}$.

It is a trivial task when *P* does not located at the origin: our function $f(z) = \frac{z^2}{z}$ becomes equal to *z*, and thus, the limit equals to $\lim\limits_{z \to P} \frac{z^2}{z} = P^{'}.x + i \cdot P^{'}.y$.

Redefine in *VisuMatica* f1 to w=z^2/z. The Domain and Range views become identical. The point's P and P' and the point's distributions around them are also identical. Select point P and drag it to different locations. Press F5, change the *rate*.

- *Does this change the conclusions made?*

z = 0 does not belong to D*f*.

- *Is there a limit when P lies at the origin?*
- *If the answer is "Yes", what is its value?*

Use the "*Geometry*" dialog box to exactly position point P at the origin. The resulting show presented in Fig.10. Surprisingly, it looks like all the previous ones! The distribution in the Range View remains identical to the one in the Domain View. Looking closely at the picture, we find that point *P*' has disappeared. –*Why*?

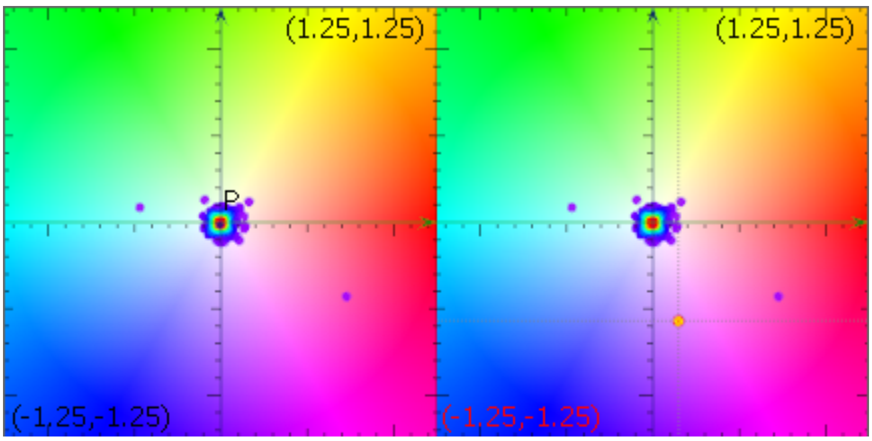


Figure 10

We go back to definition D2 and pay attention to the condition $z_n \neq z_0$. The colored points in random distribution do not coincide with 0. That is why their images are presented in Range view. And since the red dots in the Range View are concentrated at the origin, we conclude that the limit exists and is equal to 0 despite the fact that the function itself is not defined at $z = 0$.

Example 3. Consider $\lim\limits_{z \to P} \frac{z}{|z|}$.

Dividing a complex number by its modulus, we normalize it: the *result* obtained has *modul*us 1 *and the same argument*. So the images of all points of the complex plane will lie on the unit circle. In order to receive a representative display of this mapping it would be better to switch the "*Mapping style"* to "*grid colored*" (Fig.11).

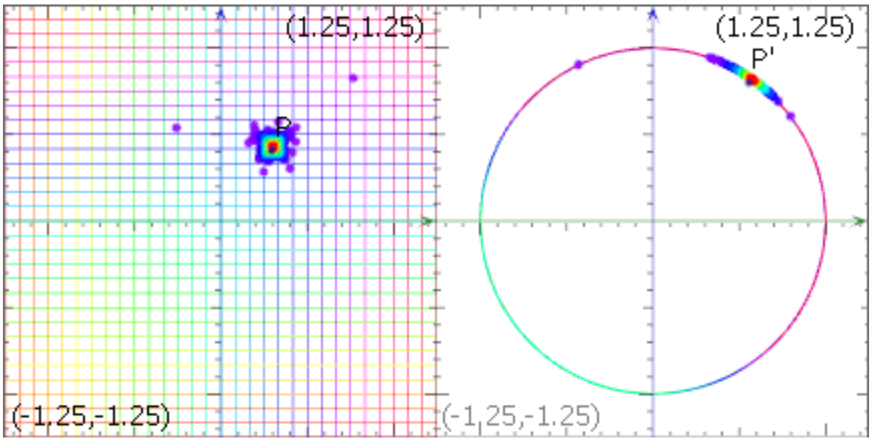


Figure 11

We were right: the whole Range View consists of the unit circle. Moreover, the images of the point P, and all images of the colored points of a random sequence are located on this unit circle at the same angles as their prototypes.

Dragging the point P along the complex plane, we make sure that the red points are concentrated around P'. That is, the limit exists and is easy to calculate, based on the received conclusions: $\lim\limits_{z \to P(x,y)} \frac{z}{|z|} = P'\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) = \frac{x}{\sqrt{x^2+y^2}} + \frac{y}{\sqrt{x^2+y^2}} i$.

Like in the *Example 2* also here $z = 0$ does not belong to D_f. As a result of the exact positioning of point P at the origin, we get a qualitatively different display. Trying to change the value of the *rate* parameter we get the same result: the red points are scattered around the circle, not concentrated near one point of the Range View, and P' has disappeared (Fig.12 a),b)). Conclusion: the $\lim\limits_{z \to 0} \frac{z}{|z|}$ does not exist.

