Strong Algebraic Manipulation Skills Are Not Adequate For Cultivating Creativity And Innovation

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Abstract

In this paper, we use some college entrance exam practice problems from China to highlight some essential algebraic manipulation skills that are required by high school students from China. Next we explore various scenarios by assuming if technological tools are available to learners, how we may see many unexpected surprising outcomes. For many countries, requiring college entrance exams is inevitable and sometimes is the only fair channel of selecting qualified students to enter a college. However, we hope examples provided in this paper can serve a purpose of advising the decision makers in education systems by allowing students to explore mathematics with available technological tools. After all, creativity and innovation do not come by giving one correct answer alone.

1 Introduction

Finding a curve defined by the locus of a moving point has been popular and often asked in Gaokao (a college entrance exam) in China. Typically students should not spend more than 10 minutes to solve one problem. Under such circumstances, it is not hard to imagine that many students may lose interest or even decide to give up solving these types of problems all at once. There have been several exploratory activities (see [4] [6], and [8]) derived from Chinese college entrance exam practice problems ([7]). In this article, we typically start with a practice problem originated from ([7]) to initiate our discussions. We demonstrate how a problem can be solved by hand first, which shall demonstrate those crucial algebraic manipulation skills that are required by high school students from China. Indeed those algebraic manipulations are very much appreciated and needed for solving even more challenging problems. One question, however, many educators or researchers may ask naturally is as follows: Assume students are not pressured in a fixed time frame to solve a problem, and if both DGS (Dynamic Geometry System) and CAS (Computer Algebra System) are available to students to explore, how can technological tools assist students' learning in this case? We conjecture the outcomes of the question can be summarized as follows, which will be validated by many examples presented in this paper:

1. Students would have many opportunities to ask 'what if' scenarios and properly make their conjectures before proving them analytically.

2. Many students will not feel frustrated either when they cannot complete or make mistakes on some complicated algebraic manipulations.

3. Students would have been given credits by knowing how to make geometric constructions and knowing what to do next by relying on a technological tool to complete the complicated algebraic manipulations.

Author believes that seeing pictures first before asking complete analytic or algebraic solutions is much more accessible, convincing and intuitive to students. In this paper, in addition to solving simple cases by hand, we typically construct a potential solution geometrically using the trace feature of a DGS. Or we ask for a symbolic answer if possible (for example using [2]). Finally, we use a CAS (such as [5]) to verify that our analytic solutions are identical to those obtained by using the DGS [2].

2 Locus When Fixing Two Points On A Curve

We start with the original entrance practice problem as follows:

Example 1 *We are given a fixed ellipse, say* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$*,* BE *is the major axis and* F *is a moving point on the ellipse. We construct two lines passing through* B *and* E *respectively and two lines intersect at* I *such that* $\measuredangle IBF = \measuredangle FEI = 90°$*.If* J *is the midpoint of* BI*, find the locus of* J*.[Note that the red curve in Figure 1(a) or 1(b) represents the scattered plot of* J *that can be traced by using [1]]*

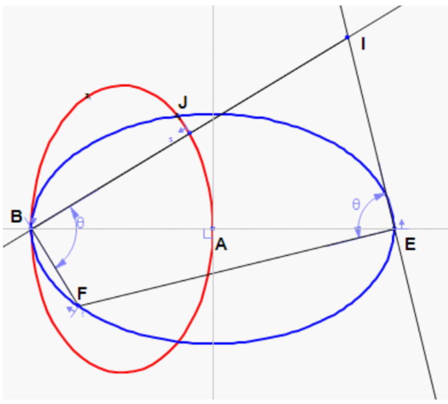

Figure 1(a). Locus, ellipse and two fixed points when $t = t_1$

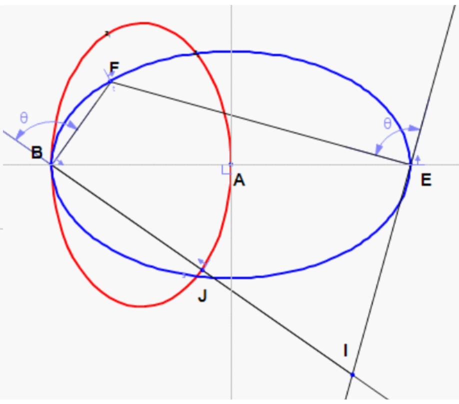

Figure 1(b). Locus, ellipse and two fixed points when $t = t_2$

We remark here that if this problem is presented as a mathematics experiment class instead, more students would have enjoyed exploring it if technological tools are available to learners. For example, before answering this question analytically, they can play and learn how a locus might look like, which makes the learning process much more enjoyable. In fact, one may adopt the following steps when exploring a problem:

1. Start with a DGS (say [1] in this case) for necessary geometric constructions and next use the scattered plot to conjecture what the locus should look like. Further experiment with a symbolic DGS such as [2] to generate a possible symbolic solution.

2. Solve the problem analytically by hand for simple scenario or solve it analytically with a CAS such as [5] if the problem becomes algebraically intensive.

We now present how one may solve this simple case by hand without the presence of technological tools. We let $B=(-a,0)$, $E=(a,0)$ and the moving point on the ellipse $F=(x_0,y_0)$. We denote the slopes of FB and FE to be k_{FB} and k_{FE} respectively, then $k_{FB}=\frac{y_0}{x_0+a}$ and $k_{FE}=\frac{y_0}{x_0-a}$. Thus, the the line equations for BI and EI are

$$y=-\frac{x_0+a}{y_0}(x+a) \tag{1}$$

and

$$y=-\frac{x_0-a}{y_0}(x-a) \tag{2}$$

respectively. We substitute 1 into 2 and yield the followings:

$$\begin{aligned}-\frac{x_0+a}{y_0}(x+a) &= -\frac{x_0-a}{y_0}(x-a)\\ (x_0+a)(x+a) &= (x_0-a)(x-a)\\ 2ax+2ax_0 &= 0.\end{aligned}$$

We see if $a\neq 0, x=-x_0$. In other words, if we assume $a\neq 0$, then $I=(-x_0,-\frac{1}{y_0}(a+x_0)(a-x_0)$. The midpoint for BI is thus

$$J=(X,Y)=\left(\frac{-x_0-a}{2},-\frac{1}{2y_0}(a+x_0)(a-x_0)\right).$$

This implies that $Y=\frac{1}{y_0}X(a-x_0)$. To obtain the parametric form for the locus J, we note that $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, which implies that $x_0=a\cos t$ and $y_0=b\sin t$. Thus we obtain the parametric equation for the locus to be

$$\begin{cases} X=\dfrac{-a\cos t-a}{2}=\dfrac{-a(\cos t+1)}{2}\\ Y=\dfrac{X(a-a\cos t)}{b\sin t}=\dfrac{aX(1-\cos t)}{b\sin t}\end{cases}.$$

