Enhancing Conceptual Understanding through Modeling and Multiple Representations in Problem Solving

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Abstract: *In this paper, we consider the notion of modeling in mathematics through multiple representations including technology in problem solving that can be employed in the K-12 classrooms by teachers to improve student learning. The focus of the work is to explain through a real-world problem on how teachers can encourage students to solve problems using several different approaches and integrate technology in the learning process to make the mathematics meaningful.*

1. Introduction

Modeling in mathematics is fundamental to many important mathematical concepts and is often regarded as the pathway to performing well in algebra. One framework that is helpful for mathematics educators in K-12 education to help students gain appreciation for modeling are the NCTM process standards of problem solving, reasoning and proof, communication, representation and connections. Related to these process standards are the five strands of mathematical proficiency from the National Research Council report "Adding it up" which includes adaptive reasoning, strategic competence, conceptual understanding, productive disposition and procedural fluency (National Research Council, 2001). In particular the strand on *conceptual understanding* is essential for students in all grade levels in order to comprehend the mathematical concepts, perform operations and identify appropriate relationships.

There is a great need to promote the awareness of *modeling in mathematics* across the curriculum in order for students to gain a conceptual understanding of the methods employed to solve real-world problems. Whether it the solution to a multiplication problem of two digits in elementary grades or solving a real-world problem, the notion of modeling the mathematics must be introduced in teaching that helps the students to appreciate what they are solving for. Modeling of real-world problems in mathematics has different interpretations at multiple levels of learning. For example, modeling mathematics in elementary grades might simply refer to writing an equation to describe a word problem. In the middle school grades modeling may mean employing concepts from proportional reasoning and reasoning through patterns to analyze a problem in the community. In secondary school, it may refer to using principles from geometry to create an architectural design or use calculus to obtain marginal cost and revenue in economics. In the college or university, modeling typically helps students to research an important problem in their field that can impact the society. To develop students to become mathematically and conceptually proficient in modeling it is important to teach them how to apply their prior knowledge to observe and theorize from a situation; how to make suitable assumptions and formulate a problem; how to identify essential inputs and outputs to describe the problem; how to make educated approximations to simplify complex problems and perform related analysis; how to implement the approximated problem via simulations and validate against benchmark solutions and; how to validate the model by comparing with true experimental data that can help predict the evolution of the model better.

Teaching modeling in mathematics through problem solving requires depth of mathematical knowledge for teaching that not only includes understanding of general content but also having domain specific knowledge of students. Consequently, there is a need for teachers who understand the concepts so deeply that their conceptual knowledge is impacted. In this paper, we will consider a simple modeling activity involving a real-world problem that can be introduced in the classroom to enhance conceptual understanding as well as to engage students in problem solving.

2. Engaging students through modeling in mathematics

An example of a modeling activity can begin by asking the students sitting at the various tables to introduce themselves by shaking hands and at the same time asking them how many different handshakes happened at their table. It is very common to hear about misconceptions that may involve the total number of handshakes at the table being twice of what it should be or the total number being the square of the number of people. For instance, a table of three students may instantly say, since there are three people and since it takes two people to shake hands, the answer is three times two, which is six. Another response might be that the answer is three squared, which is nine. Such misconceptions introduce great teaching moments. For example, one can ask the table of three students to stand up and start shaking their hands while the rest of the class keeps a count. They will quickly realize that they are engaged in *an active learning exercise* that helps to identify that they had double counted the number of handshakes.

As reinforcement, another geometric connection can be made at this point by drawing three dots that would refer to the students at the table. If an edge between the dots would represent a handshake then it is easy to connect to the numerical answer obtained through the active learning process to the number of edges in the picture (see Figure 1). Another useful approach to help the students think about the problem would be to create an organized list to record the handshakes, which illustrates the fine difference between a permutation and a combination (see Figure 1).