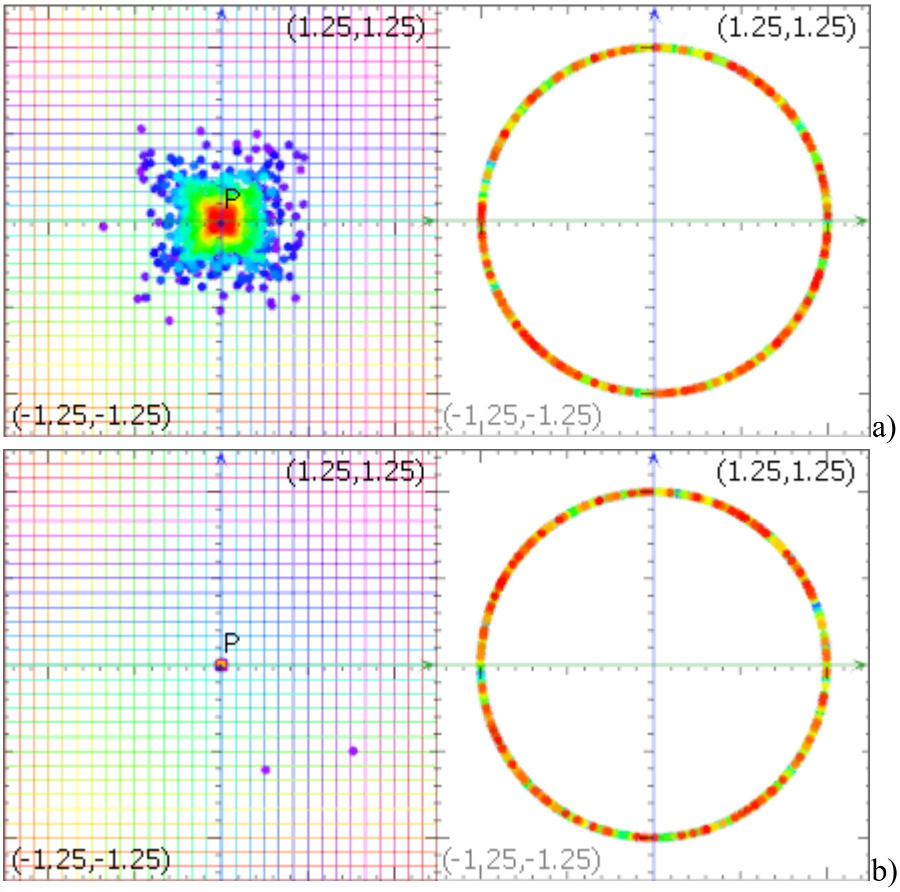


Figure 12. (a) *rate* = 0.5, (b) *rate* = 0.005

Example 4. Consider $\lim\limits_{z \to P} \frac{1}{2i}\left(\frac{z}{\overline{z}} - \frac{\overline{z}}{z}\right)$.

After redefinition of $f_1(z)$ by expresion "w=1/(2*i)*(z/conj(z)-conj(z)/z)" we receive a surprising result (Fig.13). The *Range*(*f*) looks as a segment [-1, 1]. Dragging the point *P* along the complex plane, we make sure that the red points are concentrated around *P*'. By pressing F5 and changing the *rate* value we conclude, that the limit exists and equals to *P*'.

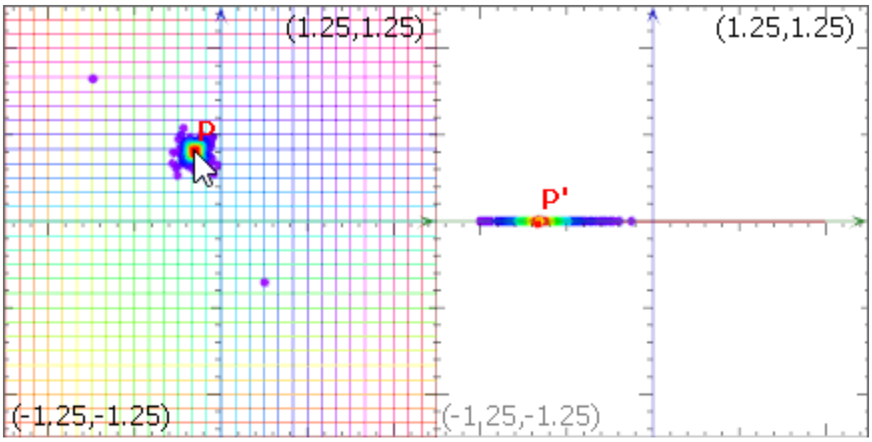


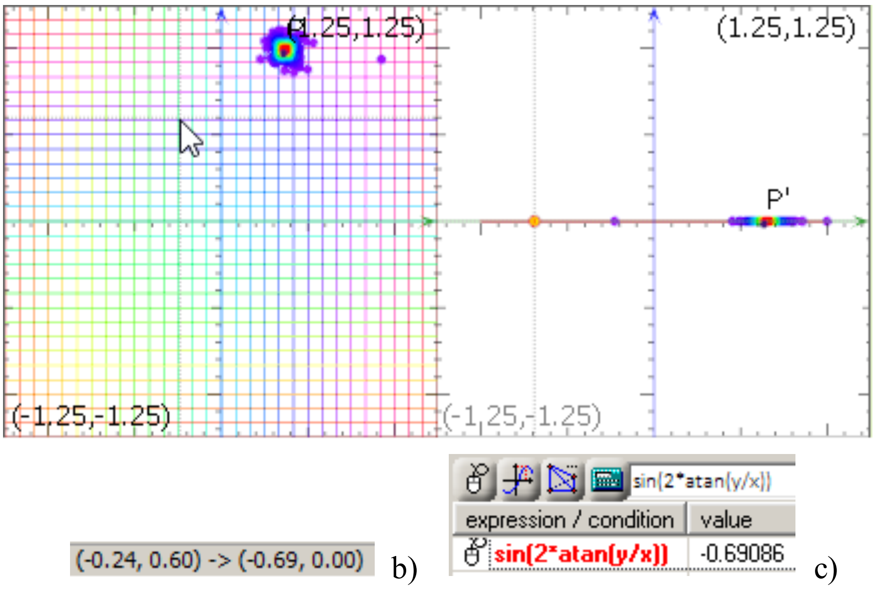
Figure 13

- *We can see the P' coordinates in the StatusBar, but what is the exact limit's value? At least some guess about its dependence on the position of P…*

