Counterexamples in Mathematics Education: Why, Where, and How? – Software aspect.

Vladimir Nodelman
nodelman@hit.ac.il

Department of Computer Sciences,
Holon Institute of Technology
52 Golomb St., Holon, 58102, ISRAEL

Abstract

For a mathematician, constructing counterexample is a common way to disproof mathematical conjectures. Counterexamples also help her to establish the constraints imposed on theorems.

This report shows that in mathematics education counterexamples can and should be intensively applied at the earliest stages - in the study of concepts, long before the first acquaintance with the theorems and proofs. Herewith, the use of software becomes an organic element of the learning process.

Studies of concepts

1. Propositional definitions

1.1. From the very early moments of their life children meet mathematical objects and acquire skills of their recognition. Pedagogical psychology provides the *principle of variation of non-essential features* (the essential features are kept invariable) [2],[3].

Which of the non-essential attributes should be varied, and which ones can be omitted? In fact, unlike the finite number of essential features, there are "infinitely" many nonessential ones.

The development of software denoted to facilitate the concept formation cannot implement such an unconstructive pedagogical approach.

This principle has been convincingly criticized in psychological literature [4], and is applicable in situations of uncontrolled or poorly managed assimilation of concepts only.

1.2. We will treat this principle as a secondary way of learning, and consider instead an ***active recognition, based on the definition of mathematical concept*.**

In fact, each definition consists of *contextual* and *logical* parts. While the contextual side (features of the concept) is specific, the logical structure of definition is extremely stable and very often coincides in the definitions of various notions. Thus, the majority of concepts studied in elementary, secondary, and high school have a conjuctive definition with two or three attributes.

There are 2^n combination of values in the truth table, which describes the logic of definition with n attributes. Each case presents a ***type of task*** - *objects with definite combination of truth values of the* ***essential-only features***. Any conceivable recognition object necessarily represents and only one of these types.

Fig.1 shows the truth table and the illustrating case of concept "*right angle*" with conjunctive definition[1]. It helps to grasp the following general conclusions:

- **Counterexamples are essential types of tasks to be studied**. From been an exotic educational tool, they becoming a vital learning element.
- There is a clear mechanism of task construction, which, been based on the combination of logical values, takes into account the concrete contents of the studied concept's features.
- There is a place to use the principle of non-essential features variation to provide numerous objects, representing the same type of task.

The selection of tasks no longer depends on the teacher's tastes.

In case of difficulties in recognizing the object students are offered another object of the same type in a pedagogically justified manner.

1	2	$\in, \notin$
f	f	f
f	t	f
t	f	f
t	t	t

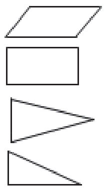

Figure 1. <u>*Right triangle*</u>: (a) triangle, **AND** (b) has a right angle

*We got a **constructive way of software development** with a very limited amount of concrete types to support. Fortunately, this types are universal and cover a wide spectrum of studies of mathematical concepts.*

1.3. The *activity approach to formation of concepts* [4] mainly asumes organization of a <u>controlled process</u> *of recognizing the belonging of an object to the studied concept*. Naturally, this process is established by <u>*an algorithm, based on the logic of the definition*</u>. Instead of guessing students make their conclusion on the basis of a coherent check of the presence of attributes.

Surprisingly, *following this approach we obtain an additional type savings* (Fig.2).

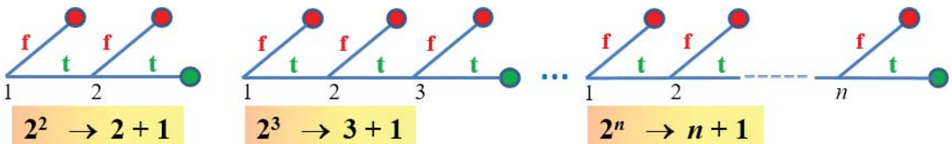

Figure 2. Green dots - examples, Red dots - counterexamples.

Motivation of conclusion, founded on algorithm, draws students' attention to the specifics of the logical structure of the definition, and stimulates their logical thinking.

1.4. <u>Human logic is not binary</u>. We have a wide "***Unknown***" between "*Yes*" and "*No*". Logic becomes ternary with *true, unknown,* and *false* possible values of trueness. The

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[1]We confine ourselves to considering only conjunctive definitions.

formal truth table and recognition algorithm are shown in Fig.3. As before, each dot presents a definite type of task-object for recognition.