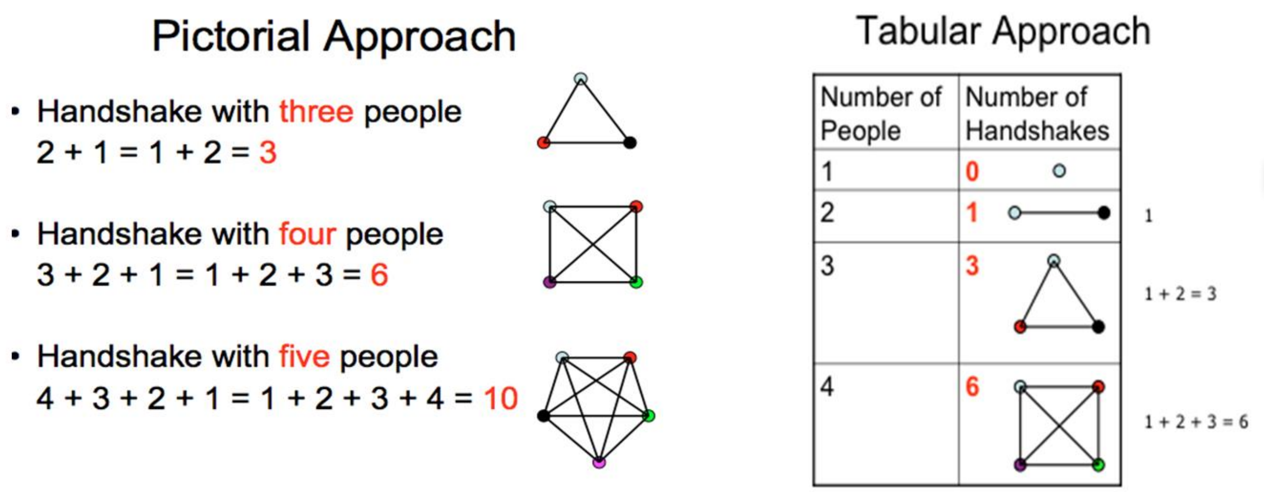


Figure 1: Pictorial Approach (Left) and Tabular Approach (Right)

The active learning exercise where the students physically shook hands, the geometric approach of drawing dots with edges indicating handshakes, as well as creating the organized list brings out another important approach, which is verbalizing the process. For instance, saying that the first person shook hands with two people and then the second person shakes only one more mathematically translates to the total number of handshakes being 2 + 1. This *verbal reasoning* not only helps the students to make a connection to the active learning, geometry, and organized list approaches, but also helps them to think ahead and start seeing a pattern when there are more people at the table. For instance, a table with four students may immediately notice that this pattern leads to 3 + 2 + 1. This then could lead to a *tabular approach* where one could record the relationship between the number of students at a table and the corresponding handshake count as a sum of natural numbers (see Figure 1 - Right).

Although we want the students to ultimately converge upon the fact that there is some hidden formula that this investigation process is going to lead them to, we want to help them discover this through their mathematical work. Naturally we want them to next *abstract from the computation* that they have worked out. For instance, we next ask how many different handshakes happen if the room had 25 people in it. The students at this point recognize the importance of being able to learn how to abstract. It is common to see them verbalize or argue that the answer is the sum of all the natural numbers 1 + 2 + 3 ….. + 24, but not know how to go about computing this. This is a good place to help them learn and discover *triangular numbers* and their connection to the Gauss formula through efficient algebraic approaches (see Figure 2).

$$\begin{aligned} S &= 1 + 2 + 3 \\ S &= 3 + 2 + 1 \\ \hline 2S &= 4 + 4 + 4 \\ &= 3 \times 4 \end{aligned}$$

$$\Rightarrow S = \frac{3 \times 4}{2} = 6$$

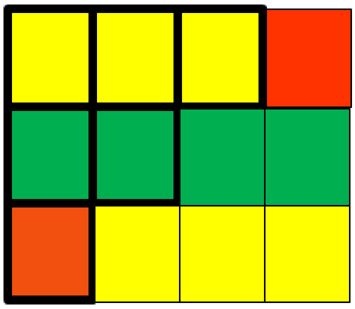
- $S = 1 + 2 + 3 + 4 + \ldots + n = \dfrac{n(n+1)}{2}$

