Synchronization of Chaos with the CAS Maxima

Guillermo Dávila-Rascón
davila@mat.uson.mx
Departmento de Matemáticas
Universidad de Sonora
Hermosillo, 83000
México

Abstract

Synchronization phenomena pervades our daily lives. Lots of instances confirming this could be cited. For example, many of our bodies physiological functions are synchronized to the day-night cycle (circadian rhythm); thousands of pacemaker cells in a cluster called sinoatrial node, fire at unison in order to maintain the regular beats of our hearts; around the world, thousands of fireflies come together along riverbanks and synchronize their flashes in an amazing spectacle that has been noticed and reported for over three centuries; laser beams are also examples of perfect synchronization of trillions of atoms; electrons flowing in a superconductor is another.

Here we present, by using the free CAS Maxima, some examples of synchronization of dynamical systems. We focus on chaotic dynamical systems, since as chaos is characterized by its extreme sensibility to initial conditions, at first glance, it is kind of surprising how two chaotic dynamical systems can be synchronized. Our main goal here is to explore the complete synchronization scenario between two chaotic systems through a coupling mechanism that results in a common behaviour and the same evolution of both systems.

1 Introduction

Synchronization is, roughly speaking, a process where two or more systems interacting with each other happen to evolve jointly, in the sense that each one is capable to adjust its own evolution, "pulled by the others", and thus all together having the same behaviour over time of some on their dynamical properties. This phenomena occurs mainly between self-sustained oscillators, coupled together in some way.

Although there has been written records describing examples of biological rythm since fourth century B. C., as cited by Strogatz [21], the first documented account on studies and experimentation about synchronization phenomena was that of the Dutch scientist Christiaan Huygens around the year 1665. It was by serendipity that Huygens, whilst confined to his bedroom and recovering from an ailment, observed a very remarkably phenomena that happened between two of his pendulum clocks hung in the same wooden beam: "*Being obliged to stay in my room for several days and also occupied in making observations on my new two newly made clocks, I* *have noticed an admirable effect which no one could have ever thought of. It is that these two clocks hanging next to one another separated by one or two feet keep an agreement so exact that the pendulums always oscillate together without variation ...* " (as cited in [21].) See also [8].

Huygens was, by the time, experimenting with (identical) pendulum clocks, in order to tackle the then famous *longitude problem*. Governments like those of Spain, Portugal, and The Netherlands, had offered sustancial prices to whom could solve this problem [17, 21], that was tantamount of constructing very precise clocks, capable of great redundancy during navigation in any weather conditions: "*... in case one of the clocks stopped or needed to be cleaned, the other could still keep accurate time ...* " [21].

Eventually, Huygens discovered that the synchronized movement of the pendulums of his clocks was due to the hardly perceptible vibrations caused by both mechanisms on the beam they were attached to, and having the effect of a mutual feedback on the clocks, which resulted in an anti-phase synchronization. In his premiere explanations, Huygens attributed this miraculous phenomena to a *kind of sympathy* of the clocks but not having better arguments than that of their mutual feedback, he no longer continued his investigations on this subject [8, 21].

Since then onwards, many more descriptions of synchronization phenomena have been reported elsewhere, see for example [10, 20] and references therein. Nevertheless, a spectacular and very striking one is that of the synchronous behaviour that exhibit some species of fireflies that congregate by the thousands in the trees along riverbanks in many countries [15, 21], creating a marvelous night spectacle by flashing in unison.

From the mathematical viewpoint there were no interest in studying collective synchronization aside from that of Norbert Wiener around 1961 who recognized its importance and conjectured that the *alpha rhythm* is a byproduct of the cooperative work of millions of specialized neurons, as if in our brains we had an inner clock coordinating the important functions of the brain. However, the mathematical treatment of Wiener on this problem lead to nowhere and was necessary another approach to model synchronization phenomena.

This was done by Arthur Winfree in his 1967 seminal paper [23] that drew a lot of attention from the scientific community. Winfree proposed a model describing the collective behaviour of a population of self-sustained generic oscillators in which each individual is sensitive to the influence of the others by means of an influence and sensitivity function that depends on the phase of each oscillator. The Winfree synchronization model is the system of N first order differential equations [7, 11, 19],

$$\dot{\theta}_i = \omega_i + \frac{\varepsilon}{N} \sum_{j=1}^{N} P(\theta_j) Q(\theta_i), \tag{1}$$

where $N \gg 1$ is a natural number. Here, $\dot{\theta}$ denotes the derivative of θ with respect to time t; ω_i and θ_i are, respectively, the natural frequency and phase of each oscillator; P and Q are 2π-periodic functions, and ε is a parameter that controls the global coupling strength. Notice how each oscillator receives an input of the *mean field* which has the form $(1/N) \sum_{j=1}^{N} P(\theta_j)$ and responds through the function $Q(\theta)$.

Winfree investigated this model numerically and among his conclusions was the fact that for some scenarios, synchronization is inevitable, although for some others, chaos emerges. Nevertheless, when the frequencies ω_i are fairly homogenous, and for certain choices of P, Q and ε, the oscillators get synchronized and evolve locked in time.