Exploration 1. It is not difficult to extend our result if we ask for the locus $J=(X,Y)$ satisfying $\overrightarrow{BJ}=s\overrightarrow{BI}$ for some $0<s<1$. In view of $\overrightarrow{BJ}=s\overrightarrow{BI}$, we have

$$\begin{aligned}(X+a,Y) &= s\left(-x_0+a,-\frac{1}{y_0}(a+x_0)(a-x_0)\right)\\ J &= (X,Y)=(-s(x_0-a)-a),\frac{s}{y_0}(x_0+a)(x_0-a)\end{aligned}$$

Since $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, we set $x_0 = a\cos t$ and $y_0 = b\sin t$, where $t \in [0, 2\pi]$, to obtain the parametric equation for the locus J to be

$$\begin{aligned} X &= -s\,(a\cos t - a) - a \\ Y &= \frac{s\left((a\cos t)^2 - a^2\right)}{b\sin t}. \end{aligned} \tag{3}$$

Remarks:

1. We use the DGS Geometry Expressions [2] to construct the locus J above through geometry constructions. We depict some screen shots when $s = 0.25$ and 0.75 in the following Figures 2(a)-(b).
2. We use the CAS [5] and the analytic derivation 3 to verify that the Figures 2(a)-(b) obtained by using [2] are identical to the corresponding ones obtained by using [5].

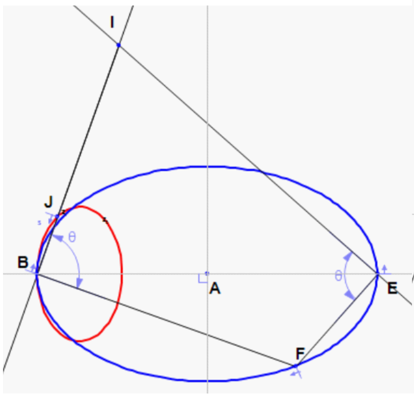

Figure 2(a). Locus when $s = 0.25$

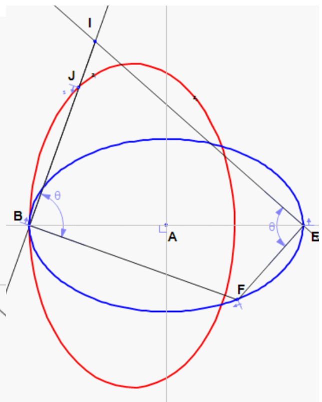

Figure 2(b). Locus when $s = 0.75$

Exploration 2. We leave it as an exercise to readers to verify that if we set $\measuredangle IBF = \measuredangle FEI = \theta$, the parametric equation for the locus should be proved to be the one obtained by Geometry Expressions [2] as follows:

$$\left(\begin{array}{l} X = -(1-s)\,|a| - \\ \dfrac{2a^2 s\sin(t)\cos(t)\,|b|}{-2a^2\sin(t)^2\sin(\theta)\cos(\theta) + 2b^2\sin(t)^2\sin(\theta)\cos(\theta) - 2\sin(t)\,|a|\,|b| + 4\sin(t)\sin(\theta)^2\,|a|\,|b|} \\ Y = \dfrac{s\left(-a^2\sin(\theta)^2\,|a| + a^2\sin(\theta)^2\cos(t)^2\,|a| - b^2\sin(t)^2\cos(\theta)^2\,|a| - 2a^2\sin(t)\sin(\theta)\cos(\theta)\,|b|\right)}{\left(-a^2\sin(t)\sin(\theta)\cos(\theta) + b^2\sin(t)\sin(\theta)\cos(\theta) + \left(-|a| + 2\sin(\theta)^2\,|a|\right)|b|\right)\sin(t)} \end{array}\right).$$

The DGS Geometry Expressions [2] has the capability of linking to a CAS [5] computation, an implicit equation can also be displayed by Geometry Expressions [2] for this example. Since it is too long to display, we will omit it here. Therefore, one may think that with a powerful symbolic geometry software such as Geometry Expressions [2] at hand, learners can visualize how graphs may change according to various parameters a, b, s, t and θ.

2.1 When we replace the ellipse by a cardioid

Assuming technological tools are available to learners, it is natural to ask what if the ellipse, discussed earlier, is replaced by another curve, say a cardioid. In particular, we consider the following

Example 2 *We are given the cardioid $r = 1 - \cos t, t \in [0, 2\pi]$ in Figure 3. Suppose the moving point C is on the cardioid and two lines passing through $B = (0,0)$ and $A = (a, 0)$ respectively, and intersect at G so that the angles $\measuredangle CAG = \measuredangle CBG = 90°$. If J is the midpoint of AG, find the locus for J. [Note the red curve is a scattered plot of the locus of J, when $A = (-2, 0)$, and has been obtained using [1]]*

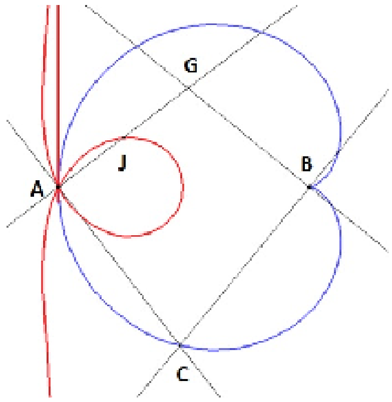

Figure 3. Locus and a cardioid

Since $A = (a, 0)$ and $B = (0,0)$, and the moving point $C = (x_0, y_0)$, we denote the slopes for CB and CA to be $k_{CB} = \frac{y_0}{x_0}$ and $k_{CA} = \frac{y_0}{x_0 - a}$ respectively. Thus, the the line equations for CB and CA are respectively

$$y = -\frac{x_0}{y_0}x, \tag{4}$$

and

$$y = -\frac{(x_0 - a)}{y_0}(x - a) \tag{5}$$

respectively. We substitute 4 into 5 and yield $\frac{x_0}{y_0}x = \frac{(x_0 - a)}{y_0}(x - a)$. By assuming $a \neq 0$, we see $x = a - x_0$, then $y = \left(\frac{x_0}{y_0}\right)(x_0 - a)$, in other words, the intersection $G = (a - x_0, \left(\frac{x_0}{y_0}\right)(x_0 - a))$. Then midpoint for AG is

$$J = (X, Y) = \left(\frac{2a - x_0}{2}, \frac{x_0(x_0 - a)}{2y_0}\right).$$

We note that C is a point on $r = f(t) = 1 - \cos t$, which implies that $x_0 = f(t)\cos t$ and $y_0 = f(t)\sin t$. Thus we obtain the parametric equation for the locus to be

$$\begin{aligned} X &= \frac{2a - (1 - \cos t)\cos t}{2} \\ Y &= \frac{(1 - \cos t)\cos t\,((1 - \cos t)\cos t - a)}{2(1 - \cos t)\sin t}. \end{aligned}$$