The movement of *P'* only within the real segment [-1, 1] as well as the fact that R_f is this segment, pays attention to the coordinates of its ends. We are aware of two functions, *sinus* and *cosin*e, also taking values in this range. Both are functions of an angle. So, let us try to play with some angle… Instead of relocating point *P* it is sufficient to move the mouse in the role of $z \in D_f$, its image - yellow-red point - will show the corresponding value of the function.
Voila! The red-yellow point in some way "periodically" runs the segment [-1, 1], when the mouse pointer is moving around the origin. Within one complete mouse bypass this point has time to pass over the entire segment twice, moving in each direction.
In addition, when we move away or approach the mouse pointer to the origin along the ray, which starts at the origin, the position of the yellow-red point on the segment does not change. Really, all complex numbers on the ray, which starts in the origin, differ by a real constant multiples, and thus,

$$f(cz) = \frac{1}{2i}\left(\frac{cz}{\overline{cz}} - \frac{\overline{cz}}{cz}\right) = \frac{1}{2i}\left(\frac{z}{\overline{z}} - \frac{\overline{z}}{z}\right) = f(z).$$

While moving the mouse pointer and looking to the changing values in the StatusBar one can discover an interesting relation between the argument of complex number *z* the mouse points to and the abscissa of the correspondent yellow-red point on the segment. It behaves like *sinus* but twice quicker, i.e. like sin 2arg z. To check this hypothesis, we introduce expression "sin(2*atan(y/x))" as based on the mouse pointer position (x,y) in the "*expression/condition*" window of *VisuMatica*, and move the mouse again.
Our assumption confirmed! Fig.14 presents the situation of some mouse pointer location (a), and the proper mapping details (b) and the expression value (c).



a) b) c)

Figure 14

So, we assume that $\lim\limits_{z \to z_0} \frac{1}{2i}\left(\frac{z}{\overline{z}} - \frac{\overline{z}}{z}\right) = \sin 2 \arg z_0$.

It is clear that $f(0)$ does not exist. Let us determine whether there is a limit at $z = 0$ and, if so, what is its value?

Recall that, by definition D2, the function $f(z)$ tends to the limit, regardless of the way of approaching the point z to z_0. In other words, if a limit exists, then at z, tending to z_0 according to any rule (for example, along any line) $f(z)$ will approach this limit. We've already used this feature while studying limits of a real function with two real variables. We have forced random points to be located along a certain straight line with We have forced random points to be located along a certain straight line with a certain angle of slope and varied this angle [3]. Our task also deals primarily with angles. So it seems that this "old" mechanism can help in our case. The simples way is to redefine the series s_1 by expression "$s(n)$=*Randomly*To(a*x,0,0.5)" and trying different values of parameter a.
The result is as expected: for each value of a, the images of all points coincide, but the position of red point each time is different. -There is no limit[6].

2. The concept of Continuity

We will limit our consideration to only real functions of real argument. In all the examples of in [3], the equality $\lim\limits_{x\to a} f(x) = f(a)$ is violated only in specific, isolated values of the argument.

D3 | *A function f: A* → R *is called* ***continuous at a point*** $a \in A$ if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$, whenever $|x - a| < \delta$.

Drawing students' attention to the similarities and differences between this concept and the concept of the limit of function, we conclude that the previously used models and methods are suitable for studying the concept of continuity at point, and its specifics.

D4 | *A function f: A* → R *is called* ***continuous on A*** *if it continuous at all points* $a \in A$.

2.1 Classification of discontinuities

Not every function is continuous at any point. It is common to distinguish between the following three types of discontinuity[7].

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[6] *We could offer students to express this mapping analytically, using presentation forms of a complex number with modulus r and argument φ:*

$$f(r(\cos\varphi + i\sin\varphi)) = \frac{1}{2i}\left(\frac{r(\cos\varphi + i\sin\varphi)}{r(\cos(-\varphi) + i\sin(-\varphi))} - \frac{r(\cos(-\varphi) + i\sin(-\varphi))}{r(\cos\varphi + i\sin\varphi)}\right) = \ldots = \sin 2\varphi$$

or

$$f(re^{i\varphi}) = \frac{1}{2i}\left(\frac{re^{i\varphi}}{re^{-i\varphi}} - \frac{re^{-i\varphi}}{re^{i\varphi}}\right) = \ldots = \sin 2\varphi$$

The solution of exercise after such "boring" and not easy algebra, trigonometry and calculus will become theoretic and rather simple. But the presented above an active way of visual exploration seems us not less useful, especially at the beginning of studies of a new subject.

[7] Let function $f(x)$ be defined in some (may de punctured) neighborhood of point a.

1. *Removable discontinuity:*
   In this case:
   a) The *one-sided limits are both exist, equal and finite.*
      Thus, there exist the limit of $f(x)$ at point a, and $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a} f(x)$.
   b) *But* $\lim_{x\to a} f(x) \neq f(a)$ or $a \notin D_f$.
2. *Jump discontinuity* or *discontinuity of the first kind:*
   In this case the both *one-sided limits exist and finite, but are not equal*
   $$\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x). \text{ Thus, } \lim_{x\to a} f(x) \text{ does not exist.}$$
   The function f may have any value or not be defined at a.
3. *Essential discontinuity* or *discontinuity of the second kind:*
   In this case one of *one-sided limit does not exist or infinite.*

- Define examples of functions with each type of discontinuities. Check your guesses using the *Limit Machine*[8] [3].

- Find points of discontinuity and their type of the function, graphed in Fig.15[9]. Check your answers with *VisuMatica*.