Figure 2: Algebraic Approach

Even after it is evident by applying this formula to the original problem with twenty-five people, resulting in a solution of $(24 \times 25) \div 2$, teachers have the opportunity to talk about taking advantage of important arithmetic operations that can help compute such answers without having to rely on a calculator. For instance, talking about associative property of multiplication helps us to think of one half times (24 x 25) as (½ x 24) times 25, which yields the final answer of 12 x 25. At this point, we once again help the students to make a real world connection to what they are computing. For example, 12 x 25 could be thought of as 12 quarters, which by proportional reasoning up and down can be seen to be $3.00 or 300 cents (since 4 quarters is a dollar). Such simplicity in solving problems helps teachers to recognize the power of using real-world examples to reinforce computation. Along with simplicity, it is also important for students to learn to appreciate and use mathematical formulas to help provide greater insight into solving complex problems. Revisiting this newly discovered formula for the original example of three people yields

$1 + 2 + 3 = (3 \times 4) \div 2$. While this equation may seem obvious, it provides a natural challenge to prove that an additive representation on the left side of the equation equals a multiplicative representation on the right hand side. One may think of proving this using mathematical induction for the general case, but how does one convince students that this equation makes sense using a simple mathematical model?

Consider the illustration shown that denotes a pictorial proof. If each addend on the left can be thought of as a square, the 1 + 2 + 3 is denoted by the group of squares on the left that have boundary denoted by solid lines. Adding an exact copy to this makes a rectangular grid whose area is 3×4 and since we only wanted the sum of one of those copies, we can divide this area by 2. Such simple pictorial illustrations not only help to provide a simple proof of the mathematical statement, but also provide opportunity for students to test hypotheses by doing the mathematics using manipulatives (see Figure 3).



$$1 + 2 + 3 = \frac{3 \times 4}{2}$$

Figure 3: Concrete Approach

While the approaches that we have taken so far to solve the handshake problem may seem elementary, one can also take advantage of advanced concepts students learn in middle school and beyond involving combinations to solve this problem. For example, the handshake problem can be simply solved with some basic knowledge of combinations, as it only takes two people to shake hands and since there are *n* people to choose these two people from, the answer is the formula $\binom{n}{2}$ which may also be referred to as the number of combinations of *n* people selected two at a time. Note that in order to increase the cognitive demand of the task, the students may be asked to make predictions on the function that represents the number of handshakes from the values obtained from the tabular approach.

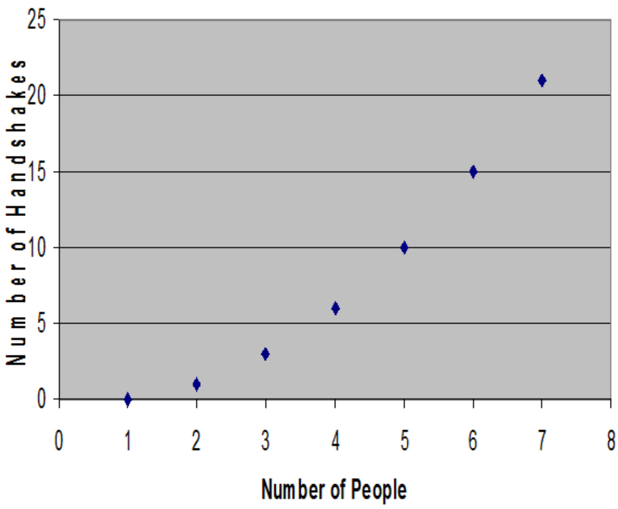


Figure 4: Graphical Approach

One approach to accomplish this may be to plot the set of ordered pairs obtained from the tabular approach on a graph to determine the nature of the function (see Figure 4). This will later lead to connections to high school standards, where one may also combine this with observations of the change in the y-values, which can then help to determine the quadratic function for H(n) that we are looking for via finite differences as shown below (see Figure 5). Having such flexibility in problem solving not only helps the students to engage in multiple representations in problem solving, but also helps the teachers to scaffold the task to accommodate cognitive demand at multiple levels.

n	H(n)	First Difference	Second Difference
1	0		
2	1	1	
3	3	2	1
4	6	3	1
5	10	4	1
6	15	5	1
7	21	6	1

$$H(n) = an^2 + bn + c$$

$$H(1) = a(1)^2 + b(1) + c = 0$$
$$H(2) = a(2)^2 + b(2) + c = 1$$
$$H(3) = a(3)^2 + b(3) + c = 3$$

$$a = \frac{1}{2},\ b = -\frac{1}{2},\ c = 0$$

Figure 5. Finite Difference Approach

Once the students are taken through the journey of doing the problem and abstracting from computation, it is important to also teach them the skill of *undoing a problem*. For example, we go on to ask the following question: "*If you were at a party where 190 different handshakes happen, how many people were there*?" At this point, the students recognize that the answer to the handshake problem is given and they must now work backwards to find out the number of people in the room. Going through this, they recognize the power of being able to work backwards to solve problems, which is a very powerful problem solving strategy. The work in Figure 6 is based on reflection from a teacher from a professional development that we offered along with a poster their team created that summarizes the discussion on the multiple solution strategies to the handshake problem where the star was used as a way to assess their representational fluency.