Needless to say that Winfree never presented analytical solutions to (1) since it is very untractable in its full generality, and for many years, researchers were looking for simplified integrable models similar to (1). However, in 1975, Yoshiki Kuramoto proposed a model that became a paradigm in the study of collective synchronization [1, 19, 22]. Kuramoto's model departs from that of Winfree's, but in the mean field interaction all the oscillators contribute equally and act equally upon them. The equations are the following:

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i), \tag{2}$$

for $i = 1, \ldots, N$, and where A_{ij} are the coupling parameters. If we set $A_{ij} = K/N$, then parameter $K \geq 0$ measures the coupling strength of the system and equation (2) can be written as

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i). \tag{3}$$

Explicit solutions can be drawn for this system [1, 14, 19] and has served as a model for applications ranging from biology, chemistry, and physics to finances [1, 14].

It is worth noting that for the case of $P(\theta) = 1 + \cos\theta$ and $Q(\theta) = -\sin\theta$, the Winfree model can be solved explicitly, a remarkable work that was done by Ariaratnam and Strogatz [3].

In another vein, chaotic dynamics is characterized by its sensibility to initial conditions. That is, the behaviour of a chaotic dynamical system is unpredictable for long periods of time. Usually, this is demonstrated by running two identical chaotic systems starting at very close initial conditions and observing how their evolutions diverge exponentially with time. Thus, it was striking when the first works on synchronization of chaotic dynamical systems appeared in several journals, which gave rise to a new field of research on synchronization phenomena that is very active since the early 1990s.

Although we can trace back the synchronization of chaos to the paper of Fujisaka and Yamada [6] and that of Afraimovich, Verichev and Rabinovich [2], those works were almost unnoticed and it was until several years later when the scientific community realized of their significance. A very influential article on the subject was that of Pecora and Carroll [12] that starts with the following words: *Chaotic systems would seem to be dynamical systems that defy sinchronization* . . . . Therefore, the fact that it is possible to synchronize chaos . . . *has deepened our understanding of synchronicity itself. From now on, sync will no longer be associated with ryhthmicity alone, with loops and cycles in repetition* . . . [21].

On the other hand, there are different scenarios for synchronizing two (or more) dynamical systems. According to [4, 5, 13, 14, 18], we can distinguish the following: identical (or complete), phase, lag, and generalized synchronization. Two of the most popular schemes of complete synchronization are the so called *master-slave* and *partial replacement* syncrhonization, but in this paper we only explore an example of the last one. Nevertheless whatever the case, if $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n$, and $\mathbf{y}(t) = (y_1(t), \ldots, y_n(t)) \in \mathbb{R}$ denote the evolution trajectories of two dynamical systems

$$\dot{\mathbf{x}} = f(t, \mathbf{x}), \qquad \dot{\mathbf{y}} = g(t, \mathbf{y}),$$

for smooth functions f, and g defined in some domain of $\mathbb{R} \times \mathbb{R}^n$, then one has to verify that

$$\|\mathbf{x}(t) - \mathbf{y}(t)\| \to 0, \quad \text{as} \quad t \to \infty,$$

where, $\| \, \|$ denotes the Euclidean norm in $\mathbb{R}^n$.

We will not present here the theoretical foundations of synchronization. Instead, we refer the reader to the cited references and to the excellent books [14, 18], and references therein. We remark that, as this paper is just an introduction into this topic, we do not require the use of plenty of sophisticated mathematics and it can be taught at undergraduate level for a deeper understanding of chaos and synchronization.

The paper is organized as follows. In Section 2, we introduce some Maxima commands by studying a particular case of a driven and dumped oscillator. This system is not chaotic but offers an opportunity for introducing an example of synchronization of two, non chaotical, dynamical systems, which is done in Section 3. Next, in Section 4 we present some generalities of chaotic systems and explore some properties of chaos with a paramount example, namely, the Lorenz system. Finally, we demonstrate with Maxima how chaos can be synchronized by showing an example of two Rösler systems coupled through the partial replacement mechanism.

We remark that everything that was done here with Maxima, and its graphical interface wxMaxima, can also be carried out with other computer algebra systems like Mathematica®, Maple®, and Matlab®. We have chosen Maxima since it is a free and open source software, that runs in most used platforms, such as Android©, Linux©, Windows©, and Mac OS©. See [9].

2 The driven and damped oscillator

It is well known that the equation of motion for a simple harmonic oscillator is given by the second order linear differential equation [24],

$$\ddot{x} + \omega_0^2 x = 0. \tag{4}$$

Here $x = x(t)$ is the displacement of the oscillator from the equilibrium ($x = 0$) and ω_0 its natural frequency, where $\omega_0^2 = k/m$, with $m > 0$ being the mass of the oscillator and k the Hooke's constant of the spring sustaining the system. As is well known, solutions of (4) are of the form

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t),$$

for constants c_1 and c_2 depending on initial conditions.

In the case of a damped and driven harmonic oscillator, the equation of motion is given by

$$\ddot{x} + \lambda \dot{x} + \omega_0^2 x = \frac{1}{m} F(t), \tag{5}$$

where $\lambda > 0$ is the damping coefficient and the driving force is given by a smooth function $F = F(t)$. This is an excellent model to illustrate periodic harmonic motion in the classroom as well as several key facts of the theory of second order differential equations, such as *resonance*. Analytical solutions for (5) are well known and can be found elsewhere, see for example [24].