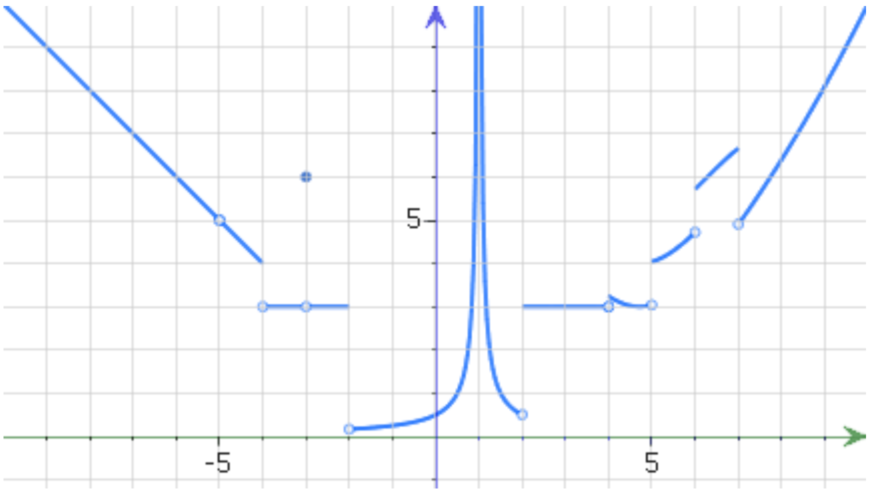

Figure 15

2.2. Local properties of continuous function

The local properties of the function $f(x)$ are those properties that are satisfied in an arbitrarily small neighborhood of the point $a \in D_f$, for example, the continuity of the function at a point. Local are the properties of continuity at the point a of functions $f_1(x) + f_2(x)$, $f_1(x) - f_2(x)$, $f_1(x) * f_2(x)$, $f_1(x) / f_2(x)$ (если $f_2(a) \neq 0$), in the case of continuous functions $f_1(x)$ and $f_2(x)$ at a.
Let us slightly alter our model of *Limit Machine* in order to search for other local properties:
- "Restore" the definition of the variable L. Select and redefine it back as $L = f_1(a)$.

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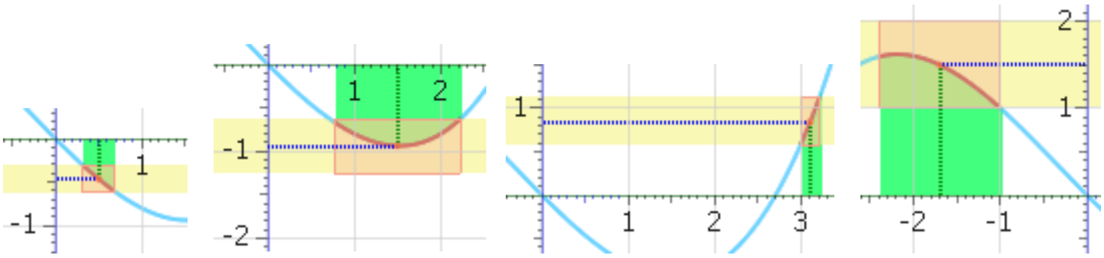
[8] A model that automatically chooses the maximum possible value of δ and presents an adequate visualization based on the given value of ε, the expected limit L of function $f_1(x)$ at point a.

[9] The function defined in *VisuMatica* as follows
"y=|0.5/(x-1)|, for -2<x<2; x^2/10,for x>7;-(5*x+x^2)/(x+5), for x<=-4;[x]+sin(x), for x>=4;6, for x=-3 else 3"

- Make our mechanism even more automatic: instead of the "manually" setting of value ε, we will adjust it to a specific value of $f_1(a)$. Namely, we define $\varepsilon = |f_1(a)|/3$.

As the values of the parameter a, we will consider only those for which the function f_1 is continuous. Students are invited to compare the sign of function f_1 at the continuity point a with the sign of function f_1 at the adjacent points of the "automatically" selected interval - a neighborhood of point a.

Fig.16 shows some results of modeling for the case of $f_1(x) = \dfrac{x^3 + x^2}{10} - x$, and Fig.17 for the case of $f_1(x) = [x] + \sin x$.



$a = 0.5$ $a = 1.5$ $a = 3.1$ $a = -1.7$

Figure 16

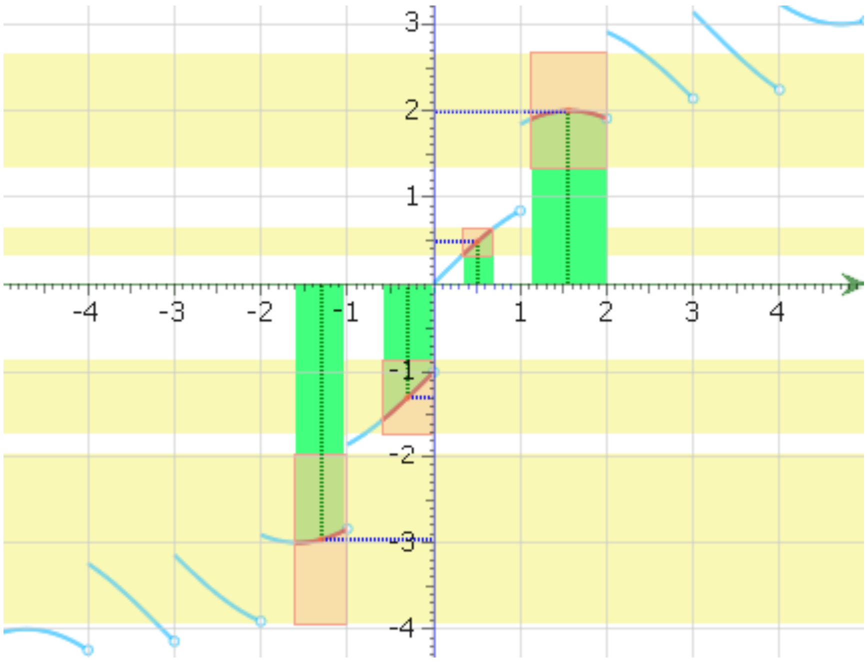


Figure 17. a = -1.3 -0.3 0.5, 1.56 (from left to right)

The explanation of constant signs of the function leads students to the formulation (and proof) of the *theorem of stability of the sign of a continuous function*:

If a function $f(x)$ is continuous at point a and $f(a) > 0$ ($f(a) < 0$), then there exists a neighborhood of a at which $f(x) > 0$ ($f(x) < 0$).