After organizing the data in a table it was possible to find the rule. The rule written in the form of a formula is used to find a given number to determine the handshakes. The rule consists of Symbolic representation

n(n-1)/2=h

where n is the number of people shaking hands and h the total number of handshakes. The 1 is subtracted from the number since there will be no handshake by one person. When the class consists of 20 students the rule makes it easier to calculate the number of handshakes rather than drawing pictures or filling in the numbers in a chart. Over all, the formula shortens the continuous computation of numbers. For example,

n(n-1)/2=h

20(20-1) =h

20 (19) =h

380/2 =h

When there are 20 students there are 190 handshakes.

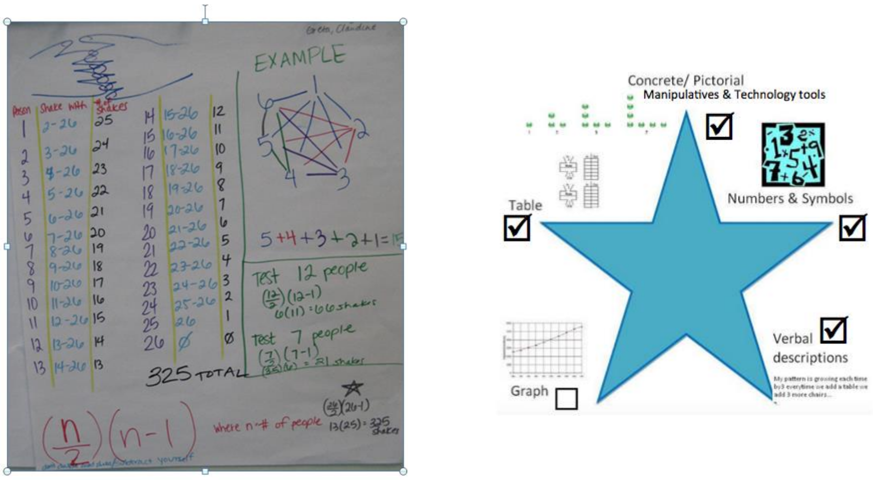


Figure 6. Assessing Multiple Representations in Problem Solving

As mentioned earlier, the choice of the problem that we want the students to grapple with should not only be mystifying but also be interesting. This means that besides learning multiple approaches to solving the handshake problem, it is important to discuss related problems where such triangular number patterns naturally emerge. For example, as a follow-up to our discussion, we introduce the following Christmas song, *The 12 days of Christmas*. The first few lines are: *On the first day of Christmas, my true love sent to me, A partridge in a pear tree. On the second day of Christmas, my true love sent to me, Two turtle doves, and a partridge in a pear tree. The song continues with three French hens* on the 3rd day, four calling birds on the 4th day, five golden rings on the 5th, and so on, up to the 12th day. As the teachers read through this problem, they seem to immediately notice a connection to the triangular number patterns. These connections can also be seen in the Pascal's triangle (see Figure 7).