Exploration 1. It is not difficult to extend our result if we ask for the locus $J = (X, Y)$ such that $\overrightarrow{AJ} = s\overrightarrow{AG}$ for some $0 < s < 1$. In view of $\overrightarrow{AJ} = s\overrightarrow{AG}$, we see $X = a - sx_0$ and $Y = s\left(\frac{x_0}{y_0}\right)(x_0 - a)$. Since (x_0, y_0) is a point on the cardioid $r = f(t) = 1 - \cos t$, the locus J in this case is

$$\begin{aligned} X(a,s,t) &= a - s(1-\cos t)\cos t \\ Y(a,s,t) &= \frac{s(1-\cos t)\cos t}{(1-\cos t)\sin t}((1-\cos t)\cos t - a) \end{aligned} \tag{6}$$

We show some screen shots when $a = -2$ and $s = 0.3$, 0.7 and 1.5 respectively in Figures 4(a)-(c) by using [2] below, which we have verified that they are identical to those corresponding ones when using the CAS [5].

Further Remarks:

1. We notice that the curve of $r = 1 - \cos t$ has a point of non-differentiabilty at $B = (0, 0)$, what will be the corresponding point for the locus J?

2. In view of the derivation in equations 6, we encourage readers to explore how the graphs varies according to the parameters a, s, and t respectively.

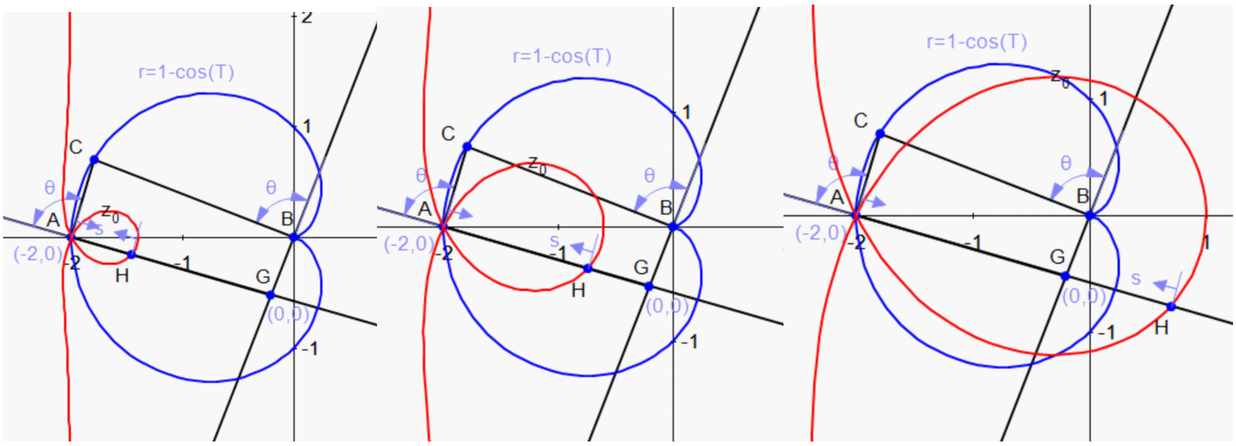

Figure 4(a). Locus and cardioid when $s = 0.3$

Figure 4(b). Locus and cardioid when $s = 0.7$

Figure 4(c). Locus and cardioid when $s = 1.5$

Exploration 2. We leave it as an exercise to readers to verify that if we set $\measuredangle CAG = \measuredangle CBG = \theta$ and $a = -2$, the parametric equation for the locus is shown as follows, which is obtained from Geometry Expressions [2]:

$$\begin{pmatrix} X = -2 + 2s - \\ \dfrac{\begin{array}{c} 2s(-(-2-(1-\cos(t))\cos(t))\sin(\theta) + (1-\cos(t))\sin(t)\cos(\theta))(-(1-\cos(t))\sin(t)\sin(\theta) \\ -(1-\cos(t))\cos(t)\cos(\theta)) \end{array}}{2\sin(t) - 4\sin(t)\sin(\theta)^2 - 2\sin(t)\cos(t) + 4\sin(t)\sin(\theta)^2\cos(t) - 2\sin(t)^2\sin(\theta)\cos(\theta)} \\ Y = \dfrac{-s\begin{pmatrix} 2\sin(\theta) + \sin(\theta)\cos(t) - \sin(\theta)\cos(t)^2 \\ +(\sin(t) - \sin(t)\cos(t))\cos(\theta) \end{pmatrix}(-\sin(\theta)\cos(t) + \sin(t)\cos(\theta))}{\sin(t) + (\sin(\theta) + \sin(\theta)\cos(t))\cos(\theta) - 2\sin(t)\cos(\theta)^2}. \end{pmatrix}$$

3 Locus And Combinations of Shifting And Scaling

Here we are given two fixed points and a moving point on a smooth convex curve, and we need to find the locus of a point lying on the line segment connecting one fixed point and a moving point. We first present the original problem and solve it by hand first and see how the problem can be extended to other scenarios in 2D. Next, we summarize how the problem is related to the translation and scaling of figures.

3.1 Generating a circle with two fixed points

The following locus problem is when we fix a point on a circle and fix another point that is not on the circle. In particular, we consider the following Example 3, which has been slightly modified from the original practice problem (see [7]).

Example 3 *We are given a fixed circle in blue (see Figure 5), the circle is of the form* $(x-a)^2+(y-b)^2=r^2$, *the moving point* $D=(x_0,y_0)$ *is on the circle. Furthermore if we choose the fixed point* $C=(c,d)$ *that is not on the circle and let* E *be the midpoint of* CD. *Find the locus of* E.