- *How does the model illustrate this fact?*

Our observations suggest the presence of at least two more local properties:

If function $f(x)$ is continuous at point a and $f(a)>0$ ($f(a)<0$), then there are such number $b>0$ $(b<0)$ and such neighborhood of point a at which $f(x)>b$ $(f(x)<b)$.

Define such b and add straight line $y = b$ to our model to visually confirm this property.

- *How many possible values of b? Suggest three versions.*

If function is continuous at point a, then there exists a neighborhood of this point at which the function $f(x)$ is bounded.

- *Give an example of "obvious" boundaries above and below* (refer to Fig.17).

- *How many possible boundary values exist? Offer a few.*

2.3 Global properties of continuous function

Functions defined and continuous on a segment [a, b] have a number of remarkable global properties, the significance of which is confirmed by theorems named after their discoverers.

The first Weierstrass theorem.
The function $f(x)$, continuous on a segment [a, b], is bounded on it, i.e.

$$(\exists m_1)(\exists m_2)(\forall x \in [a,b])\ m_1 \le f(x) \le m_2$$

- *Is this theorem true if not required the continuity of the function f (x)?*
  In case of a positive answer, bring proof, otherwise, a counterexample, confirmed by *VisuMatica*.

- *Is this theorem true, if instead of the segment [a, b] the interval (a, b) is considered?*
  In case of a positive answer, bring proof, otherwise, a counterexample, confirmed by *VisuMatica*.

D4 The *function f*, defined on a segment [a, b], is said to *reach its upper (lower) boundary* on it: $M = \sup\limits_{x\in[a,b]} f(x)$ $(m = \inf\limits_{x\in[a,b]} f(x))$, if

1. $(\forall x \in [a,b]) f(x) \le M$ $(f(x) \ge m)$
2. $(\forall \varepsilon > 0)(\exists x_0 \in [a,b]) f(x_0) > M - \varepsilon$ $(f(x_0) < m + \varepsilon)$

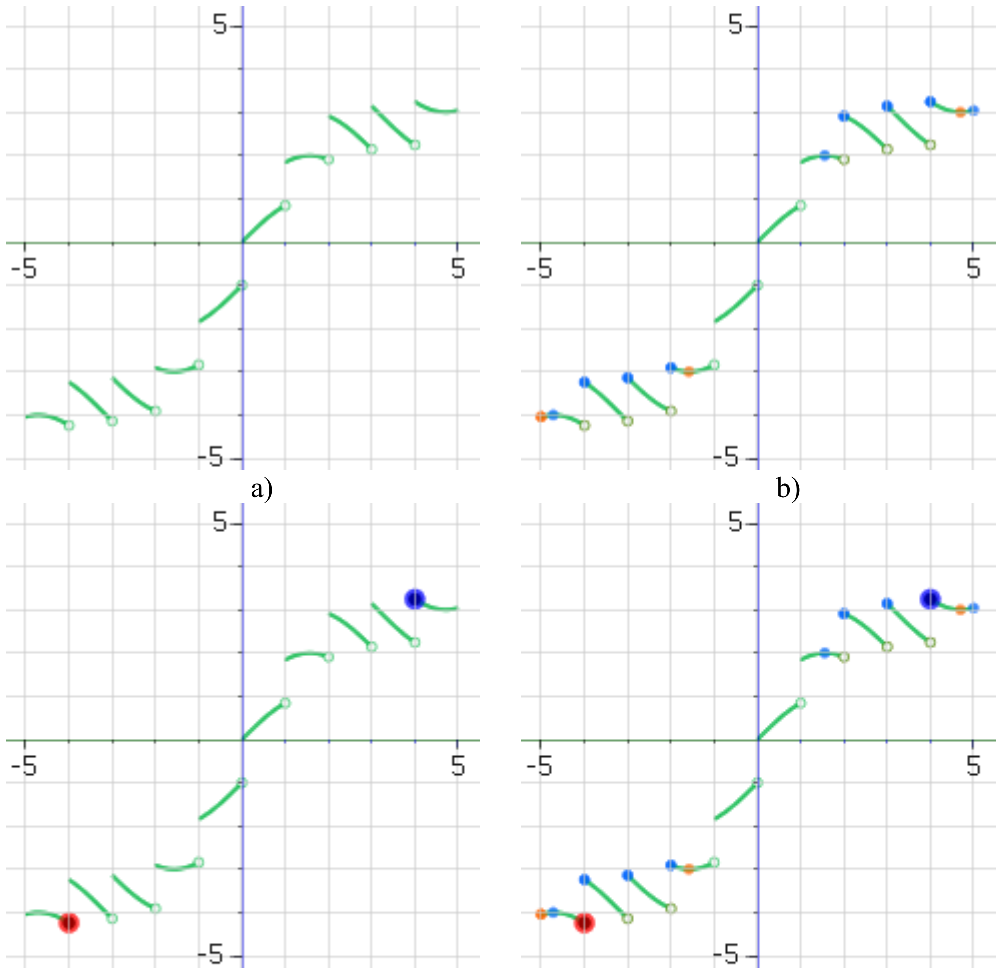
If M belongs to the R_f then M is called *maximum* of f on a segment [a, b].
If m belongs to the R_f then m is called *minimum* of f on a segment [a, b].

$$M = \max_{x\in[a,b]} f(x), m = \min_{x\in[a,b]} f(x)$$

The second Weierstrass theorem.
The function $f(x)$, continuous on the segment [a, b], reaches its exact upper and lower bounds $(\exists x_1, x_2 [a,b])\ (f(x_1) = \sup\limits_{x\in[a,b]} f(x), f(x_2) = \inf\limits_{x\in[a,b]} f(x))$