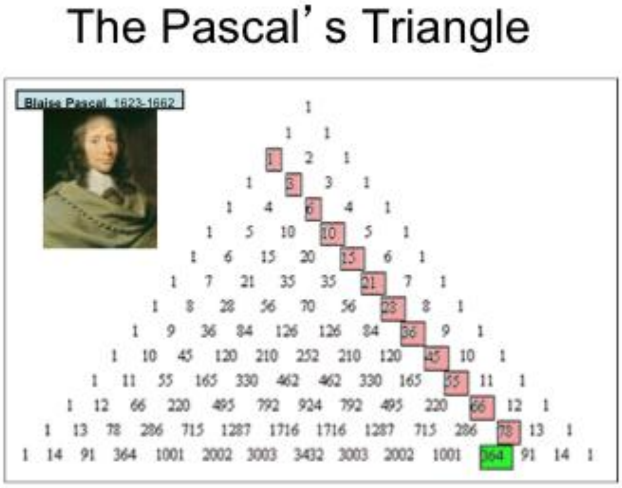


Figure 7. Connecting to Pascal's Triangle

As shown here, the handshake problem is not only a great example of a task that engages the participants whether they are students or teachers in rich problem solving, but it also helps them to expand their *algebraic habits of minds*, namely, doing and undoing, building patterns from representations, and abstracting from computation (Driscoll, 1999). For educators, it also provides the opportunity to learn about *five practices* (Smith & Stein, 2011) including:

- **Anticipating**: The teacher can anticipate the type of strategies that students might employ, along with misconceptions that can be turned into opportunities.
- **Monitoring**: It is important for teachers to provide their students with the opportunity to engage in problem solving, to monitor students to identify the various strategies that are discussed, and to understand how students think as they solve the problem.
- **Selecting**: This step requires teachers to select the order in which the strategies will be shared so everyone benefits from learning all strategies.
- **Sequencing**: This allows the teacher to sequence the approaches being presented, whether that be from simple to complex or in a sequence that supports the lesson agenda. This might mean starting with the acting out strategy, moving to looking for patterns, and concluding with finding an algebraic formula, or it might mean starting with a common misconception so that students can repair their understanding.
- **Connecting**: As a last step, the teacher has the opportunity to facilitate a discussion that helps to connect all of the various strategies displayed and shared during the discussion. This not only helps students to see connections within and across the strategies, but also connections

to related problems. For example, this problem could be changed to accommodate the distribution of valentine cards in an elementary classroom or to find the number of diagonals of a n-sided regular polygon.

One way to deepen one's conceptual understanding is to make connections among important conceptual models within and between mathematics topics by unpacking the vertical learning progressions. Learning progression describes students' reasoning as it becomes more sophisticated, and as "…hypothesized descriptions of the successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time" (Corcoran, Mosher & Rogat, 2009, p. 37). In the mathematics education, the notion of learning trajectories has been important in trying to understand the progression of mathematical concepts and how students' learning progresses and becomes more advanced and sophisticated. This is illustrated below for the handshake problem.

Table 1: Mapping the learning progression for the Handshake Problem

<table>
<tr><td colspan="3">Learning progression for the Handshake Problem
<u>Benchmark Problem:</u> At a neighborhood party there were a total of 25 guests. All 25 guests shook hands to introduce themselves. How many distinct handshakes happened at this party? Show the different ways you can represent your thinking.</td></tr>
<tr><td>Grades 3-4</td><td>Grades 5-6</td><td>Grades 7-8</td></tr>
<tr><td>Use the four operations with whole numbers to solve problems.</td><td>Analyze patterns and relationships.</td><td>Solve real-life and mathematical problems using numerical and algebraic expressions and equations.</td></tr>
<tr><td><u>Related problems:</u>
Every student in the second grade classroom exchanged a valentine card with each other. If there were 30 students, how many valentine cards were exchanged?

How many handshakes do you have in at your table groups of 3, 4, and 5 friends?</td><td><u>Related problems:</u>
If everyone in your class (30 students) shakes hands with everyone else, how many handshakes would there be?
Use words, pictures, numbers, and express the pattern or rule for the problem.</td><td><u>Related problems:</u>
There are nine justices on the Supreme Court. How many handshakes occur if each of them shakes hands with every other justice exactly once?

At a birthday party, each guest shakes hands with every guest. If 190 different handshakes take place, how many guests were at the party?</td></tr>
<tr><td colspan="3">Strategies, representations, and misconceptions</td></tr>
<tr><td colspan="3">To introduce the Handshake Problem across the vertical grades, teachers in grades 3-4 might begin with the related problem of the Valentine Exchange. This is a real-world scenario that most students can relate to regardless of the holiday. It is a problem about each student (30) giving something/item to each person (29) in the class. The problem begs for a multiplication problem structure of 30x29 because you do not have to give yourself a card or something. The simpler problem is of course to start with a smaller number of guests at the party to see if there is a pattern. As shown in the various examples above from Figures 4 to 12, there are a variety of strategies and approaches to this rich problem. The common misconception for the handshake problem is sometimes counting oneself or double counting which is allowed in the Valentine Exchange because there is a give and take of the cards, where as in the handshake the give and take of the shake is counted as one.</td></tr>
</table>