1	2	∈, ∉
f	f	f
f	?	f
f	t	f
?	f	f
?	?	?
?	t	?
t	f	f
t	?	?
t	t	t

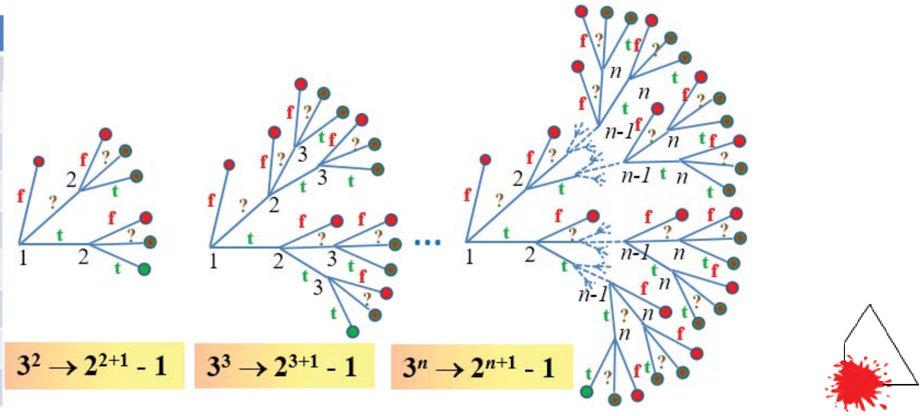

Figure 3. Green dots - *examples*, Red & green-red ("*unknown*") dots - *counterexamples*.

The right-side image in Fig.3 shows counterexample-object with value of "unknown" (blur) for some of features: the negative result of checking the first feature ("triangle") immediately leads to the concrete conclusion, that the object is not a right-angled triangle, although it is impossible to find out whether it has a right angle or not. It just does not matter. This activity models the path to left red dot in Fig.3.

1.5. This approach is implemented in our Microsoft Excel application (Fig.4).

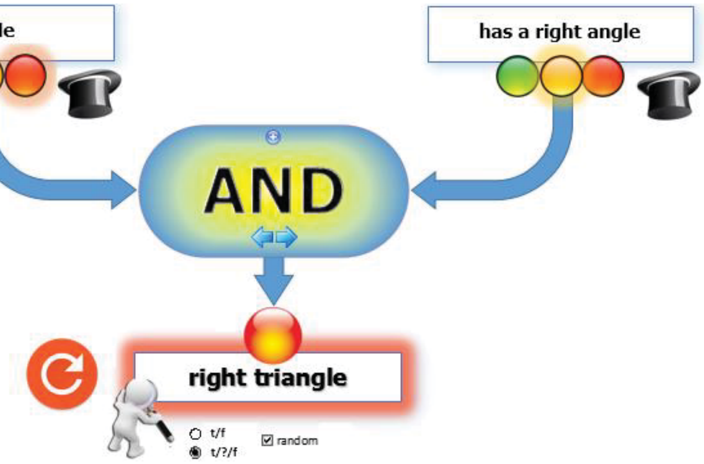

Figure 4

The following fragment presents one of the possible ways of application of this tool:

1. Consider definition of the concept. Identify *term, properties, type of properties' link* (logical operator: AND, OR).
2. Select proper logical operator by means of arrows, and amount of properties.
3. Input features and term descriptions in text boxes.
4. Disable automatic conclusion mechanism - press the term's button to make it blue and to remove the colored halo around the term's text box.

5. Set all triples of input buttons to the “*unknown*” state (Press the “*reset*” button).

Task #...:[2] Is the image with blot in Fig.3 a right triangle? (*Blot covers a part of figure*).

Solution:

- Check the first property (*Is it a* ____________?).
  It__________. So, press the ___________________________________ button.
  passes, fails *green(“yes”), yellow(“unknown”), red(“no”)*
- *Can you conclude whether the object is an example of right angle or not*? _____.
  yes, no

  If your answer is “*yes*”, check it as follows:

  a) Press the blue button of automatic answering. It becomes red.
  b) Try pressing each one of three unput buttons below the text of second feature and observe the color around the text of term.
     - *Is the color remains the same*? _____.
       yes, no
     - *What does it mean: the same or different halo colors around the term's box of the need to continue checking*?__________________________________.

     Actually, the color remains _______________.
     green, yellow, red

     So, we stop and conclude that the object is ___________ a *right triangle.*
     not, definitely

The **system of tasks types**, which examples presented in Fig.3, **is optimal: *necessary and sufficient*.** In fact, the absence of study of the path guiding to some dot leads to knowledge gaps. On the other hand, all possible logical routes of the recognition process are modeled by the tree. Surprisingly, ***the vast majority of tasks are counterexamples***.