VisuMatica allows to ***see*** local maxima and minima, supremum and infimum of a function, if any. This ability is provided by the presence of two checkboxes in the "*Features*" tab of the "*Properties*" dialog: "*sup/inf*" and "*min/max*". Their functionality illustrated in Fig.18 with graph of function $y = [x] + \sin(x),\ if\ |x| \le 5$. The four images display:

a) The graph,
b) The result of checking the "*min/max*" checkbox: small light-blue points show the local maxima, while the light-red points – the local minima.
c) The result of checking the "*sup/inf*" checkbox: big blue point shows the supremum, while the big red one – infimum.
d) Both checkboxes are checked.



a) b) c) d)

Figure 18

By overriding the function $f_1(x)$ in this model, students confirm their answers to the following questions:

- *Is this theorem true if not required the continuity of the function f (x)?*

- *Is this theorem true if instead of the segment [a, b] is considered the interval (a, b)?*

- *Can a function, discontinuous on the segment [a, b], reach its exact upper and lower bounds on this segment?*

- *Can a function continuous on the interval (a, b) reach its exact upper and lower bounds on this interval?*

The Bolzano theorem.
Let function $f(x)$ be continuous on the segment $[a, b]$, and at the ends of this segment take values of different signs ($f(a) \cdot f(b) < 0$). Then there is a point inside the segment ($c \in (a, b)$) such that $f(c) = 0$[10].

This theorem is illustrated in Fig.19 with a light blue graph of function $y = x^2 - 4$ in the role of $f_1(x)$. Parameter b is defined here as $a + c$ (c is positive), dark magenta graph of function $f_2(x)$ – as $y = f_1(x), \mathit{if}\ a \le x \le b$, and equation e_1 is defined as $f_2(x) = 0$. In the "*View*" menu were checked the "*Equation: solution as a set of points on axis*" and "*show equation root's values*" items.

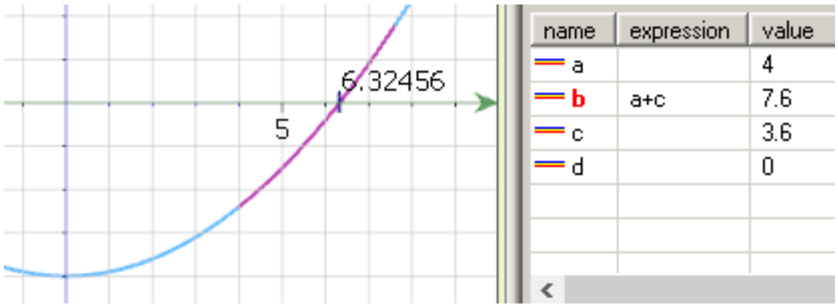


Figure 19

By redefining function $f_1(x)$ and changing the values of parameters, students consider possible cases of continuous and discontinuous functions $f_2(x)$ at different segments and intervals and confirm their answers to the following questions.

- *Is this theorem true if we do not require the continuity of function f(x)?*

- *Can a function that is discontinuous on segment [a, b], which takes values of different signs at the ends of this segment, be equal to zero at some point inside the segment?*

- *Can a function continuous on segment [a, b], which takes the values of the same signs at the ends of this segment, be equal to zero at some point inside the segment?*

This theorem underlies the simplest algorithm for finding root of equation $f(x) = 0$ in the case of continuous function[11].
It suffices to find two values of variable x, say a and b (without losing generality, let $a<b$), where $f(x)$ takes values of different signs. Consider it as $[a, b]$ and start:

---

[10] In other words, the equation $f(x) = 0$ has at least one root in the segment $[a, b]$.

[11] If do not stop at step 3, we come to the proof of the theorem itself.
Indeed, if the value of $f(c)$ is never 0, then will be constructed a sequence of nested segments $\{[a_n, b_n]\}$, such that $f(a_n) < 0 < f(b_n)$ if $f(a) < 0$, or $f(b_n) < 0 < f(a_n)$ if $f(a) > 0$. These segments are tightened $b_n - a_n = (b-a)/2^n \to 0$. By the Cauchy-Cantor theorem on contracted segments, there is a unique point c belonging to all of them ($a_n \to c$ and $b_n \to c$). $f(c) \le 0 \le f(c) \Rightarrow f(c) = 0$, c $\in (a, b)$.

1. Find the function value in the middle $c = (a + b) / 2$ of the segment.

2. If $f(c) = 0$, then c is the desired root. END.

3. If the length of the segment $b - a$ or $|f(c)|$ is less than specified accuracy, we accept the value of c as the approximate value of the desired root. END.

4. If the signs of $f(a)$ and $f(c)$ are opposite, we equate $b = c$, otherwise, the signs of $f(c)$ and $f(b)$ are opposite, therefore we equate $a = c$.

5. Go back to step 1.

The mechanism of the algorithm explains its name: "*Bisection method*".

The following model demonstrates the algorithm. Fig.20, where $f_1(x)=\sin(x-0.5) + x/2$, includes images after 1st (a), 2nd (b), 3rd (c), and 10th (d) iteration. The number of iterations controlled by parameter *steps*. Parameters a, b and c define the interval as in the previous model. Parameter *epsilon* defines the value of specified accuracy. The blue curve is the graph of $f_1(x)$. The magenta curve corresponds to $f_2(x) = f_1(x)$, for $a \le x \le b$. Each step is emphasized here by:

- The two vertical segments of different color $[[a_n, 0], [a_n, f_2(a_n)]]$ and $[[b_n, 0], [b_n, f_2(b_n)]]$ (blue and red), are located below and above the x-axis.
- The dark red vertical segment $[[c_n, 0], [c_n, f_2(c_n)]]$ shows the current approximation. Its red endpoint lies at c_n and at each step approaches to the root. Its green endpoint lies on the graph, and not necessarily goes closer to the x-axis at a sequential step, but tends to it anyway. In the next step this segment becomes one of the two red/blue vertical segments at the ends of the contracting segment $[a_n, b_n]$.
- The horizontal red segment shows the position of the proper half-segment $[a_n, b_n]$.