3. Integrating technology in modeling of mathematics

Technology should be used to "amplify the mathematics" bring out the mathematics concepts to the forefront using specific affordances. Specific opportunities that technology rich mathematics environments afford teachers and students are the abilities to:

- build representational fluency by making connections among multiple representations;
- experiment and test out conjectures which efficiently develop reasoning and proof; and
- facilitate the communication of mathematical ideas through problem solving.

In the handshake applet below, we see how one can employ technology to enhance the understanding of the solution approaches to the handshake problem. We will now describe an applet that has been custom created that can be employed as a tool to discover the associated pattern. A technology tool is also helpful to generalize a pattern observed for larger numbers.

Let us consider seven members shaking hands with each other. The technology tool allows students to enter the number seven for the "number of people" and as they proceed to solve for the handshakes, the tool can help prompt the steps in the process. For instance, the illustration below shows that the first person has performed six handshakes (denoted by yellow bold lines), and the last one by red bold line. Note that besides a visual confirmation of the handshakes on the left, the number of handshakes is recorded as circles in the center and also as a part of the lower row in the 6 x 7 rectangle shown in the right.

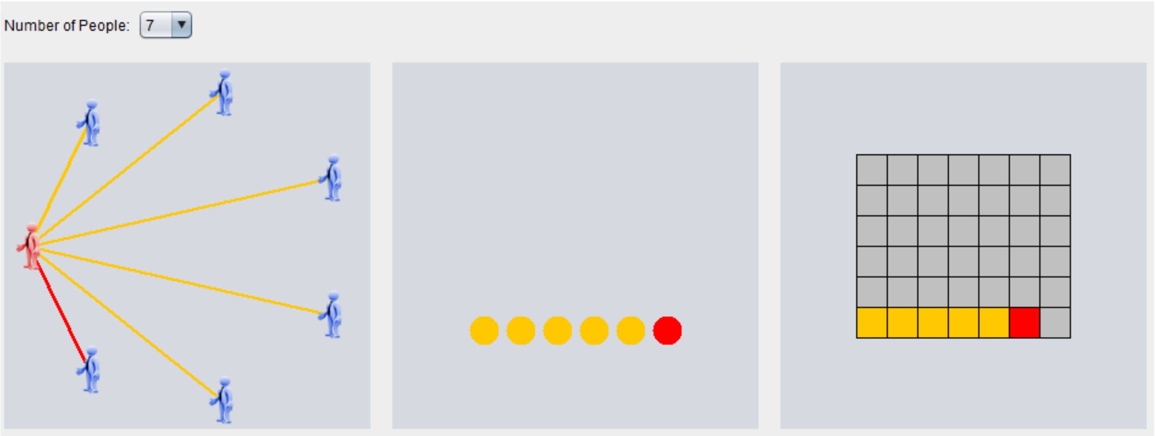


Figure 8. Using Technology to Represent the Handshake problem

As the student continues to navigate through the process, one can obtain intermediate steps like the one shown below that helps the student to make a connection between the representations and guess a growing pattern. Eventually, as the students go through the process, they would observe not only the connections but also build a representation that makes sense algebraically. The final step is shown next.