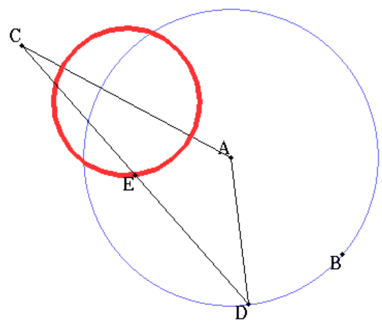

Figure 5. Circle and two fixed points

We first see how students solve this problem by hand in an exam. We note that the midpoint E of CD can be written as $E=\left(\frac{c+x_0}{2},\frac{d+y_0}{2}\right)$. Let $x=\frac{c+x_0}{2}$ and $y=\frac{d+y_0}{2}$. Then we see $((2x-c)-a)^2+((2y-d)-b)^2=r^2$, which implies that

$$\begin{aligned}(2x-c)^2-2a\,(2x-c)+a^2+(2y-d)^2-2b\,(2y-d)+b^2 &= r^2\\ 4x^2-4cx+c^2-4ax+2ac+a^2+4y^2-4dy+d^2-4by+2bd+b^2 &= r^2\end{aligned}$$

After simplifying, we see

$$\left(x-\left(\frac{a+c}{2}\right)\right)^2+\left(y-\left(\frac{b+d}{2}\right)\right)^2=\frac{1}{4}r^2,$$

Indeed the locus is a circle with center $\left(\frac{a+c}{2},\frac{b+d}{2}\right)$ and radius $\frac{r}{2}$.

Exploration. Following the discussions from Example 3, if we let the point E satisfying $\overrightarrow{CE} = s\overrightarrow{CD}$ with $s \in (0,1)$ and we would like to find the locus of E, then it is easy to verify that the locus for E will be as follows:

$$\left(x - s^2(a+c)\right)^2 + \left(y - s^2(b+d)\right)^2 = (sr)^2 .$$

3.2 Locus as a result of simple translation and scaling

In fact, we may view the discussions in the preceding Example 3, as a simple translation and scaling from a given curve to the other. For instance, if we consider a given fixed point $C = (c,d)$ and the circle C_1, centered at $A = (a,b)$ (another fixed point) with radius r and $C \neq A$. By taking the moving point D to be on the circle C_1, our objective is to find the locus of the midpoint F of CD. If O denotes the origin $(0,0)$,then the locus $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF} = \overrightarrow{OA} + \overrightarrow{AC} + \overrightarrow{CF} = \overrightarrow{OA} + \overrightarrow{AC} + \frac{1}{2}\overrightarrow{CD} = \overrightarrow{OA} + \overrightarrow{AC} + \frac{1}{2}\left(\overrightarrow{AD} - \overrightarrow{AC}\right) = \overrightarrow{OA} + \frac{1}{2}\left(\overrightarrow{AC} + \overrightarrow{AD}\right)$. Suppose A is at the origin O, we see the locus $\overrightarrow{OF} = \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OD}\right)$ will be a translation of a circle from the center at $A = O$ to the center at $\frac{1}{2}\overrightarrow{OC}$, and the radius is being scaled to $\frac{1}{2}$ of the original radius. (See Figure 6). It is clear now that if we were to find the locus of the point F satisfying $\overrightarrow{CF} = s\overrightarrow{CD} \in (0,1)$, it is equivalent to asking the locus of $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$, where

$$\begin{aligned} \overrightarrow{AF} &= \overrightarrow{AC} + s\overrightarrow{CD} \\ &= \overrightarrow{AC} + s\left(\overrightarrow{AD} - \overrightarrow{AC}\right) \\ &= \overrightarrow{AC}(1-s) + s\overrightarrow{AD}, \end{aligned}$$

we see the locus of $\overrightarrow{OF}$ can be viewed as a result of the combination of translation and scaling (see Figure 7 for translation and scaling of $s = \frac{1}{4}$).

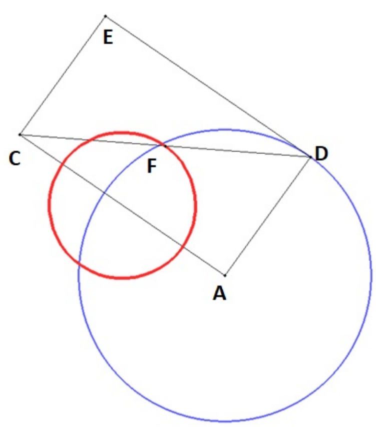

Figure 6. Locus F when $s = \frac{1}{2}$

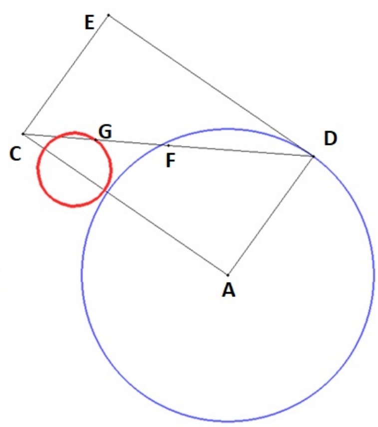

Figure 7. Locus G when $s = \frac{1}{4}$

We encourage readers to explore when we replace the circle with an ellipse as follows:

Exercise 4 *Suppose we are given an ellipse and two fixed points A and C respectively (see Figure 8 or 9), where A is the center of the ellipse and $C \neq A$. We let D be a moving point on the ellipse and F be a point such that $\overrightarrow{CF} = s\overrightarrow{CD}$, $s \in (0,1)$. Then the locus $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AF}$ is a result of simple translation and scaling from the original ellipse. The Figures 8 and 9 shows the locus $\overrightarrow{OF}$ in red when $s = \frac{1}{2}$ and $\frac{3}{4}$ respectively.*

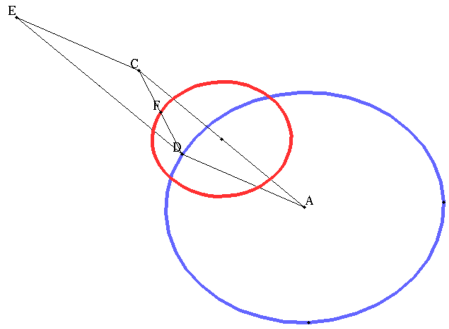

Figure 8. Ellipse, translation and $s = \frac{1}{2}$

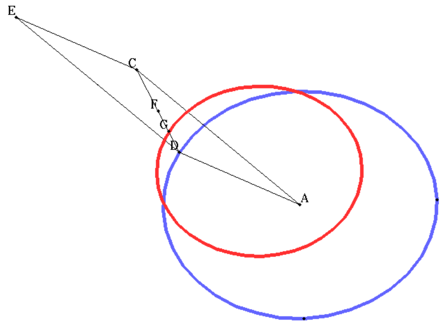

Figure 9. Ellipse, translation and $s = \frac{3}{4}$

4 The Locus Of Lines Passing Through A Fixed Point

The original problem from [7] has been modified slightly from the following general setting. Again, we start with the original locus problem with an algebraic solution and explore other 2D scenarios in the subsequent sections when technological tools are available.