> *At a certain stage, students begin to disturb the sequence of checking the presence of attributes, starting with an absent feature. This means that the recognition algorithm is learned, "curled up" and the use of software should be stopped.*

1.6. While the recognition of belonging to the concept is a universally recognized mental operation, there is another, equally important ***mental action of drawing conclusions from the fact of object's belonging or not belonging to the concept***. Really, the definition of a concept can be seen as an *equivalence* of the *term* and the *logical function of its characteristics*. This equivalence means that:

- ✓ On the basis of the value of the trueness/falseness of the logical function, a conclusion is made about the applicability/inapplicability of the term. - A*ct of concept recognition*.
- ✓ Based on the belonging to the term, a conclusion is made about the trueness of the logical function and the linked features. - *Act of drawing conclusions from the fact of belonging/not belonging to the concept*.

[2] We show here only one example. In fact, students are offered several exercises of each type. In the first of them hints are maximal and texts are very concrete: "Is the figure a triangle?" etc. In the following examples of the solution, the terminology becomes more general: "Is there a first property?" ... The prompt level becomes reduced.

The act of drawing conclusions, like the act of recognition, is also based on examples and counterexamples, more precisely: on "*imaginary*" *examples* and *counterexamples*.
Fig.5 presents table of task types in cases of concepts with conjunctive definitions with two properties. Characters *t, ?, f (true, unknown, false)* denote the "*given*" in the task. Sign ■ denote features, whose trueness should be discovered. Thus, the following example presents the outlined 6th raw in table:

- A figure is not a right triangle, although it contains a right angle. *What can you say about it*? (In case of difficulty: *Is it a triangle*?)

Microsoft Excel helps to make these tasks and the process of their solution more sensible and to "materialize" mental activity, thereby facilitating the process of interiorisation.
By pressing the proper *hat* image one hides (next press - unhides) the feature trueness input buttons (Fig.5 right image) shows elements of the model). All these buttons remain accessible. Pressing them is simulated by typing the corresponding number. Thus, the numbers 1-3 correspond to the left triple. One can sequentially type 1↵, 2↵, 3↵ and emphasize this way the difference of halo color around the term's text box – the trueness of belonging to concept.

- *Button of what color is highlighted in the hidden group?*

Second press on the *hat* image unhides the "secret".

1	2	∈, ∉
■	■	t
■	■	f
f	■	f
■	f	f
t	■	f
■	t	?
?	■	f
■	?	f
t	■	?
■	t	?

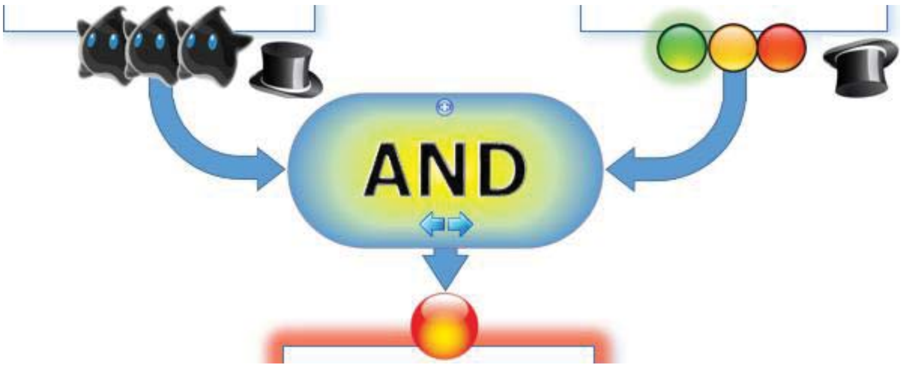

Figure 5

Summarizing, it should be noted that the process of finding the proof of theorems and the proof itself consists of chains of actions of recognition and drawing conclusions. The proposed methodology for the study of concepts forms and develops the necessary skills, often long before the students' first encounters with theorems and their proofs thereby removing the stress of novelty and facilitating understanding.

2. Predicative definition with quantificators

In high school students meet concepts with quantificators in definition.
Consider the notion of *limit of a function* f: $R \to R$ to illustrate the methods of teaching and learning such concepts in terms of using counterexamples and software. In general,

the types of student's activities and their reasons in studies of this concept do not depend on the specifics of functions (whether they are real or complex, uni- or multivariate, etc.), whereas, the supporting models are dramatically different.

2.1. Intuitive understanding

A popular intuitive definition of limit of a function looks as follows:

D1 | The *limit of f*(*x*), *as x approaches a*, *equals L* if we can make the values of $f(x)$ arbitrarily close to (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Answering the question: "*When the limit of f(x), as x approaches a doesn't equals L*?" students construct negation of D1 by naturally changing "*can*" to "*cannot*" there.

In terms of teaching, this dichotomy is not very fruitful. However, the assortment of counterexamples uncovers considering the prospect of the subsequent study of another fundamental mathematical concept - continuity, based on the notion of a limit. The discontinuity, as negation to continuity, provides us with a rich and concrete typology of counterexamples.

