# Learning and Teaching of Group Theory through Visualization using Graphs

*Khawaja M Fahd and Faqir M Bhatti*[*]
{muhammad.fahd, fmbhatti}@riphah.edu.pk
Riphah Institute of Computing and Applied Sciences
Riphah International University, Lahore
Pakistan

**Abstract**

Group Theory is one of the difficult courses in undergraduate level. This is not only difficult for the students from learning point of view, but it is also difficult for the instructors from teaching perspective. The main difficulty in both teaching and learning is due to the abstract nature of this course. The topics are mainly based on proving theorems about abstract structures without having any visual representation. This lack of visualization limits the capability of most students to understand basic concepts of group theory.

In this paper we discuss a model that provides visualization in the form of a directed labeled graph. Using this graph it becomes easier for the instructors to teach and for the students to learn fundamental concepts of group theory. Moreover, in this paper we explain how different group axioms like closure property, identity element and the inverse can be visualized using the proposed graph representation.

**Keywords** : Group Theory, Graphs, Group Automata, Visualization, Labeled Directed Graph

## 1 Introduction

Group theory is considered as one of the difficult courses to teach at undergraduate level. This is mainly because of its abstract nature and has been reported in several research papers [1, 2, 3, 4, 5]. Moreover there are many preliminary knowledge areas and concepts [3, 4] that are required prior to starting any formal and rigorous discussion on group theory. Many researchers and mathematicians [2, 4, 5, 6] have suggested that an alternate model should be used or designed to teach group theory in a better way.

The other reason why group theory is difficult to teach and learn is that it is purely a theoretical course; the main feature of this course revolves around proving different theorems.

[*]Corresponding author: email: fmbhatti@riphah.edu.pk

Proving theorems require expertise and practice and it is not a straight forward task [5, 7].

In this paper, we describe how to transform any group into a graph. Using this graph several properties of group can be visualized with relative ease. This has been discussed in detail in the Ph.D. thesis[8]. Thus our focus is on representing closure property, identity element and inverse of an element. Before we discuss our idea, we first discuss the related work of other researchers in understanding groups visually using some other techniques.

# 2 Related Work

The concept of using different or alternative models in teaching mathematical concepts, especially group theory, is not new, it has been suggested by many researchers [2, 4, 5, 6, 9] in the past. Thus several new models [4, 9] were developed to teach group theory.

These models also emphasised on the fact that using visual techniques rather than just discussing the abstract concepts will significantly enhance the learning of students. We briefly discuss two models here. Both these models give some visual representation for the groups.

## 2.1 Groups as Graphs

In 2009, Kandasamy and Smarandache[9] wrote a book titled "Groups as Graphs". In this book they described a simple transformation from groups to graphs. The resultant graph is a simple undirected unweighted graph having no self-loops and no parallel edges. In this graph they described how different axioms can be visualized. The technique to construct this graph is simple. Every element from the group will correspond to a vertex. All the vertices will be connected to the vertex representing identity element, and there will be an edge between inverse elements.

Although different properties of groups can be visualized, but they are not intuitive. For example the binary operator of the group is not well represented. The cyclic groups do not transform into a cyclic graph. Different cosets cannot be identified easily, and hence it is not easy to understand normal subgroups in this case.

Nonetheless this transformation provided a visual representation of abstract groups.

## 2.2 Teaching Group Theory using Cayley Table

Uri Rimon presented an idea in 1984 in his paper titled "Teaching Group Theory Visually". He described how multiplication table (or Cayley table) can be used along with the rectangle property to teach various properties and theorems of the group. This gives visualization to groups to some extent, and consequently it helps in learning. But this transformation, or visualization is again not obvious. It takes some time for the students to understand and absorb this transformation, and to visualize the entries of a table as vertices.

Now we discuss automata with the help of an example before discussing the transformation of group into group automata. An automata can be visualized as a directed labeled directed graph that we use to discuss various properties of groups.

## 2.3 Deterministic Automata

A deterministic automata, D is a 4-tuple $D = (Q, \Sigma, q_0, \delta)$, where

1. $Q$ is the set of states,
2. $\Sigma$ is set of alphabets,
3. $q_0 \in \Sigma$ is the start state and
4. $\delta$ is the transition function

Here $\delta$ is defined as follows.

$\delta$: $QX\Sigma \rightarrow Q$.