Example 5 *We are given a fixed circle in black and a fixed point A in the interior of the circle $(x-a)^2 + (y-b)^2 = r^2$ (see Figure 10(a)). A line passes through A intersects the circle at C and D respectively, and the point E is the midpoint of CD. Find the locus E.*

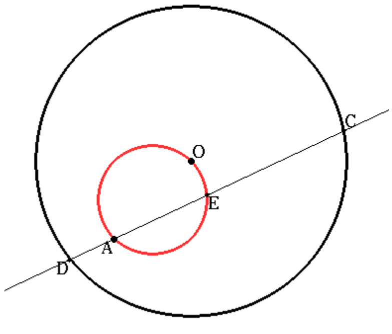

Figure 10(a). Locus and lines passing through a fixed point

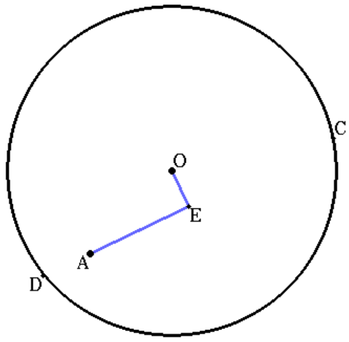

Figure 10(b). Locus, circle and perpendicular

We let the fixed point A to be (x_0, y_0), and let the line passing through A and intersect the circle at C and D respectively (see Figure 10(a)). We set $E=(x,y)$ to be the midpoint of CD. If we denote the center for the circle to be $O=(a,b)$, then we using the fact that

$$\overrightarrow{AE}\cdot\overrightarrow{OE}=0,$$

(see Figure 10(b)) and we see

$$(x-x_0)(x-a)+(y-y_0)(y-b)=0. \tag{7}$$

The equation 7 can be reduced as

$$\left(x-\frac{a+x_0}{2}\right)^2+\left(y-\frac{b+y_0}{2}\right)^2=\frac{1}{4}a^2-\frac{1}{2}ax_0+\frac{1}{4}b^2-\frac{1}{2}by_0+\frac{1}{4}x_0^2+\frac{1}{4}y_0^2,$$

which shows that the locus is a circle.

Exploratory Activity. Suppose we change the scenario to find the locus of E satisfying $\overrightarrow{ED}=s\overrightarrow{CD}$, where $s\in(0,1)$, then the problem becomes more complicated, which may not be suitable as an exam question. However, it is a perfect example for students to explore with technological tools. Precisely, suppose C, D are two points on a circle, say $x^2+y^2=r^2$, and the fixed point $A=(u_0,v_0)$ is not the center of the circle. We let E be the point on the line CD passing through A, we want to find the locus of $E=(x,y)$ satisfying $\overrightarrow{ED}=s\overrightarrow{CD}$, where $s\in(0,1)$. We note that the parametric solution for the locus E obtained from Geometry Expressions [2] can be obtained as follows:

$$\left(\begin{array}{l} X(r,s,t,u_0,v_0)=(1-s)\left(\dfrac{-2\left(-v_0+\sin(t)\left|r\right|\right)\left(u_0\sin(t)\left|r\right|-v_0\cos(t)\left|r\right|\right)}{-r^2-u_0^2-v_0^2+2v_0\sin(t)\left|r\right|+2u_0\cos(t)\left|r\right|}-\cos(t)\left|r\right|\right)\\ +s\cos(t)\left|r\right| \\ Y(r,s,t,u_0,v_0)=(1-s)\left(\dfrac{2\left(-u_0+\cos(t)\left|r\right|\right)\left(u_0\sin(t)\left|r\right|-v_0\cos(t)\left|r\right|\right)}{-r^2-u_0^2-v_0^2+2v_0\sin(t)\left|r\right|+2u_0\cos(t)\left|r\right|}-\sin(t)\left|r\right|\right)\\ +s\sin(t)\left|r\right| \end{array}\right).$$

We shall show how this formula is derived:

Step 1. We label $C=(r\cos t, r\sin t), D=(x_1,y_1)$, and E as (x,y), and observe from $\overrightarrow{DE}=s\overrightarrow{CD}\in(0,1)$ that

$$\begin{aligned} x-x_1 &= s\left(x_1-r\cos t\right) \\ x-rs\cos t &= x_1-sx_1 \\ &= x_1\left(1-s\right), \\ y-rs\sin t &= y_1-sy_1 \\ &= y_1\left(1-s\right). \end{aligned}$$

Step 2. Next we note that $D=(x_1,y_1)$ lies on the line equation AC of

$$y-v_0=\left(\frac{r\sin t-v_0}{r\cos t-u_0}\right)\left(x-u_0\right),$$

that is passing through the fixed point $A=(u_0,v_0)$. Therefore, we see

$$\begin{aligned} y_1 - v_0 &= \left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)(x_1 - u_0), \\ y_1 &= v_0 + \left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)(x_1 - u_0). \end{aligned}$$

Step 3. Now, since the line AC intersects the circle of $x^2+y^2=r^2$, we rewrite the equation of the circle by using the line equation AC as follows:

$$x^2 + \left(v_0 + \left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)(x-u_0)\right)^2 = r^2.$$

We label $k = \dfrac{r\sin t - v_0}{r\cos t - u_0}$ and rearrange the quadratic equation as follows:

$$\begin{aligned} k^2u_0^2 - 2k^2u_0x + k^2x^2 - 2ku_0v_0 + 2kv_0x - r^2 + v_0^2 + x^2 &= 0, \\ \left(1+k^2\right)x^2 - \left(2k^2u_0 - 2kv_0\right)x + \left(k^2u_0^2 - 2ku_0v_0 - r^2 + v_0^2\right) &= 0. \end{aligned} \tag{8}$$

Step 4. We solve the quadratic equation involving x and notice that both $x_1, r\cos t$ are two roots of the quadratic equation 8 above, thus

$$\begin{aligned} x_1 + r\cos t &= \frac{2k^2u_0 - 2kv_0}{1+k^2} \\ x_1 &= \frac{2k^2u_0 - 2kv_0}{1+k^2} - r\cos t. \end{aligned}$$