Here, we will be interested in a particular case of (5), namely that of a periodic driving force,

$$\ddot{x} + \lambda \dot{x} + \omega_0^2 x = \mu \cos(\omega t), \tag{6}$$

for a constant μ [8]. We notice that the right hand side of (6) is a periodic driving force with its own frequency ω. We have chosen the model in (6) since it has two simple attractors: a point attractor for $\mu = 0$ (unforced case) given by the stationary solution $x = 0$ and that for the case $\mu \neq 0$ having the periodic solution

$$x(t) = A\cos(\omega t - \varphi), \tag{7}$$

where the *amplitude* A and the *phase shift* φ are given, respectively, by

$$A = \frac{\mu}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\lambda\omega)^2}}, \qquad \varphi = \frac{\lambda\omega}{\omega_0^2 - \omega^2}.$$

Setting $v = \dot{x}$ and $\dot{s} = 1$, equation (6) becomes the first order autonomous system

$$\dot{x} = v, \tag{8}$$
$$\dot{v} = \mu\cos(\omega t) - \omega_0^2 x - \lambda v, \tag{9}$$
$$\dot{s} = 1, \tag{10}$$

for which, numerical solutions can easily be obtained by means of the fourth order Runge-Kuta method available in Maxima. We solve system (8)-(10) for two different settings of initial conditions in order to illustrate an interesting fact: any solution of (6) converges in the long time limit to the attractor (7), which is a *limit cycle* [8].

We set the values for the frequencies w_0 and w, starting from $t = 0$ and advancing with a step $h = 0.05$ for a total time $T = 50.0$. We run the Runge-Kuta routine, which is implemented in Maxima through the `rk` function, for initial conditions: $(x_0, v_0, s_0) = (0.5, 0, 0)$, and $(x_0, v_0, s_0) = (-0.2, 0, 0)$.

Let us solve the system for the first initial condition, namely, $(0.5, 0, 0)$. The following syntax should be clear:

```
(%i1) w0:1.0$ w:1.5$ T:50$
(%i4) trajectory1 : rk([v, 0.5*cos(w*s)-(w0^2)*x-0.7v,1],
                      [x,v,s],[0.5,0,0],[t,0,T,0.05])$
```

Since the output of the `rk` function is a list whose elements are of the form `[t,x(t),v(t),s(t)]`, and we are interested mainly in the displacement $x(t)$ and velocity $v(t)$, it is necessary to obtain those values from the entire list:

```
(%i5) data_traj1 : makelist([p[2],p[3]],p,trajectory1)$
```

This gives a list whose entries are of the form $[x(t_i), v(t_i)]$, for each time t_i, $i = 1, \ldots, 1001$, resulting from the Runge-Kutta method. Next we do the same for the second initial condition, $(-0.2, 0, 0)$:

```
(%i6) trajectory2 : rk([v, 0.5*cos(w*s)-(w0^2)*x-0.7v,1],
                      [x,v,s],[-0.2,0,0],[t,0,T,0.05])$
(%i7) data_traj2 : makelist([q[2],q[3]],q,trajectory2)$
```

Now we set the explicit list for time points t_i, since we are going to evaluate the velocity $v(t) = \dot{x}(t)$ of (7), and also for graphical purposes. Also, introduce the amplitude A and phase shift φ:

```
(%i8) time : makelist(T*(i-1)/1000,i,1,(T/0.05)+1)$
(%i10) A : 0.5/sqrt((w0^2 - w^2)^2 + (0.7*w)^2)$  phi : 0.7*w/(w0^2 - w^2)$
```

Next, we introduce the periodic solution (7) and calculate its derivative in order to define the velocity function $v(t) = \dot{x}(t)$ and to create a list with entries of the form $[x(t), v(t)]$:

```
(%i11) xt : A*cos(w*t - phi)$
(%i12) diff(xt,t);
(%o12) -0.4594229188737667 sin(1.5*t + 0.8399999999999999)
(%i13) v(t) := -0.4594229*sin(1.5*t + 0.8399999)$
(%i14) xxt : create_list(A*cos(w*time[i] - phi), i, 1, length(time))$
(%i15) vvt : makelist(v(time[i]), i,length(time))$
(%i16) attractor : makelist([xxt[i],vvt[i]],i,lenght(time))$
```

Finally we plot the evolution of $[x(t), v(t)]$ in phase space, together with the graph for the attractor, Figure 1:

```
(%i17) wxplot2d([[discrete,trajectorya],[discrete,trajectoryb],
                 [discrete, attractor]],[color, blue,red,black],
                 [style,[lines,1], [lines,1],[lines,1]],
                 [xlabel,"Position"], [ylabel, "Velocity"],
                 [legend,"Trajectory 1","Trajectory 2", "Attractor"]);
(%o17)
```

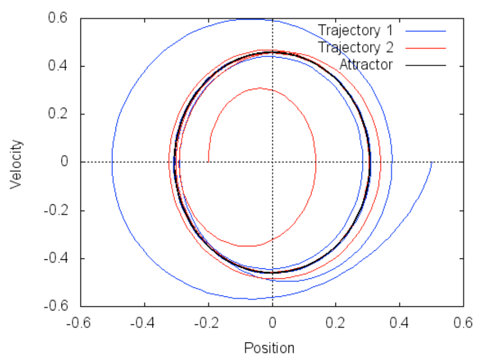


Figure 1: Two trajectories for the damped and driven oscillator.