Here $Q$ is the set of states and every state holds some information. $\Sigma$ is a set of alphabets. Any automata is typically formed to process a word, and the word is analyzed character by character. The automata needs to start from some special state, which is called the start state, and it is represented by $q_0$. It is the transition function $\delta$ that actually process the character and move the automata to the appropriate state.

After all the characters are processed, the last state gives some meaningful information about the input string. Following is an example to clarify the concept of an automaton.

Consider the following Deterministic Automata $D = (Q, \Sigma, q_0, \delta)$.

Let $Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}$, $\Sigma = \{a, b\}$, start state $= q_0$ and $\delta$ is the transition function that can also be described through a table, named appropriately as transition table. The transition table is given below.

| $\delta : QX\Sigma \to Q$ | a | b |
|---|---|---|
| $q_0$ | $q_1$ | $q_2$ |
| $q_1$ | $q_3$ | $q_4$ |
| $q_2$ | $q_5$ | $q_6$ |
| $q_3$ | $q_3$ | $q_4$ |
| $q_4$ | $q_5$ | $q_6$ |
| $q_5$ | $q_3$ | $q_4$ |
| $q_6$ | $q_5$ | $q_6$ |

Table 1: Transition Table for Automata

The goal of this deterministic automata is to determine the last two characters of the string. In case there are less than two characters in the string then what are those strings. Although automata itself, similar to group theory, is not easily understood. An easier way to understand the automata is through the help of state diagrams. These state diagrams are drawn like a directed labeled graph. The vertices of these graph represents states and the directed labeled edges define the transition function between states as a result of transition function with inputs as the label on the state and the label on the edge. State diagram corresponding to this automata is shown in Fig 1.

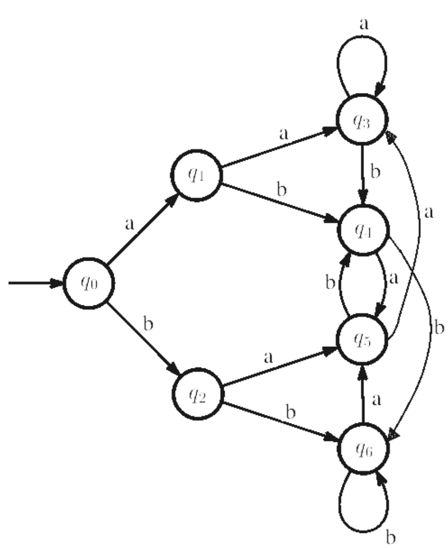


Figure 1: State Diagram for a Simple Automata

Here all the vertices are representing states and the state $q_0$ is a start state and therefore the state corresponding to $q_0$ have an arrow pointing towards it. The alphabets that are being processed are a and b. This implies that the set of states $\Sigma = a, b$. The transition function $\delta$ is shown on the edges that are labeled with either a or b.

Every state means something, e.g. if a string ends in state $q_3$ then this will mean that the last two characters of this string is aa. Similarly if a string terminates at state $q_5$, then this means that the string has ba as its last two characters.

Suppose that the string **aaba** is given to this automata. Initially the automata start at state $q_0$ which is the start state. At this state the alphabet a is processed and this changes the state of this automata to $q_1$. This information about the next state is available in the definition of $\delta$, the transition function, but it can be seen more clearly in Fig 1.

At $q_1$, the next alphabet of the string, which is again a, is processed. This leads to state $q_3$. At $q_3$ the next alphabet is b, so the automata move to state $q_4$. Now, at this state again the next character is a, so the state changes to $q_5$.

Since this was the last character of the string, the automata halts at state $q_5$. This indicates that the last two characters of the string was ba. The transition between the states can be seen in Fig 2.

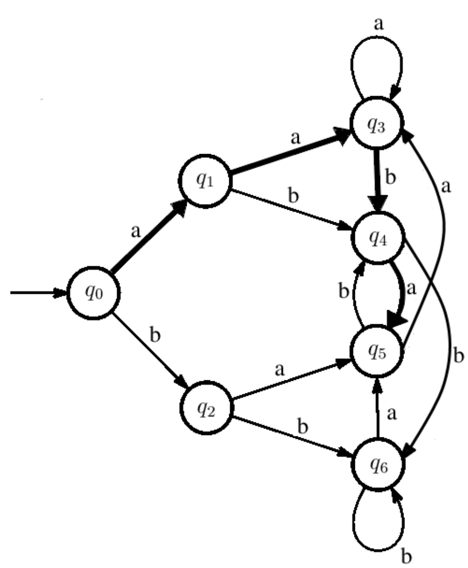


Figure 2: Transitions for Input String aaba

Here we describe how to represent a group in such an automata.