Step 5. We write x by using x_1 and substitute this into $x - rs\cos t = x_1(1-s)$ yields

$$\begin{aligned} x(r,s,t,u_0,v_0) &= rs\cos t + \left(\frac{2k^2u_0 - 2kv_0}{1+k^2} - r\cos t\right)(1-s) \\ &= rs\cos t + \left(\frac{2\left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)^2 u_0 - 2\left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)v_0}{1+\left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)^2} - r\cos t\right)(1-s) \\ &= rs\cos t + \left(\frac{2\left(r\sin t - v_0\right)^2 u_0 - 2\left(r\sin t - v_0\right)\left(r\cos t - u_0\right)v_0}{\left(r\cos t - u_0\right)^2 + \left(r\sin t - v_0\right)^2} - r\cos t\right)(1-s) \\ &= rs\cos t + \left(\frac{2\left(r\sin t - v_0\right)\left(\left(r\sin t - v_0\right)u_0 - \left(r\cos t - u_0\right)v_0\right)}{r^2\cos^2 t - 2ru_0\cos t + u_0^2 + r^2\sin^2 t - 2rv_0\sin t + v_0^2} - r\cos t\right)(1-s) \\ &= rs\cos t + \left(\frac{2\left(r\sin t - v_0\right)\left(ru_0\sin t - rv_0\cos t\right)}{r^2 + u_0^2 + v_0^2 - 2ru_0\cos t - 2rv_0\sin t} - r\cos t\right)(1-s) \end{aligned}$$

Step 6. We find y_1 express y accordingly: We substitute x_1 into $y_1 = v_0 + \left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)(x_1 - u_0)$ to get y_1 and yield

$$y_1 = v_0 + \left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)\left(\frac{2k^2u_0 - 2kv_0}{1+k^2} - r\cos t - u_0\right),$$

we then substitute y_1 into

$$\begin{aligned} y - rs\sin t &= y_1 - sy_1 \\ &= y_1(1-s). \end{aligned}$$

to get y as follows:

$$y(r,s,t,u_0,v_0) = rs\sin t + \left(v_0 + \left(\frac{r\sin t - v_0}{r\cos t - u_0}\right)\left(\frac{2k^2u_0 - 2kv_0}{1+k^2} - r\cos t - u_0\right)\right)(1-s).$$

We have used [5] to verify that, when setting $r > 0$, indeed we have $[X(r,s,t,u_0,v_0), Y(r,s,t,u_0,v_0)] = [x(r,s,t,u_0,v_0), y(r,s,t,u_0,v_0)]$. We show some screen shots of the locus in red, with respective values of s, in the Figures 11(a)-(c) when using [2]

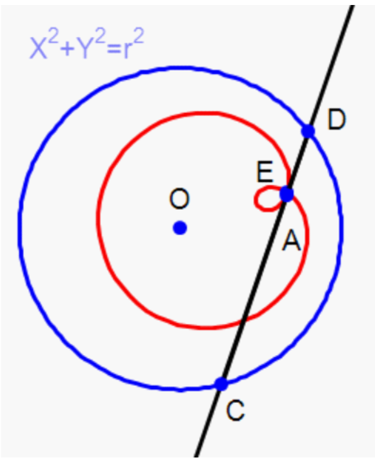

Figure 11(a) Locus and $s = 0.25$

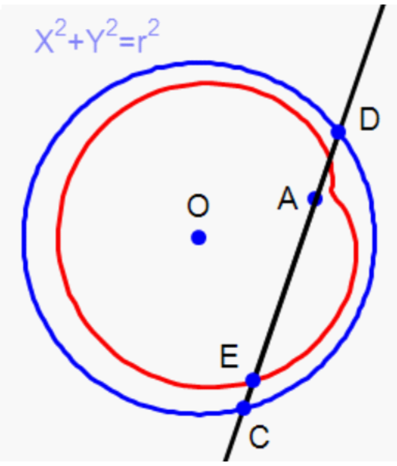

Figure 11(b) Locus and $s = 0.9$

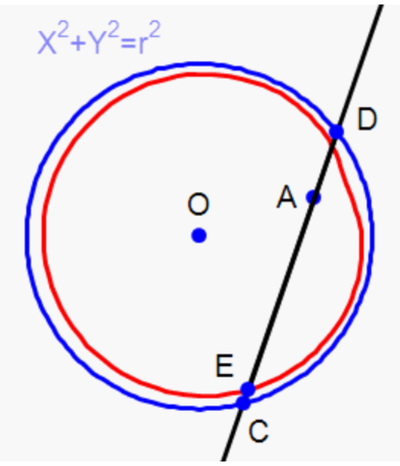

Figure 11(c) Locus and $s = 0.95$

4.1 Exploration with a DGS for the case of an ellipse

Here we turn to the scenario when we replace a circle with an ellipse.

Example 6 *We are given a fixed ellipse in blue and the fixed point A is in the interior of the ellipse. A line passes through A and intersects the ellipse at C and D respectively. If the point E is the midpoint of CD. Then find the locus of E (see Figures 12(b)).*

Without loss of generality, we consider the case when the ellipse is in the standard form of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We let the line pass through the fixed point $A = (x_0, y_0)$ and intersect the ellipse at C and D respectively. In addition, we assume CD is not a vertical line perpendicular to the x axis. If we let the slope CD to be k, then the line equation of $\overleftrightarrow{CD}$ is $y - y_0 = k(x - x_0)$. If we write $C = (x_1, y_1)$ and $D = (x_2, y_2)$, then we use the technique of **difference of squares** to find the equation of the locus. Since C, D are points on the ellipse, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \qquad (9)$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} = 1. \qquad (10)$$

We subtract 10 from 9 and see

$$\begin{aligned}
\frac{(x_1 - x_2)(x_1 + x_2)}{a^2} &= -\frac{(y_1 - y_2)(y_1 + y_2)}{b^2} \\
&\implies \frac{y_1 - y_2}{x_1 - x_2} = \frac{-b^2}{a^2}\frac{(x_1 + x_2)}{(y_1 + y_2)} \\
&\implies k = \frac{-b^2}{a^2}\frac{(x_1 + x_2)}{(y_1 + y_2)}.
\end{aligned}$$

If we denote the midpoint E as (X, Y) then $k = \frac{-b^2}{a^2}\frac{X}{Y}$. Since the midpoint E satisfies the line equation $\overleftrightarrow{CD}$ passing through the fixed point A, we see $Y - y_0 = \frac{-b^2}{a^2}\frac{X}{Y}(X - x_0)$. Hence, we have $a^2Y(Y - y_0) = -b^2X(X - x_0)$ or $a^2(Y - \frac{y_0}{2})^2 + b^2(X - \frac{x_0}{2})^2 = \frac{a^2y_0^2 + b^2x_0^2}{4}$, which yields the followings:

$$\begin{aligned}
\frac{(Y - \frac{y_0}{2})^2}{b^2} + \frac{(X - \frac{x_0}{2})^2}{a^2} &= \frac{a^2y_0^2 + b^2x_0^2}{4a^2b^2}, \\
\frac{(Y - \frac{y_0}{2})^2}{\frac{a^2y_0^2 + b^2x_0^2}{4a^2}} + \frac{(X - \frac{x_0}{2})^2}{\frac{a^2y_0^2 + b^2x_0^2}{4b^2}} &= 1.
\end{aligned}$$

Therefore, the locus E is an ellipse centered at $\left(\frac{x_0}{2}, \frac{y_0}{2}\right)$ with major and minor lengths $\frac{\sqrt{a^2y_0^2 + b^2x_0^2}}{2b}$ and $\frac{\sqrt{a^2y_0^2 + b^2x_0^2}}{2a}$ respectively.