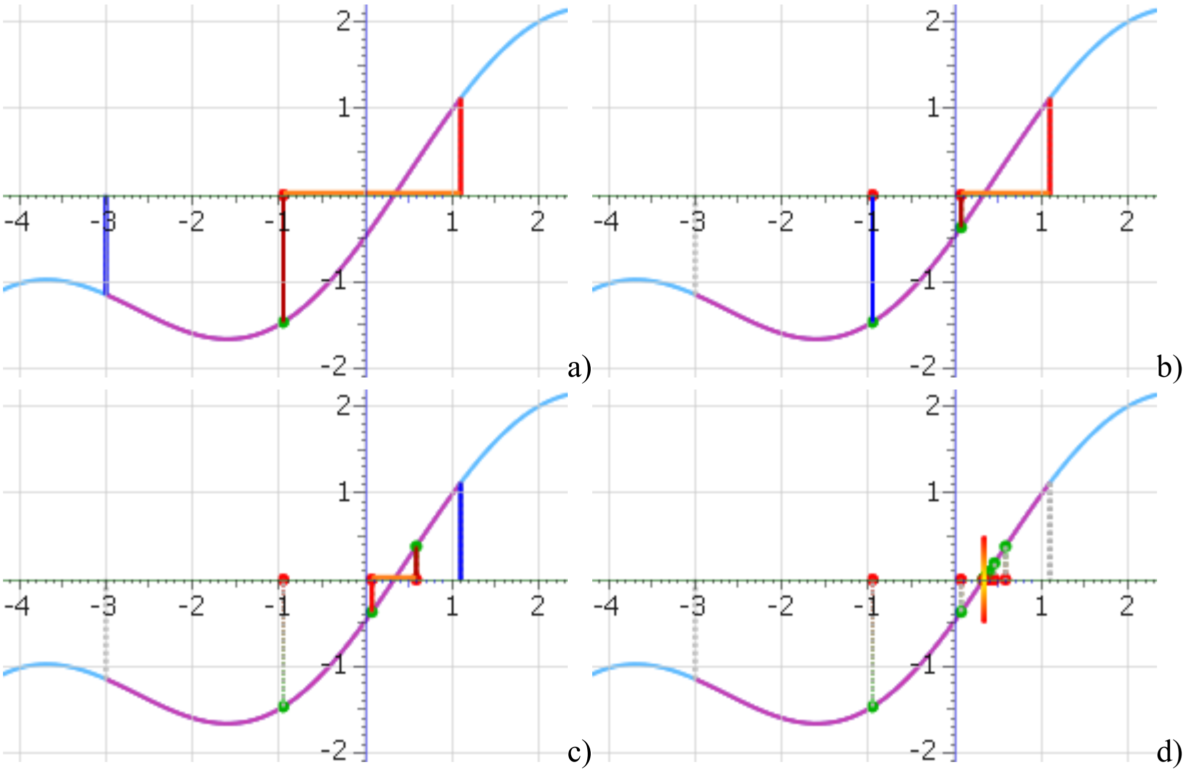


Figure 20

The algorithm stops when $b - a < \varepsilon$ or $|f(c)| < \varepsilon$. The discovered approximated/exact root's value becomes highlighted by a vertical red-yellow-red segment.

By playing with this model student answer the questions on the Bolzano theorem.

- In what cases the model remains empty with not more than two vertical gray dotted lines?
- Does the model execute exactly this algorithm or it behaves smarter in some way? Explain.

Return to Fig.20. Throughout all steps, the three red segments and the part of graph f2 between them resemble the configuration of two similar triangles. This circumstance suggests: instead of approaching by dividing the segment in half $c=(a_n+b_n)/2$, divide it in proportion. $c = a_n + \dfrac{(b_n-a_n)|f(a_n)|}{|f(a_n)-f(b_n)|}$.

Our model supports this mechanism by changing the value of the parameter d from 0 to 1. Fig. 21 shows the results of the same steps as in Fig. 20, but in the proportional division mode. The tilted grey dotted lines highlight the way of finding the value of approximation c.