# 3 Group Automata

In groups there are only two items, a set $S$ and a binary operator $*$, whereas in automata there are four items, two sets, a function, and a starting state. Now we start mapping the group on automata.

In group automata, there are two states, namely the set of states and the set of alphabets. Both of these sets are defined to the the set of elements of the corresponding group. The transition function of automata is defined to be the binary function of the group and the identity element is marked as the start state. Consider the group of addition modulo 5 as described below.

## 3.1 Example 1: ($\mathbb{Z}_5$, addition modulo 5)

In this example the set of elements is $\{0, 1, 2, 3, 4\}$ and the result of binary operator is obtained by adding two numbers and then taking their remainder after dividing the result by 5. The transition function is shown below.

| $\delta : QX\Sigma \rightarrow Q$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 1 | 2 | 3 | 4 | 0 |
| 2 | 2 | 3 | 4 | 0 | 1 |
| 3 | 3 | 4 | 0 | 1 | 2 |
| 4 | 4 | 0 | 1 | 2 | 3 |

Table 2: Group Table for $Z_5$ under addition modulo 5

The identity element of this group is 0, so it is the start state of the group.

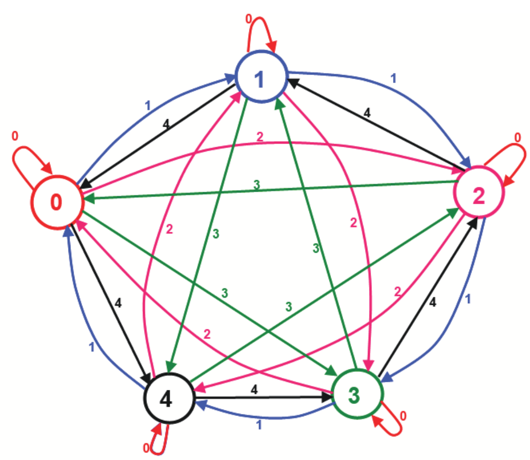

Figure 3: Representation of group ($\mathbb{Z}_5$, addition modulo 5)

The representation of this group in group automata is shown in Fig 3. On close observation of this automata it can be noticed that all the edges are directed and from every state there are many out-going edges. The number of out-going edges is the same as the cardinality of the set of elements $S$ of the group. Each edge has a different label attached to it. This is the property of deterministic automata.

In deterministic automata, for every input alphabet, there is exactly one transition from each state. This property does not hold in non-deterministic automata, as the number of transitions of an alphabet from one state can be zero, one or more than one.

We discussed the representation of group in the form of state diagram (directed labeled graph) with the help of an example. Now we discuss properties of the axioms in group automata and its state diagram.

# 4 Representation of Group Axioms in State Diagram

In the following section we describe the axioms and show how each of them can be seen easily in the state diagram. we take the group $G = (\mathbb{Z}_5$, addition modulo 5) as an example to show these axioms.

## 4.1 Closure Property in State Diagram

Closure Property states that in a group, when the binary operator operates on two elements from the set $S$, then the result of this operation must be from the same set $S$. In this automata, the result of transition function on any two elements is one of the states from the given set of states. This set of states are initially defined to be the set of elements $S$ of the group. This means that if closure property holds, then the result of every transition can be drawn in the state diagram with the transition finishing at the resultant state.

In case the closure property does not hold, and if the transition are drawn, then the transition starts from a state but it is without the destination state. In graph theory, such an edge cannot occur, as it is necessary for an edge to have both its ends attached to some vertex.

If one chooses to ignore the transitions which does not fulfill the closure property, then the automata is no longer deterministic. Recall that the deterministic automata have exactly one transition of each alphabet in all the states.

In either case, if closure property does not hold, then it is not possible to convert the group into deterministic automata.

Now let us see the identity element in a group.

## 4.2 Identity Element in State Diagram

Suppose that the identity element is $e$ and another element $a$ is from the set $S$. The identity element enforces the two laws. which are as follows:-

1. $\forall a \in S, a * e = a$. This is called the left identity.

2. $\forall a \in S, e * a = a$. This is called the right identity.