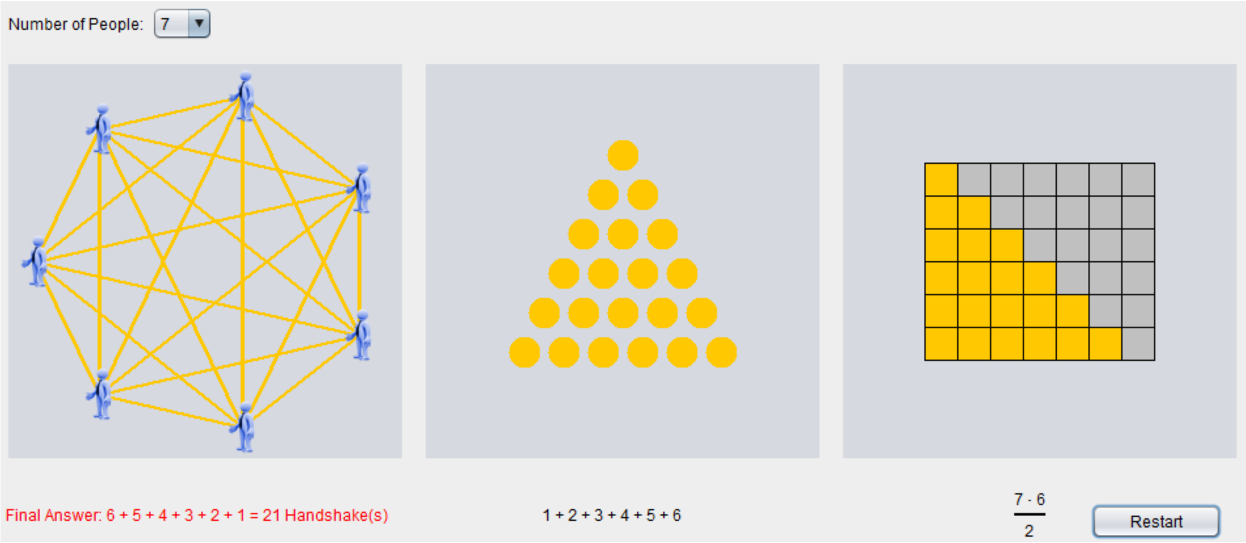


Figure 9. Final screen shot of the Handshake problem

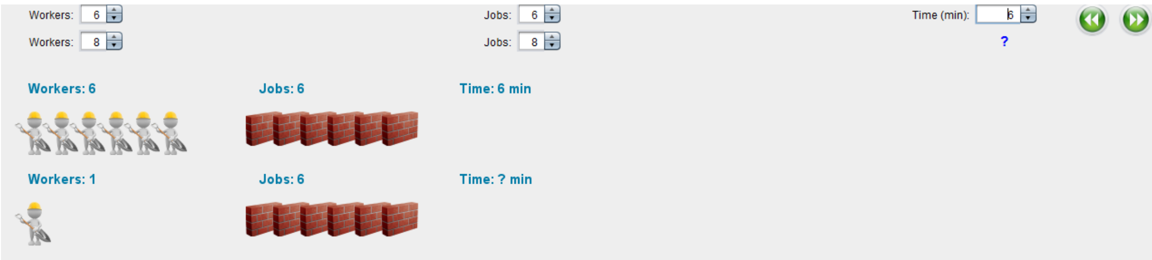
So often an approach to a mathematical problem is formulaic, totally plug and play, and without much attention given to concepts. Technology can play an important role in providing an opportunity to make the mathematics meaningful.

After all, adding ½ and ½ and getting ¼ does not make any sense if one would only take a few seconds to think about it. The emphasis of thinking, really thinking, about a problem before rushing to get a solution is often major issue for students. For example, another follow-up problem we have considered is: *If six rabbits can dig six burrows in six minutes then assuming each rabbit works at the same rate, how much time will it take eight rabbits to dig eight burrows?* When the students were not allowed too much time to think and were asked to tell what they thought the answer was, majority of them admitted that the answer they thought was eight minutes.

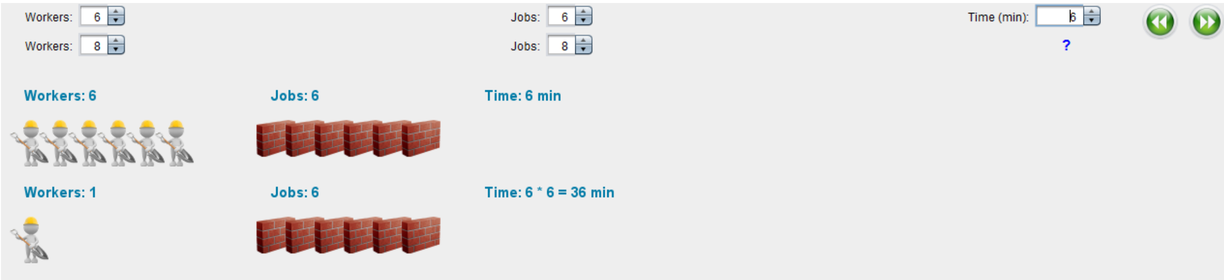
It is helpful at that point to employ a technology tool that was developed in-house as a part of this project to clarify their misconception using a pictorial demonstration of reasoning up and reasoning down strategy. First one can illustrate the given input in the problem by the following representation.