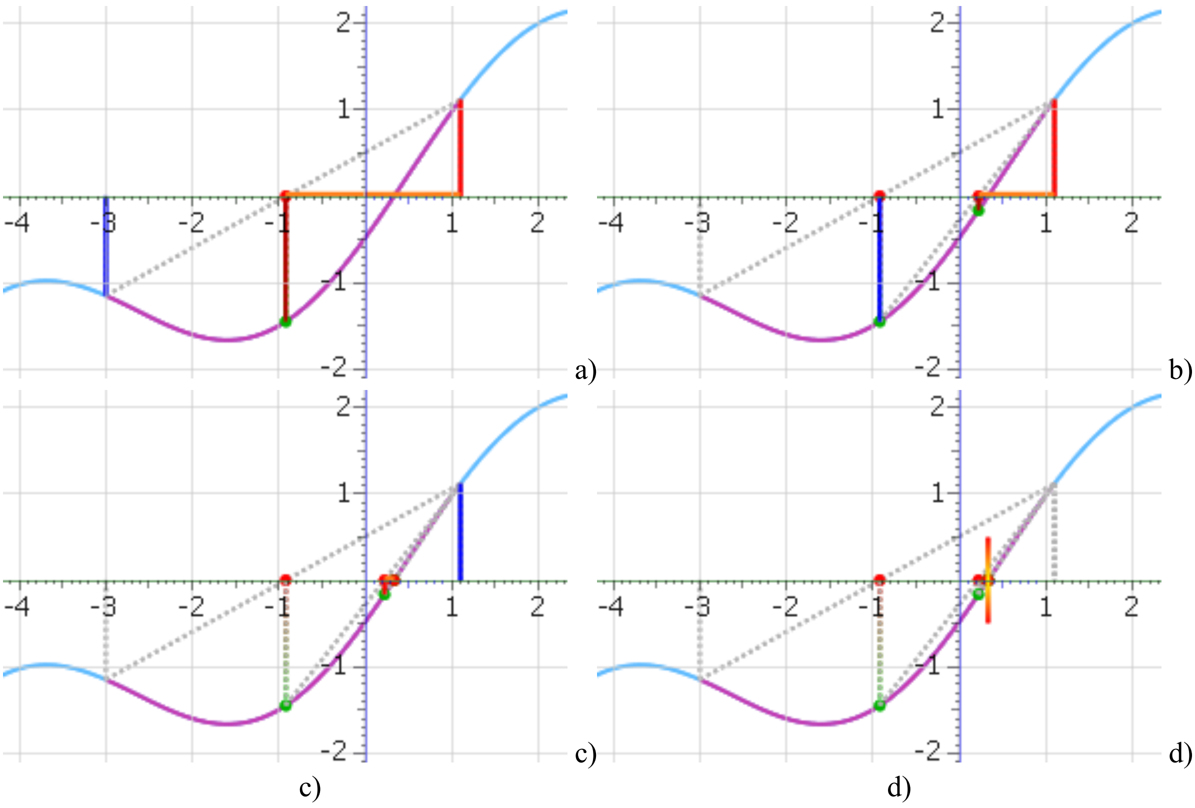


Figure 21

- *Compare Fig.20 and Fig.21. Which mode is faster? How did you find out it?*
- *Does this mode always work faster? (prove or provide a counterexample confirmed by VisuMatica)*

Bolzano-Cauchy Theorem (Intermediate Value Theorem).

Consider function $f(x)$, continuous on a segment $[a, b]$, which at the ends of this segment gets unequal values $f(a) = A$ and $f(b) = B$. Then for any C contained between A and B, there is a point inside the interval ($c \in (a, b)$) such that $f(c) = C$.

Paying students' attention to the both possible cases of A and B values ($A < B$ or $A > B$) we get more *formal description of the theorem* as follows:

$$f(x) \text{ continuous on } [a,b], f(a)=A, f(b)=B, A \neq B$$
$$\Rightarrow$$
$$\forall (C \in (\min(A,B), \max(A,B))) \exists (c \in (a,b)) f(c) = C$$

This theorem is illustrated in Fig.22 with a light blue graph of function $f_1(x)$, where $f_1(x) = \sin(x\text{-}0.5) + x/2$. In addition to the magenta graph of function $f_2(x)$ - as $y = f_1(x), \text{ if } a \le x \le b$ the values A and B are highlighted here by two horizontal red dotted lines ($y = f_1(a)$ and $y = f_1(b)$). Vertical wide green segment and the light green curvilinear trapezoid display its range R_{f_2}.

The vertical red segment connects points $(0, f_1(a))$ and $(0, f_1(b))$.

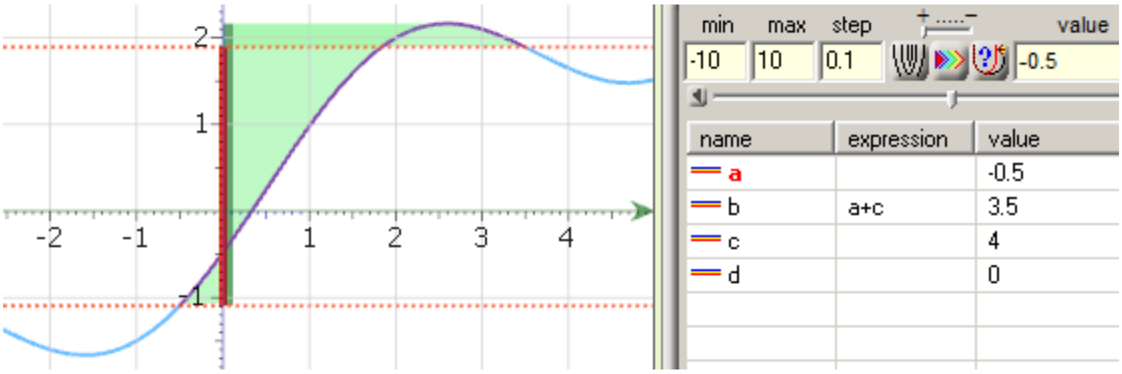


Figure 22

- *Explain, how this model illustrates the theorem.*
  Hint: Consider the relation between the vertical red and green segments.
  Check the illustration by varying the initial function $f_1(x)$ and values of a and b.

- *Why this theorem is called* ***Intermediate Value Theorem?***

- *Is this theorem true if we do not require the continuity of the function f (x)?*

- *Is this theorem true if instead of continuity on the segment [a, b] we require the continuity of f (x) in an open interval (a, b)?*

- *Is the answer to the previous question always correct?*

- *Is the converse theorem also true?*

  Construct the converse theorem by reversal of the condition and conclusion (use the formal description).

- *For what values of A, B, and C does the Bolzano theorem follow from the Bolzano – Cauchy theorem?*

- *What values should replace the ordinates of the ends of the vertical red segment, so that it coincides with the green one? Does this replacement always produce the desired result*?

- *Fill the space in the following statement:*
  The Range of function f(x), continuous on a segment [a, b], is a segment [m, M], where m and M are _____________________________ of f(x) on [a, b].

Conclusions

The above models and ways of their application show the wide potential of using educational software in the process of teaching fundamental concepts of limit and continuity of a function. Of course, these examples do not exhaust all the possibilities. For example, we have not touched on modeling of uniform and Lipschitz continuity. If there will be time, we will touch on these issues in the presentation

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