Exploratory Activity: Suppose we would like to find the locus $E = (X, Y)$ so that $\overrightarrow{CE} = s\overrightarrow{CD}$, where $s \in (0, 1)$. We invite readers to apply the algebraic techniques, which we used for the circle case, to derive the equation of the locus analogously in this case. Consequently, the derived equation of the locus should be identical to the one obtained by Geometry Expressions [2], which we show here:

$$\left(\begin{array}{l}
X = (1-s)\left(\frac{2\left(v_0 - \sin(t)\,|b|\right)\left(-v_0\cos(t)\,|a| + u_0\sin(t)\,|b|\right)}{b^2\left(-1 - \frac{u_0^2}{a^2} - \frac{v_0^2}{b^2} + \frac{2u_0\cos(t)\,|a|}{a^2} + \frac{2v_0\sin(t)\,|b|}{b^2}\right)} - \cos(t)\,|a|\right) + s\cos(t)\,|a| \\
Y = (1-s)\left(\frac{2\left(-u_0 + \cos(t)\,|a|\right)\left(-v_0\cos(t)\,|a| + u_0\sin(t)\,|b|\right)}{a^2\left(-1 - \frac{u_0^2}{a^2} - \frac{v_0^2}{b^2} + \frac{2u_0\cos(t)\,|a|}{a^2} + \frac{2v_0\sin(t)\,|b|}{b^2}\right)} - \sin(t)\,|b|\right) + s\sin(t)\,|b|
\end{array}\right)$$

We use the following screen shots to show the locus in red when $s = 0.4, 0.5$, and 0.8 respectively in Figures 12(a)-(c).

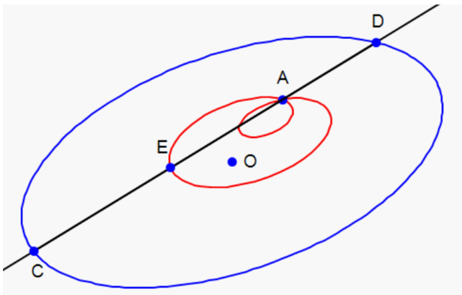

Figure 12(b). Locus, ellipse and $s = 0.4$

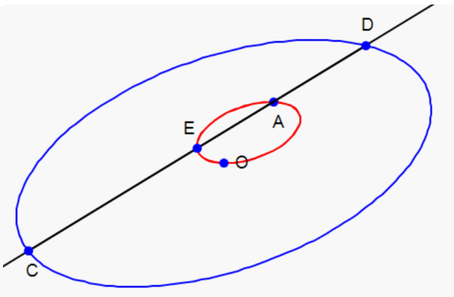

Figure 12(b). Locus, ellipse and $s = 0.5$

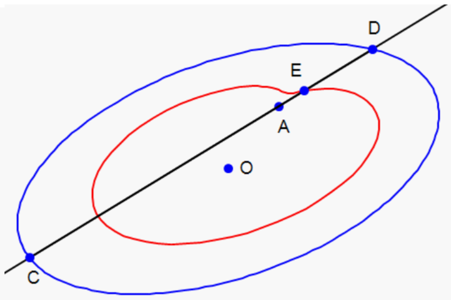

Figure 12(c). Locus, ellipse and $s = 0.8$

4.2 Special case when the fixed point is at the origin

Now we attempt to replace the ellipse by a more complex closed curve. In the following example, we show an easier case when the fixed point is at the origin, and line $\overleftrightarrow{AB}$ passes through the origin. We see in the following Example 7 how we may generalize the method we adopted earlier, when solving the circle case, to find the locus for a curve *that has both polar (or parametric) and implicit form of* $f(x, y) = 0$. Specifically, we consider

Example 7 *We recall the cardioid of* $r = f(t) = 1 - \cos t$ *has the implicit form of* $(x^2 + y^2 + x)^2 - x^2 - y^2 = 0$. *If we A and B are two points on the cardioid and the line* $\overleftrightarrow{AB}$ *passes through the fixed point* $I = (0, 0)$. *Find the locus* $M = (x, y)$ *that satisfies* $\overrightarrow{BM} = s\overrightarrow{BA}$, *where* $s \in (0, 1)$.

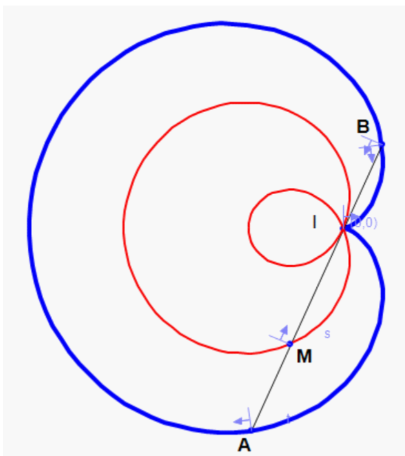

Figure 13. Locus, cardioid and the fixed point at $(0, 0)$

We first remark that the following locus can actually be derived by hand for those students who are not afraid of tedious algebraic manipulations. Next we show how the DGS such as [2] can help students visualize the interesting loci. Finally, we use the CAS such as [5] to show analytically that both answers from [2] and [5] coincide with each other.

Step 1. We label $A = (f(t)\cos t, f(t)\sin t)$ and $B = (x_1, y_1)$ to be two points on $r = f(t)$. Also we label locus M as (x, y) and observe from $\overrightarrow{BM} = s\overrightarrow{BA}$ that

$$\begin{aligned}
\begin{bmatrix} x - x_1 \\ y - y_1 \end{bmatrix} &= s \begin{bmatrix} f(t)\cos t - x_1 \\ f(t)\sin t - y_1 \end{bmatrix} \qquad (11)\\
x - f(t)s\cos t &= x_1 - sx_1 \\
&= x_1(1-s), \\
y - f(t)s\sin t &= y_1 - sy_1 \\
&= y_1(1-s).
\end{aligned}$$

Step 2. Since $\overleftrightarrow{AB}$ passes through the fixed point I at the origin, we write the line of $\overleftrightarrow{AB}$ as $y = mx$,

Step 3. Now, we plug $y = mx$ into the implicit equation of the cardioid,

$$\begin{aligned}
\left(x^2 + y^2 + x\right)^2 - x^2 - y^2 &= 0 \\
\left(x^2 + m^2x^2 + x\right)^2 - x^2 - m^2x^2 &= 0 \\
x^2\left(m^4x^2 + 2m^2x^2 + 2m^2x - m^2 + x^2 + 2x\right) &= 0
\end{aligned}$$

If $x = 0$ then $B = (0, 0)$ which implies $B = I$, the problem becomes a simple exercise to explore, and we leave it to readers to verify.