In general, the need for propaedeutic of subsequent content - one of the teaching rules - often delivers fresh pedagogical ideas, including those relating to examples and counterexamples of currently studied concepts.

2.1.1. The model **M1** of intuitive definition uses graph of function $f(x)$, and based on understanding of "*closeness*" to value a as belongings to some small interval with center at a. Fig.6 shows the model[3], where $f_1(x)$ is light blue line $y = 0.2(x^3-2x^2)/(x-2)$.

Model includes two variables x_o and *delta*, which allow to express the condition of closeness of x to x_o in form $|x - x_o| < \delta$. We express D1 by *domain* and *range* of function $f_2(x)$, defined as "$y = f_1(x)$, if $|x\text{-}x_0|<=delta$"[4] and graphed by the dark green subcurve.

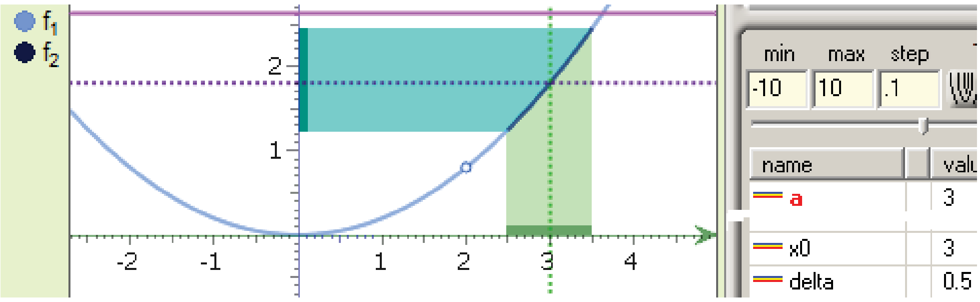

Figure 6

Fortunately, we can use the *VisuMatica*'s possibility to display *domain* and *range* of function (intervals on the coordinate axes). The optional accompany curvilinear trapezoids illustrate their source. Model also includes two dotted lines $x = x_o$, $y = f_1(x_o)$, and a magenta line $y = a$ - a spare one (until the right time it can be made invisible).

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[3] Starting from here all the models are constructed by the author's software *VisuMatica*.

[4] The inequality should be strict, but we use a legal unstrict version to eliminate the hollow circled at the ends.

Exploration tasks (Types)

Change the value of parameter *delta* (use the scroll bar). Make it as small as possible. *What happens with graph of function $f_2(x)$? …with its domain* and *range*? *How do you read it from the chart*? *What is the value of limit at* $x_o = 3$?

1. Repeat step 1 in case of x_o = **2**, -1,...
2. *Is the range symmetric with respect to the limit*?
3. *Is the biggest distance of points in range from the limit always less or equal to delta*?
   To check it - select and redefine $f_1(x)$ with a suitable expression.
4. Consider the following cases:
   a) Select function f1(x) and redefine it to $y=x+1/(x-1)$.
   - Set $x_0 = 0.5$. *What limit has* f1(x) *when x approaches* 0.5, *if any* (Fig.7 a,b)?
   - Set $x_0 = 1$. *What limit has* f1(x) *when x approaches* 1, *if any* (Fig.7 c,d)?

   What is the principal difference between these cases of $x_0 = 0.5$ *and* $x_0 = 1$?
   How one can recognize this difference just by the only graph of initial function $f_1(x)$?
   Guess 5 more values of x_0, where $f_1(x)$ has a limit. Find these limits without *VisuMatica*. *Can you guess any new* x_0, *where the limit does not exist*?

   a) Select function f1(x) and redefine it to $y=x/2+\mathrm{int}(x)$.
   - Set xo = 1.5. *What limit has* f1(x) *when x approaches* 1.5, *if any* (Fig.8 a, b)?
   - Set xo = 2.0. *What limit has* f1(x) *when x approaches* 2.0, *if any* (Fig.8 b)?

   Change the model to support exploration of one side limits (redefine $f_2(x)$ properly).