Recall that in in right identity i.e. $a * e$, the first element correspond to the state and the second element correspond to the transition. The first rule, in terms of automata, says that if we follow the transition of identity element then the state do not change after the transition. Therefore the starting and ending states remain the same. In state diagram, this is represented by self-loop. Self-loop is an edge that has both its ends on the same vertex. This is shown in Fig 4. In this group 0 is the identity element and all the states have a self-loop of 0. This is the case of right identity. Hence identity element can be visualized easily in the form of this directed labeled graph.//

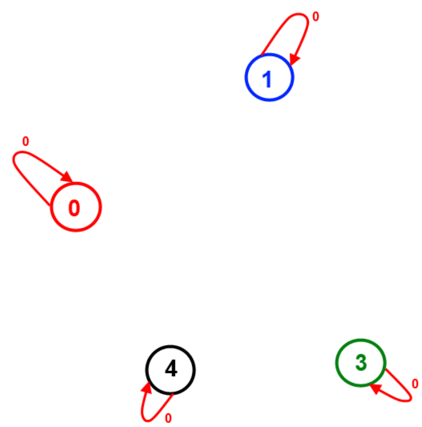


Figure 4: Representing Right Identity Element of group ($\mathbb{Z}_5$, addition modulo 5)

Now we show how to visualize the inverse of any element through state diagrams.

## 4.3 Inverse of an Element in State Diagram

An inverse of an element theoretically cancels the effect of the element under discussion. As an example if +5 adds 5 to the total then its inverse should be the number -5, as it cancels the +5 that was added to the sum. This is true for all the inverse elements in the context of groups.

Suppose the task is to find the inverse of element a. if the automata is at a state b, then on processing the element a, it should move on to the next state which is $b * a$. Now, if the next element encountered is the inverse of a, then the result should be b, as a and its inverse cancels out each other.

In state diagram, if the transition of $a$ change the state from $b$ to $c$, then its inverse change the state back to $b$ from $c$. This implies that the edges or transition of $a$ and its inverse are between the same states but in opposite directions.

Consider the number 2 in this example of the group $G = \mathbb{Z}_5$, addition modulo 5). The inverse of 2 is 3 in this example $(2 + 3)$ mod $0(mod\ 5)$, and 0 is the identity element of this group. In the state diagram, wherever the transition of 2 appears, its inverse, the number 3, is also present between the same states, but in the opposite direction, thus cancelling the effect of 2. This can be easily seen in Fig 5.

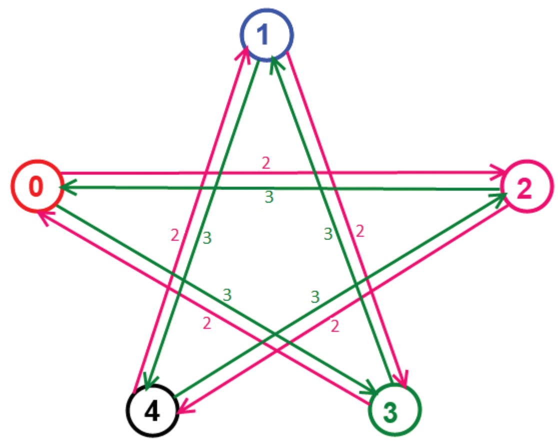


Figure 5: Representing inverse of 2 in group ($\mathbb{Z}_5$, addition modulo 5)

Hence the inverse can also be visualized very easily in state diagrams of group automata.

In this paper we discussed visual representation of groups using directed labeled graphs. We also discussed how closure property, identity element and inverse are visualized in this representation. Similarly other properties of groups like subgroups and cyclic groups can be visualized. Moreover, many theorems like Cayley's theorem and Lagrange's theorem can also be proved in this setting. These results will be published somewhere else[10] .

# 5 Conclusion

In this paper we discussed the known problem of teaching group theory which was due to the lack of any visual element. We use the concept of group automata by representing group through group automata and then visualizing it in the form of state diagram. Other visualizations suggested for representing groups lacked the representation of binary operator of the group, making the calculations very difficult. In state diagrams, the identity element and the inverse are identified without any difficulty, making the course and its axioms easy to teach and understand. Also the result of binary operator can be obtained by following successive labeled edges. We believe that if group theory is taught with the help of using group automata and the its visualization, then students will be able to learn the concepts more effectively. This new way of learning and teaching group theory will prove more beneficial in the long term as images stay in the mind longer than the equations.

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