Now we estimate, for each time t, the distance between the points $[x(t), v(t)]$ for the two trajectories, in order to have an idea if they are getting closer, and show the result in Figure 2:

```
(%i18) error_traject:makelist(sqrt((trajectorya[i][1]-trajectoryb[i][1])^2
                              + (trajectorya[i][2]-trajectoryb[i][2])^2),
                                i, 1, length(time))$
(%i19) Error:makelist([time[i],error_traject[i]],i,1,length(time))$
(%i20) wxplot2d([discrete,Error],[x,0,25],
                [style,[lines,1]], [color, blue],
                [xlabel,"Time"], [ylabel, "Error"],
                [legend,false]);
(%o20)
```

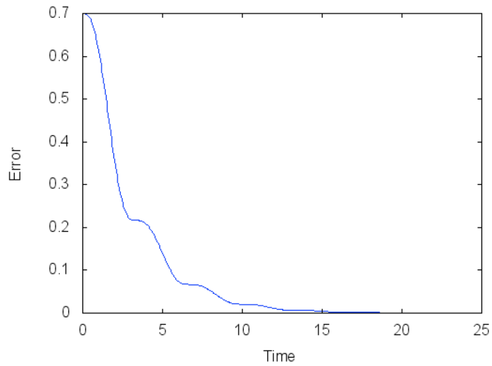


Figure 2: Distances between the two trajectories in output (%o20).

We can see that both trajectories approximate each other and the distance between them converges to zero with time.

3 Synchronization of coupled oscillators

Now, consider two damped and driven oscillators,

$$\dot{x} = v, \qquad \dot{y} = u, \tag{11}$$
$$\dot{v} = \mu\cos(\omega t) - \omega_0^2 x - \lambda v, \qquad \dot{u} = \gamma\cos(\theta t) - \theta_0^2 y - \eta u, \tag{12}$$
$$\dot{s} = 1, \qquad \dot{z} = 1. \tag{13}$$

Here, as in (5) and (6), $\mu, \lambda, \gamma, \eta$ are constants, with $\lambda, \eta > 0$ and $\mu \neq 0 \neq \gamma$. For the sake of simplicity, we assume that the frequencies ω_0, ω and θ_0, θ, for each oscillator, respectively, satisfy $\omega_0 = \theta_0$ and $\omega = \theta$.

Instead of trying to solve systems (11)-(13), we will couple them and study the whole system. Thus, we propose the following coupling:

$$\dot{x} = v + \alpha(y - x), \qquad \dot{y} = u + \beta(x - y), \tag{14}$$
$$\dot{v} = \mu\cos(\omega t) - \omega_0^2 x - (\lambda v - \rho u), \qquad \dot{u} = \gamma\cos(\omega t) - \omega_0^2 y - (\eta u - \varrho v), \tag{15}$$
$$\dot{s} = 1, \qquad \dot{z} = 1. \tag{16}$$

Notice that for $\alpha = \beta = 0$ and $\rho = \varrho = 0$, we obtain system (11)-(13), since we are assuming $\omega_0 = \theta_0$ and $\omega = \theta$. The coupling is given by the linear terms $\alpha(y-x)$, and ρu in the first system, and by $\beta(x-y)$, and ϱv in the second. This is the case of two-way *replacement synchronization* [4, 12, 13, 14, 18], in which two copies of the system are settled down by substituting some variables occurring in one or both of them with variables of the other, here the name.

In order to apply the Runge-Kuta method to study system (14)-(16), we set the following values for the parameters: $\alpha = \beta = 0.5$, $\mu = \gamma = 0.5$, $\lambda = \rho = 0.7$, and $\eta = \varrho = 0.7$. We also let the frequencies attain the values $\omega_0 = 1.0$ and $\omega = 1.5$. Set total time $T = 100.0$, with an advancing step $h = 0.05$.

```
(%i3) w0:1.0$ w:1.5$ T:100.0$
(%i4) coup_oscil : rk([v+0.5*(y-x), 0.5*cos(w*t)-(w0^2)*x - 0.7*(v-u),1,
```

```
                    u+0.5*(x-y), 0.5*cos(w*t)-(w0^2)*y - 0.7*(u-v),1],
                    [x,v,s,y,u,z],[1,0,0,-1,0,0],[t,0,T,0.05])$
(%i5) oscillator1 : makelist([p[2],p[3]], p, coup_oscil)$
(%i6) oscillator2 : makelist([p[5],p[6]], p, coup_oscil)$
(%i7) wxplot2d([[discrete,oscillator1], [discrete,oscillator2]],
              [style,[lines,1],[lines,1]],[xlabel,"Displacement"],
              [ylabel,"Velocities"],
              [legend,"Oscillator 1","Oscillator 2"]);
(%o7)
```

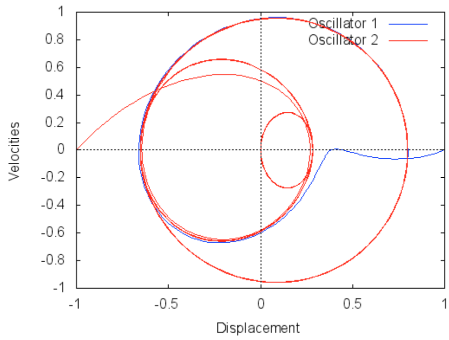


Figure 3: Two coupled oscillators

It is clear from Figure 3 that both trajectories follow each other after small lapse of time and we can see how both oscillators pull one another on. Notice that in this case there is no simple attractor as that introduced for the dumped and driven oscillator in Section 2.