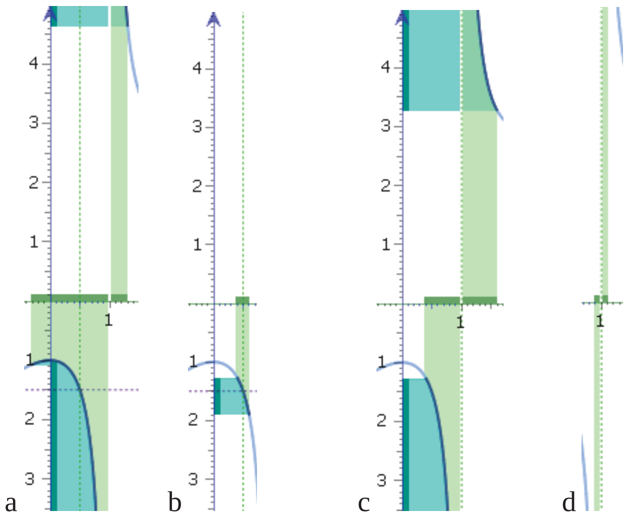

Figure 7. a) δ = 0.8, b) δ = 0.1, c) δ = 0.6, d) δ = 0.1

   b) Select function f1(x) and redefine it to $y=(x^4-x^2)/(x^2-1)$, $x_o=1$.
   What now is similar and what is different from the previous cases?
   Why the horizontal dotted line disappeared?
   Explain the meaning of the punctured circle on the curve and the two white gaps (Fig.9, a).
   "Play" with parameter *delta*. *Has function* f1(x) *a limit at* 1.0?
   If your answer is "Yes" then
   - *What is the limit's value*?
   - *Is the limit equal to* f1(x0)?
   - Change the value of parameter *a* to locate the magenta line $y = a$ in the expected position of the disappeared horizontal line y = f1(xo) (Fig.9 b).

else

Read the definition D1 once more. Pay attention to its ending condition. Have you changed your mind? *What is the limit's value*?

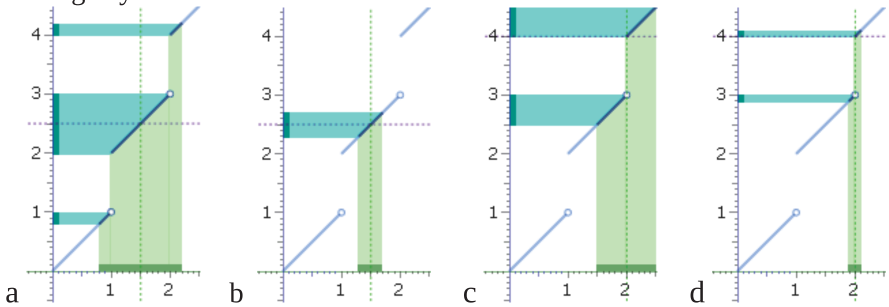

Figure 8. a) δ = 0.7, b) δ = 0.2, c) δ = 0.5, d) δ = 0.1

c) Select function f1(x) and redefine it to *y*=2 *if x*=1 *else* (*x*^4-*x*^2)/(*x*^2-1), x_o=1.
What now is similar and what is different from the previous task (c)?
Why does the horizontal dotted line appear again??
"Play" with parameter *delta*. *Has function* f1(x) *a limit at* 1.0?
What is the limit's value?
Is the limit equal to f1(x0)?

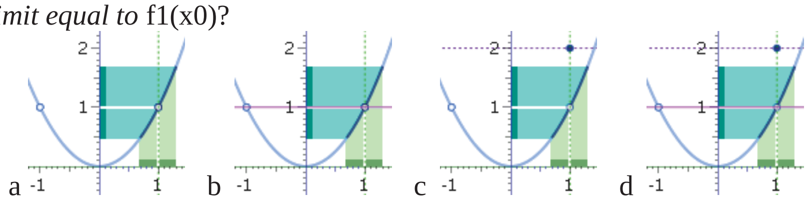

Figure 9

Although only the types a) and b) exemplify counterexamples, two other types are extremely useful in order to adequately formulate the notion of the limit of a function.

3.1 Precise definitions

3.2.1 Definition *by Cauchy*

The understanding of "*closeness*" of *x* to x_o, expressed by inequality $|x - x_o| < \delta$ with small enough δ, was sufficient in modeling of intuitive definition of the concept of limit. But, actually, we were looking for <u>two</u> "*closenesses*", related to function *f*(*x*): closeness of <u>argument *x* to x_o</u>, and closeness of the <u>correspondent *f*(*x*) to the value of limit *L*</u>.

This argumentation in class finalizes by introduction of the following **"ε – δ"** definition:

D2 | The number *L* is called the *limit* of function *f* (*x*) as $x \to a$ if and only if, for every $\varepsilon > 0$ there exists such $\delta > 0$ that $|f(x) - L| < \varepsilon$, whenever $0 < |x - a| < \delta$.

Such logically complicated construction students meet for the first time. Clarification of definition by means of the following formal logical notation, which includes both universal and existence quantifiers and implication, stresses students even more.

D3 | The number *L* is called the *limit* of function *f* (*x*) as $x \to a$ if and only if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in Dom)(0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

In case of counterexample this statement fails.

It can be useful to explain the general role of negation of expressions with quantifiers:

a) Change the quantifiers (∀ to ∃ and ∃ to ∀).

b) Negate the predicate expressions.