If $x \neq 0$, then

$$x^2\left(m^4 + 2m^2 + 1\right) + \left(2m^2 + 2\right)x - m^2 = 0, \qquad (12)$$

and we consider the discriminant of the quadratic equation in x 12 as follows:

$$D = \left(2m^2 + 2\right)^2 + 4\left(m^4 + 2m^2 + 1\right)m^2 > 0.$$

Furthermore, since two roots x_1^* and x_2^* from 12 satisfying $x_1^* + x_2^* = \frac{-\left(2m^2+2\right)}{(m^4+2m^2+1)}$, we write

$$\begin{aligned}
x_1^* &= \frac{-\left(2m^2 + 2\right)}{\left(m^4 + 2m^2 + 1\right)} - f(t)\cos t \\
&= \frac{-\left(2\tan^2 t + 2\right)}{\left(\tan^4 t + 2\tan^2 t + 1\right)} - f(t)\cos t \\
y_1^* &= (\tan t)\, x_1^* \\
&= (\tan t)\left(\frac{-\left(2\tan^2 t + 2\right)}{\left(\tan^4 t + 2\tan^2 t + 1\right)} - f(t)\cos t\right)
\end{aligned}$$

Step 4. We write x by using x_1^* in 11, in other words, we have $x - f(t)s\cos t = x_1 - sx_1 = x_1(1-s)$, which implies the following:

$$\begin{aligned}
x(s,t) &= f(t)s\cos t + x_1^*(1-s) \\
&= s(1-\cos t)\cos t + (1-s)\left(\frac{-\left(2\tan^2 t + 2\right)}{\left(\tan^4 t + 2\tan^2 t + 1\right)} - f(t)\cos t\right) \\
&= s(1-\cos t)\cos t + (1-s)\left(\frac{-\left(2\tan^2 t + 2\right)}{\left(\tan^4 t + 2\tan^2 t + 1\right)} - (1-\cos t)\cos t\right)
\end{aligned}$$

Step 5. We use y_1^* to find y 11 In other words, we have

$$
\begin{aligned}
y(s,t) &= sf(t)\sin t + y_1^*(1-s) \\
&= s(1-\cos t)\sin t + (1-s)\left(\tan t\left(\frac{-\left(2\tan^2 t+2\right)}{\left(\tan^4 t + 2\tan^2 t + 1\right)} - f(t)\cos t\right)\right) \\
&= s(1-\cos t)\sin t + (1-s)\left(\tan t\left(\frac{-\left(2\tan^2 t+2\right)}{\left(\tan^4 t + 2\tan^2 t + 1\right)} - (1-\cos t)\cos t\right)\right)
\end{aligned}
$$

Step 6. We remark that the output of the parametric equation for the locus from [2] is shown below

$$
\left(\begin{array}{l} X(s,t) = (-1+2s)\cos(t) - \cos(t)^2 \\ Y(s,t) = (-1+2s-\cos(t))\sin(t) \end{array}\right).
$$

After using `simplify` command in [5], we see $x(s,t) = X(s,t)$ and $y(s,t) = Y(s,t)$. We show various screen shots of the locus obtained from the CAS Maple [5], which corresponding to their respective value s below.

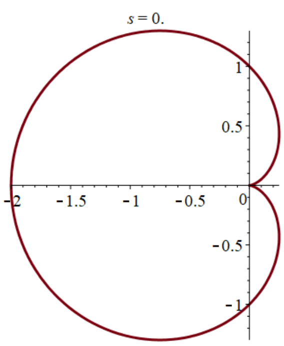

Figure 14(a) Maple plot with $s = 0$

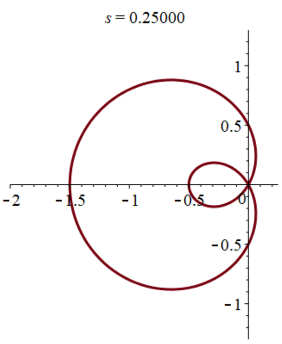

Figure 14(b) Maple plot with $s = 0.25$

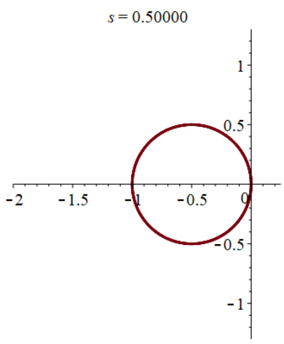

Figure 14(c) Maple plot with $s = 0.5$

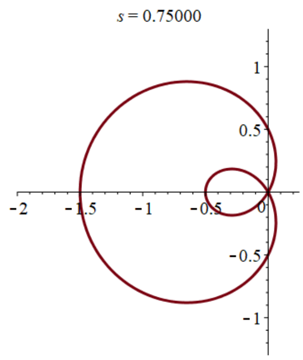

Figure 14(d) Maple plot with $s = 0.75$

5 Acknowledgements

The author would like to thank Dao Hoang and Phil Todd (founder of [2]) for creating instrumental dynamic figures using Geometry Expressions ([2]). The author also would like to thank Qiuxia Li for providing the proofs of some examples and numerous inspiring discussions about the level of high school math content knowledge from her teaching experiences in China.

6 Conclusion

It is clear that technological tools provide us with many crucial intuitions before we attempt more rigorous analytical solutions. Here we have gained geometric intuitions while using a DGS such as [1] or [2]. In the meantime, we use a CAS such as Maple [5], for verifying that our analytical solutions are consistent with our initial intuitions. The complexity level of the problems we posed vary from the simple to the difficult. Many of our solutions are accessible to students from high school. Others require more advanced mathematics such as university levels, which are excellent examples for professional trainings for future teachers.

Evolving technological tools definitely have made mathematics fun and accessible on one hand, but they also allow the exploration of more challenging and theoretical mathematics. We hope that when mathematics is made more accessible to students, it is possible more students will be inspired to investigate problems ranging from the simple to the more challenging. We do not expect that exam-oriented curricula will change in the short term. However, encouraging a greater interest in mathematics for students, and in particular providing them with the technological tools to solve challenging and intricate problems beyond the reach of pencil-and-paper, is an important step for cultivating creativity and innovation.

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