In order to have a better idea of what is going on with both oscillators is by giving some animation to the trajectories of the system, which is done through the following Maxima input:

```
(%i8) set_draw_defaults( xrange = [-1,1], yrange = [-1,1])$
(%i9) wxanimate_framerate: 5$
(%i10) with_slider_draw(d,makelist(i,i,1,length(oscillator1),5),
             point_type = filled_circle,point_size = 0.4,color=blue,
             points(rest(oscillator1,d-length(oscillator1))),
             point_type = filled_circle,point_size = 0.4,color=red,
             points(rest(oscillator2,d-length(oscillator2))))$
(%o10)
```

Maxima output (%o10) can be seen in Figure 4.

Now we compare the evolution of the displacements and velocities for the two oscillators and, for each time t_i, calculate the distance between both trajectories in phase space. Finally, the output is shown in Figure 5:

```
(%i10) time : makelist(T*(i-1)/1000, i, 1, (T/0.05)+1)$
(%i11) displac_osc1 : makelist(p[2], p, coup_oscil)$
(%i12) displac_osc2 : makelist(p[5], p, coup_oscil)$
(%i13) vel_osc1 : makelist(p[3], p, coup_oscil)$
(%i14) vel_osc2 : makelist(p[6], p, coup_oscil)$
```

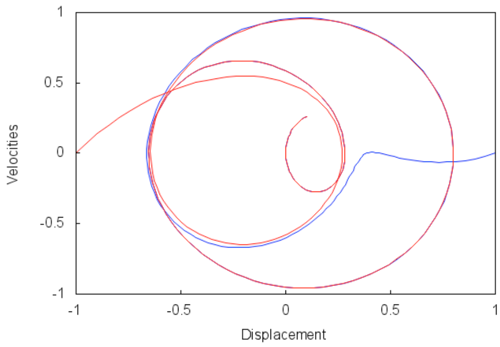


Figure 4: Animation for two coupled oscillators.

```
(%i15) error_displac : makelist(abs(displac_osc1[i]-displac_osc2[i]),
                                 i,1,length(time))$
(%i16) error_veloc : makelist(abs(vel_osc1[i]-vel_osc2[i]), i,1,length(time))$
(%i17) evolution_displ : makelist([time[i],error_displac[i]], i,length(time))$
(%i18) evolution_vel : makelist([time[i],error_veloc[i]], i, length(time))$
(%i19) distance_traj : makelist([time[i],
                                 sqrt(error_displac[i]^2 + error_veloc[i]^2)],
                                 i, 1, length(time))$
(%i20) wxplot2d([[discrete,evolution_displac],[discrete,evolution_vel],
         [discrete,distance_traj]],[x,0,13],
         [style,[lines,1]],[color,blue,magenta,red],
         [xlabel,"Time"],[ylabel, "Evolution of Errors"],
         [legend,"Difference on displacements","Difference on velocities",
         "Distance between trajectories"]);
(%o20)
```

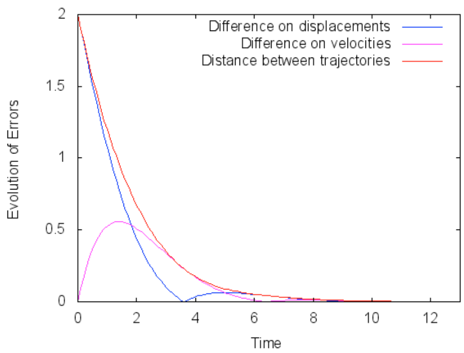


Figure 5: Comparing the evolution for two coupled oscillators.

4 Chaos and synchronization

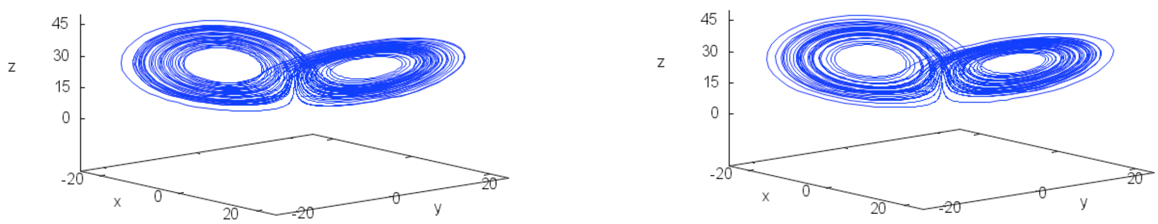
Chaotic dynamics is characterized by its sensibility to initial conditions, that is, the behaviour of a chaotic dynamical system is strikingly different even for infinitesimal variations of the initial states [8, 14, 21]. Let us illustrate this fact with the famous Lorenz system given by the equations

$$\dot{x} = 10(y - x), \tag{17}$$
$$\dot{y} = 28x - y - xz, \tag{18}$$
$$\dot{z} = xy - (8/3)z. \tag{19}$$

Solving this system with Maxima for the following two different settings on initial conditions

$$(x_0, y_0, z_0) = (-8, 5, 13) \qquad \text{and} \qquad (x_0, y_0, z_0) = (-8, 5, 13.0001), \tag{20}$$