All together:

- $\lim_{x \to a} f(x) \neq L \Leftrightarrow (\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in Dom)(0 < | x - a | < \delta \wedge | f(x) - L | \geq \varepsilon)$

In human:

The number L is *not the limit* of function $f(x)$ as $x \to a$ if exists such $\varepsilon > 0$ that for each $\delta > 0$, $| f(x) - L| \geq \varepsilon$, whenever $0 < |x - a| < \delta$.

Analysis of **"ε – δ"** definition D3 with students brings up the following "algorithm" of limit recognition:

1. Select ***a*** and an expected limit ***L***.
2. Choose some $\boldsymbol{\varepsilon > 0}$.
3. Find such $\boldsymbol{\delta > 0}$, that for all $\mathbf{0 < |x-a| < \delta}$ the condition $\mathbf{|f(x) - L| < \varepsilon}$ remains correct.
4. If a proper $\boldsymbol{\delta}$ in step 3 was found

   choose some smaller value of $\boldsymbol{\varepsilon}$, say, its half and go to step 3[5],

   else

   L is not the limit of function $f(x)$ as $x \to a$.

Fig.10 shows model **M2** for studies of **"ε – δ"** definition. It includes:

1. Variables *a*, *epsilon*, *delta*, and *l* (at the beginning defined as $l = f_1(a)$).
2. Graph of function $y=x^3/50$ in the role of $f(x)$ - light blue curve f_1.

<u>*Epsilon neighborhood* related objects:</u> f_2, defined as y=f1(x), if |f1(x) − l| < ε - dark magenta subcurve with its *domain*-light blue and *range*-light green, and light yellow bar $|f(x) - L| < \varepsilon$.
<u>*Delta neighborhood* related objects</u> - the M1 model from Fig.6. Here f_3 is defined as $y = f_1(x)$, if $|x\text{-}x_0|<=$*delta* and parameter *a* plays the role of x_0.

Student interact with model by changing values of its four parameters. As a starting point, model presents the case of *l*, defined by expression $f_1(a)$ (Fig.10). So, we have only three independent parameters *a*, *epsilon*, and *delta*. Such "limitation" of $l = f_1(a)$ is very helpful at the beginning of student's activities: it eases explorations, and it provides an important propaedeutics of the concept of *continuity*.

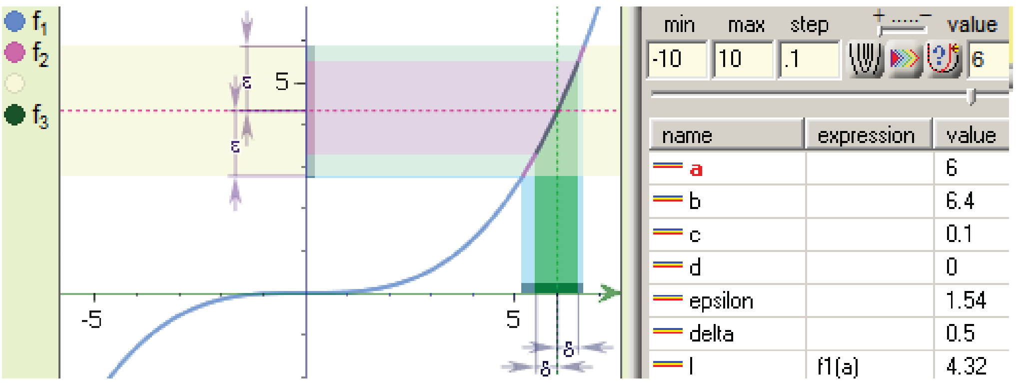

Figure 10

The work with this model on examples and counterexamples of the limit is based on the formulated algorithm and actively refers to model elements as analogs of elements of the

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[5] Pay students' attention to the infinite amount of such returns if ***l*** is really the limit.

definition[6]. In result of routine of these activities it becomes clear that we can ***facilitate*** our model. It will be sufficient to leave it only with the yellow ε-bar, graphs of the initial function f_1 and the current f_3, defined on the **δ-*interval***. We don't need any region (domains, ranges and curvilinear trapezoids). Our new model **M3**(Fig.11) becomes easier to manipulate and explore without loose of its educational potential[7]. Of course, this conclusion and transition to the "light" model is **justly** *only after detailed exploration with students of the* $\varepsilon - \delta$ *definition of limit by means of model M2.*

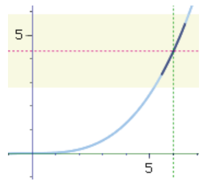

Figure 11

After solving examples and counterexamples, e.g. of mentioned ***types*** with these models let us construct a *provocative* counterexample.
We start with a very simple function "y=(x-2)^2-1" and step-by step transform it up to "y=sqrt(|(x-2)^2-1|-1)" (Fig.12)[8].