Note that the students can then use the forward green button step by step to see the following:

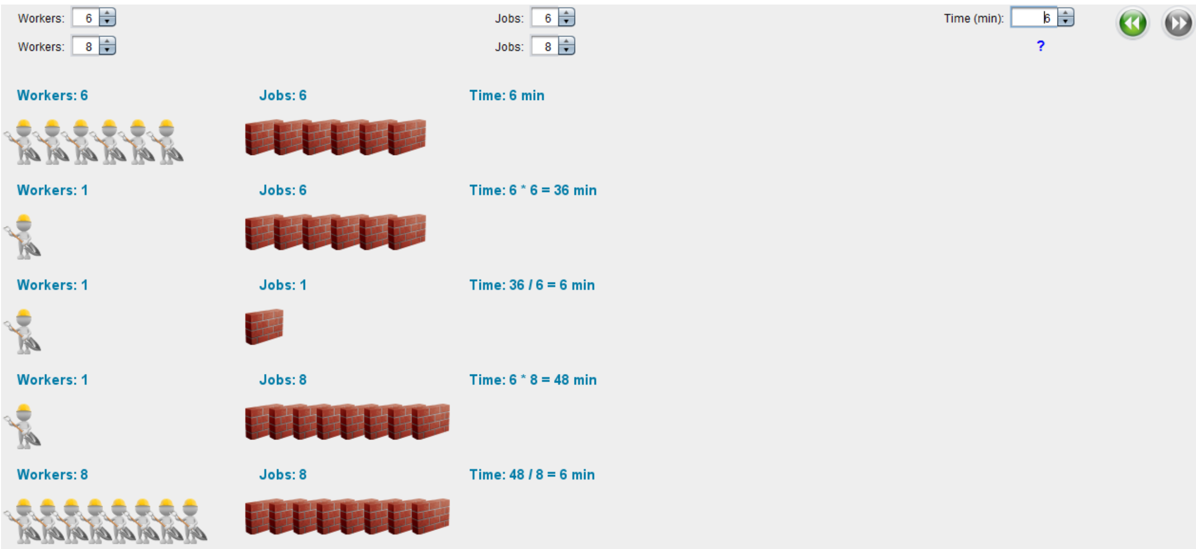
The technology tool also prompts the user to say an answer before they proceed to the next step which is a great motivation to try continuing with the process. So the next step yields:



At this point the student can guess the answer or simply go forward which then yields:



At this step, the student can pause and describe the reasoning up process where the number of jobs have been kept fixed by the workers have reduced by a factor of 6 which increases the time by a factor of 6 as well. This is clearly illustrated by the technology. This process continues until the final screen yields:



Not only is this methodical process helpful to understand the problem solving process but also provides the opportunity for the students to learn conceptually.

3. Discussion and Concluding Remarks

Coupling the two ideas of critical thinking and multiple valid approaches, teachers will be able to recognize that they need to take a closer look at their students' work. When the handshake problem was presented to teachers at several teacher workshops, we saw many instances of teachers asking others in their group to explain their reasoning. This helped the teachers to understand their colleagues' reasoning and also helped the speaker clarify her/his own thoughts through explanation. Then, the table discussed whether or not the logic was valid and if they wanted to use that approach. Teachers benefitted from these collaborations in several distinct ways. First, they began to see that real problems involving problem solving are not simply plug-and-play exercises; they are multi-layered challenges, which require analysis, sound reasoning, and understanding of the relationships among quantities. Second, they recognized the profound importance of conceptual understanding as a baseline for strategizing approaches to problem solving. And, third, they gained an acute appreciation for the frustration of their students who apply incorrect procedures and cannot understand why their answers are incorrect. Several teachers mirrored that idea in their writings. Lastly, another teacher reflected, "*I am also starting to think differently about analyzing student work. When problems have the opportunity of yielding a variety of correct answers, it is important to consider what the student is doing and what math they can do and understand.* "

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