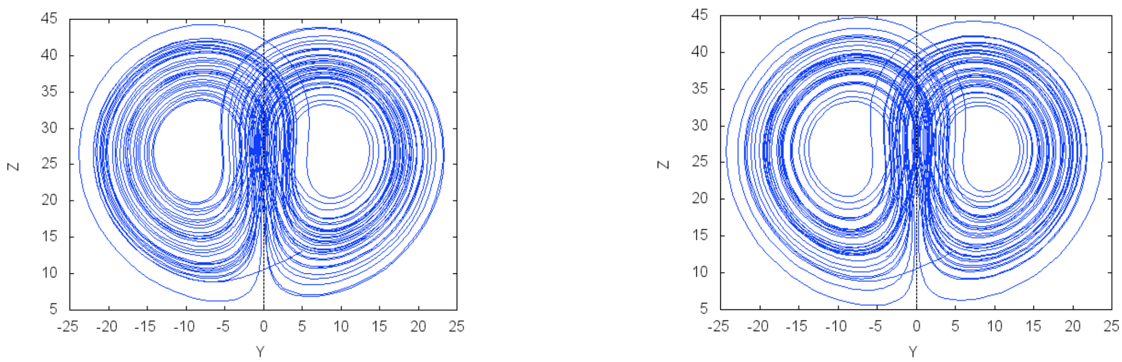
we get two Lorenz attractors which are clearly different, as can be appreciated in Figure 6.



(a) Initial conditions at $(-8, 5, 13)$ (b) Initial conditions at $(-8, 5, 13.0001)$

Figure 6: Two Lorenz attractors with initial states given in (20).

If we project the Lorenz attractors for both initial states given by (20) onto the yz-plane, we get a clearer picture showing the behaviour of the two trajectories and how different they are, despite the two systems started "almost" from the same point, Figure 7.



(a) Initial conditions at $(-8, 5, 13)$ (b) Initial conditions at $(-8, 5, 13.0001)$

Figure 7: Projections on the yz-plane of Lorenz attractors with initial states (20).

On the other hand, a better picture of what is going on with Lorenz system (17)-(19) at both initial conditions (20) is by calculating the distance between their respective trajectories at any time t_i. The results are plotted in Figure 8, where can be appreciated how a varying behaviour occurs. This is the characteristic mark of chaos.

What it is quite surprisingly is that, despite the fact of extreme sensitivity to initial conditions, two chaotic dynamical systems can be synchronized if we couple them properly. In

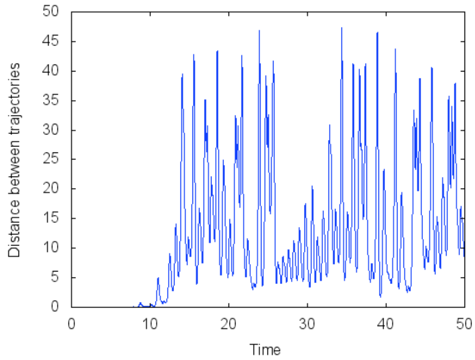


Figure 8: Distance at time t_i between the two trajectories of a Lorenz system with initial conditions (20).

what follows, we will explore with Maxima another well known example of chaotic dynamics, namely, the Rösler system, given by

$$\dot{x} = -y - z, \tag{21}$$
$$\dot{y} = x + \alpha y, \tag{22}$$
$$\dot{z} = \beta + z(x - \gamma), \tag{23}$$

where α, β and γ are the parameters of the system.

As was pointed out in Section 1, there are several scenarios for synchronization. Here we will couple two Rössler systems like (21)-(23), with $\alpha, \beta = 0.2$ and $\gamma = 5.7$, but coupled through a very simple mechanism, namely that of replacement synchronization:

$$\dot{x} = -y - z + (u - x), \qquad \dot{u} = -v - w + (x - u), \tag{24}$$
$$\dot{y} = x + 0.2y, \qquad \dot{v} = u + 0.2v, \tag{25}$$
$$\dot{z} = 0.2 + z(x - 5.7), \qquad \dot{w} = 0.2 + w(u - 5.7), \tag{26}$$

Notice that the coupling is simply given by the terms $(u - x)$ in the first and $(x - u)$ in the second.

We solve numerically this system for a total time $T = 200$ with a step $h = 0.05$, and then plot the output, which is shown in Figure 9. Notice that one system starts at initial condition $(x_0, y_0, z_0) = (5, 5, 5)$ while the other starts at $(u_0, v_0, w_0) = (-5, -5, -5)$.

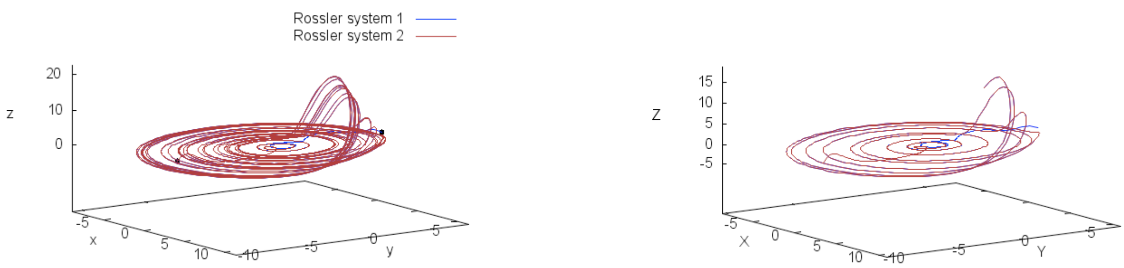
```
(%i1) T:200$
(%i2) coupled_rossler : rk([-y-z+(u-x), x + 0.2*y, 0.2 + z*(x - 5.7),
                           -v-w+(x-u), u + 0.2*v, 0.2 + w*(u - 5.7)],
                          [x,y,z,u,v,w],[5,5,5,-5,-5,-5],[t,0,T,0.05])$
(i%3) rossler1 : makelist([q[2],q[3],q[4]], q, coupled_rossler)$
(i%4) rossler2 : makelist([q[5],q[6],q[7]], q, coupled_rossler)$
```