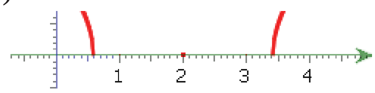

Figure 12

The received graph includes a red isolated point (2, 0). *Is* $\lim_{x\to 2} y = \sqrt{|(x-2)^2+1|-1} = 0$?

Really, for each $\varepsilon > 0$ and any $0<\delta<\sqrt{2}$ we have no x in domain of $f(x)$. The δ-*neighborhood* is empty - there is nothing to check on satisfaction of inequality $|f(x)-l| < \varepsilon$. To take into account similar situations it is accepted to include requirements to the domain in definition of the concept of limit of a function. The simplest way, is to require belonging of the whole punctured opened δ–interval to the function's domain Dom(f). But *arbitrarily closeness* is presented better by the notion of *accumulation point*:

> Point a is called an *accumulation point* of a set S if for every $\varepsilon > 0$ there exists a point $x \in$ S such that $0 < |x - a| < \varepsilon$.

If a is an accumulation point of the domain, then there are infinitely many other neighboring points in domain.

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[6] In these terms, taking into account the definition of a counterexample, l is not a limit if has been found such value of ε, that for any arbitrarily small δ we see magenta elements outside the yellow bar.

[7] In these terms, taking into account the definition of a counterexample, l is not a limit if has been found such value of ε, that for any arbitrarily small δ we see dark green parts of graph outside the yellow bar.

[8] This and the following examples illustrate the exclusive modeling strength of *VisuMatica* – an important feature of educational software.

It is time to update definition D2:

D4 | Let a be an accumulation point of the domain of the function f. Then the number L is called the *limit* of function $f(x)$ as $x \to a$ if and only if, for every $\varepsilon > 0$ there exists $\delta > 0$ so that, whenever $x \in \mathrm{Dom}(f)$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

This definition makes clear the judgement about the previous limit as a counterexample, and the following *pathological* function's behavior as a correct example of a limit.

Consider $\lim\limits_{x \to 0} \sqrt{x \sin \frac{1}{x}}$. Fig.13 shows graph of function $f(x) = \sqrt{x \sin \frac{1}{x}}$ with different levels of zooming. Function's domain does not include 0. However, the graph condenses as we are approaching 0[9]. It looks like from some zooming moment the whole "area" around the origin belongs to graph.

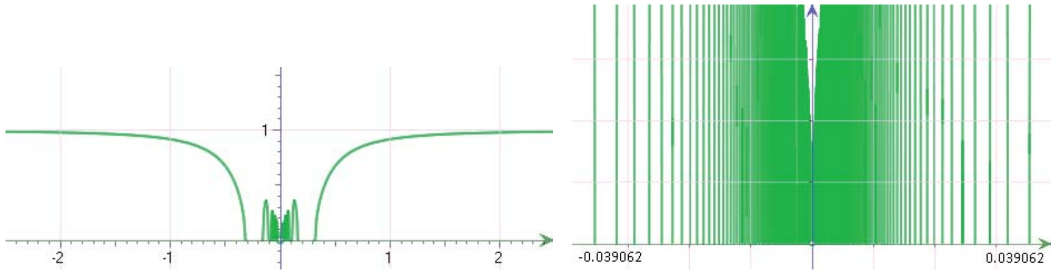

Figure 13

To catch up craftiness of graph pay students' attention to the radicand. When we graph it separately it becomes clear that radicand has negative values arbitrarily close to zero. Arguments of these values are not in the domain. So, there is no neighborhood of 0, which fully belongs to Dom(f)! Meanwhile, that radicand has positive values arbitrarily close to zero. Arguments of these values are in the domain of $f(x)$. Each neighborhood of 0 includes points that belong to Dom(f). Thus, zero is an accumulation point of the Dom(f) and consideration of the limit of $f(x)$ becomes legitimate. Graph of radicant also helps to grasp the value of limit, especially after addition of radicand envelopes $y = \pm x$ and referring to the "sandwich" theorem.

Summarizing together with students their observations of different cases we come to the following scheme (Fig.14). Cases a) - d) present examples of limit, while cases e) - g) present counterexamples.

- *What unites and what differs these cases? If limit is "closeness" then <u>closeness of what</u> is common, and <u>closeness of what distinguishes</u> a)-d) and e)-g)?*

Pay attention*: consideration <u>has no relation to the value of L and ε–interval around it</u>*!

We've "discovered (!)" the *Cauchy Criterion*:

| *The finite limit* $\lim\limits_{x \to a} f(x)$ *exists if and only if*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(0 < |x_1 - a| < \delta \wedge 0 < |x_2 - a| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon).$$

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[9] *VisuMatica* marks it by a punctured origin. *Why*? – The students' answer follows soon.