We also set explicitly the initial conditions in order to plot both satarting points, together with the results of `rk`.

```
(%i5) init_cond1:[[5,5,5]]$ init_cond2:[[-5,-5,-5]]$
```

```
(%i6) wxdraw3d(xlabel="x",ylabel="y",zlabel="z", ytics=5, ztics=10,
         line_width=1.5,point_size=1.0, point_type=filled_circle,
         color=black, points(init_cond1),
         point_size=1.0, point_type=filled_circle, color=black,
         points(init_cond2),
         color=blue, point_type=none, points_joined=true,
         key="Rossler system 1",points(rossler1),
         color=brown, point_type=none, points_joined=true,
         key="Rossler system 2",points(rossler2),
         view=[70,55]);
(%o6)
```

This Maxima output (%o6) is rendered in Figure 9a.



(a) Synchronized Rösler systems.

(b) Animation of synchronized Rösler systems.

Figure 9: Synchronization of two Rösler systems.

We can better appreciate how synchronization occurs by giving some animation to the process. This is done through the following Maxima input:

```
(%i7) wxanimate_draw3d(d,makelist(i,i,1,length(rossler1),10),
                xtics=5, ytics=5, ztics=5,view=[70,55],
                line_width=1, color=blue, point_type=none, points_joined=true,
                points(rest(rossler1,d-length(rossler1))),
                point_type=none, points_joined=true, color=brown,
                points(rest(rossler2,d-length(rossler1))));
(%o7)
```

This Maxima output (%o7) is rendered in Figure 9b, and with the wxMaxima's graphical interface we can fully appreciate how both trajectories are evolving together, in perfect synchrony, despite each one started quite appart from the other.

Now we calculate the distance between the trajectories of the two coupled Rössler systems (24)-(26), and the following code should be self explanatory:

```
(%i8) dist_traj : makelist(sqrt((rossler1[i][1]-rossler2[i][1])^2 +
                                (rossler1[i][2]-rossler2[i][2])^2 +
                                (rossler1[i][3]-rossler2[i][3])^2),
                                i,length(rossler1))$
```

```
(%i9) time : makelist(T*(i-1)/1000, i,1,(T/0.05)+1)$
(%i10) Error_traj : makelist([time[i],dist_traj[i]],i,length(time))$
(%i11) wxplot2d([discrete, Error_traj], [x,0,80], [xlabel,"Time"],
              [ylabel,"Error"],[style,[lines,1]], [color,blue]);
(%o11)
```

It is clear, from Figure 10 how the difference between the two trajectories of Rössler systems (24)-(26) converges to cero, guaranteeing synchronization.

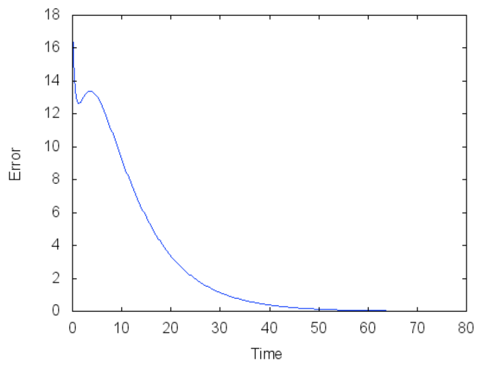


Figure 10: Error between trajectories

Finally, for a nice view of the synchronization of the two Rössler systems, we project the solutions of (24)-(26) on the xy-plane and give them some animation through the following input:

```
(%i12) rossler1xy : makelist([q[2],q[3]], q, coupled_rossler)$
(%i13) rossler2xy : makelist([q[5],q[6]], q, coupled_rossler)$
(%i14) set_draw_defaults(xrange = [-10,15],yrange = [-12,10])$
(%i15) wxanimate_framerate: 5$
(%i16) with_slider_draw(d,makelist(i,i,1,length(rossler1),5), line_width=1,
                    point_type = none,points_joined = true,color=blue,
                    points(rest(rossler1xy,d-length(rossler1xy))),
                    point_type = none,points_joined = true,color=red,
                    points(rest(rossler2xy,d-length(rossler2xy))))$
(%o16)
```

The output (%o16) is shown in Figure 11, which can be fully appreciated with the wxMaxima's graphical interface.

References

[1] Juan A. Acebrón, L. L. Bonilla, Conrad J. Pérez Vicente, Félix Ritort, and Renato Spigler, The Kuramoto model: A simple paradigm of synchronization phenomena, *Reviews of Modern Physics*, Vol. 77 (January 2005), pp. 137-182.

[2] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Stochastic Synchronization of Oscillation in Dissipative Systems, Transalted from *Izv. VUZ Radiofiz.*, Vol. 29, No. 9 (September 1986), pp. 1050-1060.

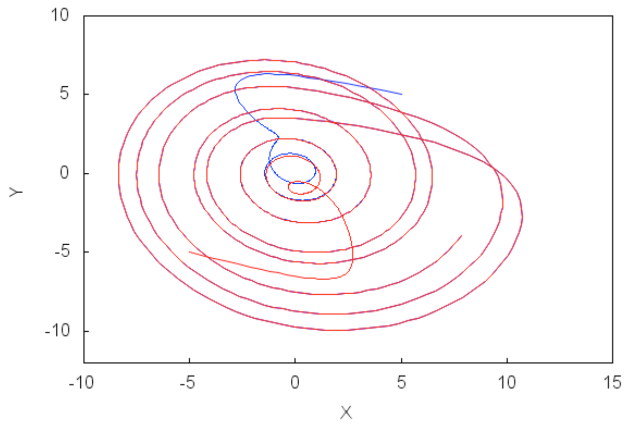


Figure 11: Projection on the xy-plane of the two synchronized Rössler systems.

[3] Joel T. Ariaratnam, and Steven H. Strogatz, Phase Diagram for the Winfree Model of Coupled Nonlinear Oscillators, *Phys. Rev. Lett.*, Vol. 86, No. 19 (May 2001), pp. 4278-4281.

[4] S. Boccaletti, J. Kurths, G. Osipov, D. L. Valladares, C. S. Zhou, The synchronization of chaotic systems, *Physics Reports*, 366 (2002), pp. 1-101.

[5] Mattia Frasca, Arturo Buscarino, Marco Branciforte, Luigi Fortuna, and Julien C. Sprott, Synchronization of two Rössler systems with switching coupling, arXiv:1507.01992v1, July 2015.

[6] Hirokazu Fujisaka, and Tomoji Yamada, Stability Theory of Synchronized Motion in Coupled-Oscillator Systems, *Progress of Theoretical Physics*, Vol. 69, No. 1 (January 1983), pp.32-47.

[7] Rafael Gallego, Ernest Montbrió, and Diego Pazó, Synchronization scenarios in the Winfree model of coupled oscillators, arXiv:1705.11065v1, June 2017.

[8] Hans Jürgen Korsch, Hans-Jörg Jodl, Timmo Hartmann, *Chaos. A Program Collection for the PC* (Third ed.), Springer-Verlag, Berlin, 2008.

[9] Maxima.sourceforge.net, Maxima, a Computer Algebra System, Version 5.41.0 (2018). http://maxima.sourceforge.net/

[10] Renato E. Mirollo, Steven H. Strogatz, Synchronization of pulse-coupled biological oscillators, *SIAM J. Appl. Math.*, **Vol. 50**, No. 6 (1990), 1645–1662.

[11] W. Oukil, A. Kessi, and Ph. Thieullen, Synchronization hypothesis in the Winfree model, *Dynamical Systems*, **32**, No. 3 (2017), pp. 326-339.

[12] Louis M. Pecora, and Thomas L. Carroll, Synchronization in Chaotic Systems, *Phys. Rev. Lett.*, Vol. 64, No. 8 (February 1990), pp. 821-825.

[13] Louis M. Pecora, Thomas L. Carroll, George A. Johnson, Douglas J. Mar, and James F. Heagy, Fundamentals of synchronization in chaotic systems, concepts, and applications, *Chaos*, Vol, 7, No. 4 (1997), pp. 520-543.

[14] Arkady Pikovsky, Michael Rosemblum, and Jürgen Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge University Press, 2001.

[15] Gonzalo M. Ramírez-Avila, Jürgen Kurths, Stéphanie Depickère, and Jean-Louis Deneubourg, Modeling Fireflies Synchronization, in *A Mathematical Modeling Approach from Nonlinear Dynammics to Complex Systems*, Elbert E. N. Macau, ed., Springer, Berlin, 2018.

[16] Michael Resemblum, and Arkady Pikovsky, Synchronization: from pendulum clocks to chaotic lassers and chemical reactors, *Contemporary Physics*, **Vol. 44**, N. 5 (2003), 401-406.

[17] Dava Sobel, *Longitude: The true story of a lone genius who solved the greatest scientific problem of his time*, Fourth Estate, 2005.

[18] Steven H. Strogatz, *Nonlinear Dynamics and Chaos. With Applications to Physics, Biology, Chemistry, and Engineering*, Westview Press, 2015.

[19] Steven H. Strogatz, From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators, *Physica D*, **143** (2000), 1–20.

[20] Steven H. Strogatz, Kimball, Hall, Spontaneous Synchronization in Nature, 1997 IEEE International Frequency Control Symposium.

[21] Steven H. Strogatz, *Sync: How order emerges froma chaos in the universe, nature, and daily life*, Theia, New York, 2003.

[22] Vladimir Vlasov, Elbert E. N. Macau, and Arkady Pikovsky, Synchronization of oscillators in a Kuramoto-type model with generic coupling, *Chaos*, **24**, 023120 (2014).

[23] Arthur T. Winfree, Biological rhythms and the behavior of populations of coupled oscillators, *J. Theor. Biol.*, **16** (1967), 15-42.

[24] Dennis G. Zill, *A First Course in Differential Equations with Applictions*, PWS Publishers, Boston, 1982.