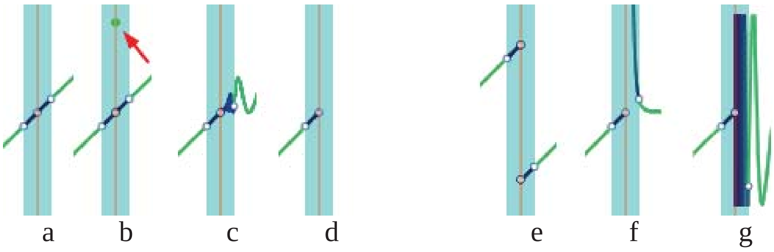

Figure 14

3.2.2 Definition *by Heine*

The sequential mechanism in studies of limit finds its definition (*by Heine*) as follows:

D5 | We call *L the limit of function* $f(x)$ as $x \to a$ if for any sequence $\{x_n\}$ converging to a with terms $x_n \neq a$ for all $n \in \mathbf{N}$, the sequence $\{f(x_n)\}$ converges to L as $n \to \infty$.

So, the software has to provide an ability to *generate various sequences* $\{x_n\}$ converging to a, and to *verify the convergence to L of the corresponding sequences* $\{f(x_n)\}$.

The concept of converging sequences seems not so convenient for visualization. Nevertheless, *VisuMatica* models them quite successfully.

M4. Fig.15 shows the model in case of $f_1(x)$, defined as “y=sin(x)”, and $f_2(x)$, defined as “s(n)=RandomlyTo(f1(x),a,0.05)”, $a = \pi/2$.

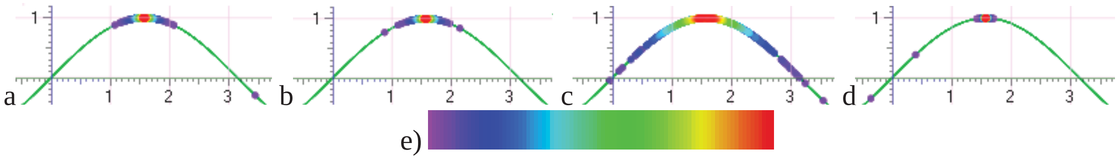

Figure 15

The second expression needs explanation. Its left side “s(n)” means that it is a definition of a sequence. The right side includes call to a function “*RandomlyTo*”, which initiates generation and drawing of a ***random sequence*** *of colored points* $\mathbf{P_n}(x_n, f(x_n))$. Function “*RandomlyTo*” has three following arguments:

1. Expression of function $f(x)$. In this model it is the initial function $f_1(x)$, i.e *sin x*.
2. Expression that defines a, to which $\{x_n\}$ converges.
3. Convergency “*speed*” - an optional argument. When avoided its value considered as 0.05. Fig.15 shows sequence for *speed* = 0.05 (a) and (b), lazy convergence for *speed* = 0.3 (c), and greedy one for for *speed* = 0.01 (d).

Thus, the abscissas of generated random sequence of points, located on graph of $f(x)$, converge to the calculated value of the second argument.

Color distribution codes the order of points in the sequence. The palette in Fig.15 e) presents this arrangement. Reading colors of the spectrum from left to right we interpret the order of sequence elements as follows:

Purple and dark blue points present the first elements, while the yellow and especially red ones – elements with the most advanced indices.

Each following point of the sequence redraws the previous image.

<u>Repeatedly pressing button F5</u> refreshes scene and this way <u>generates one more random sequence with the same parameters.</u>

As always, an important question is: *What does it mean* $\lim_{x \to a} f(x) \neq L$*?*

First, we rewrite the definition D5 in a formal logical notation:

D6 | The number L is called the *limit* of function $f(x)$ as $x \to a$ if and only if

$$(\forall \{x_n \mid x_n \neq a,\ n \in \mathbf{N}\})(\{x_n\} \to a \Rightarrow \{f(x_n)\} \to L)$$

Using the role of negation of expressions with quantifiers we get its negation::

$$\lim_{x \to a} f(x) \neq L \Leftrightarrow (\exists \{x_n \mid x_n \neq a,\ n \in \mathbf{N}\})(\{x_n\} \to a \wedge \neg \{f(x_n)\} \to L)$$

Students can gain experience in studies with some fresh tasks of previous types.

The following two questions sum up these activities.

- *What feature of the repeatedly redrawed scene can be interpreted as <u>existance of the limt</u> and <u>how to read the limit's value</u>?*
- *How to <u>ease</u> this decision and to make the show <u>clearer</u>?*

One more ***type of students activities*** consists in recognition and explanation of images, created by means of current model.

Exploration tasks (Types)

Observe and explain images in Fig.16 obtained with model M4:

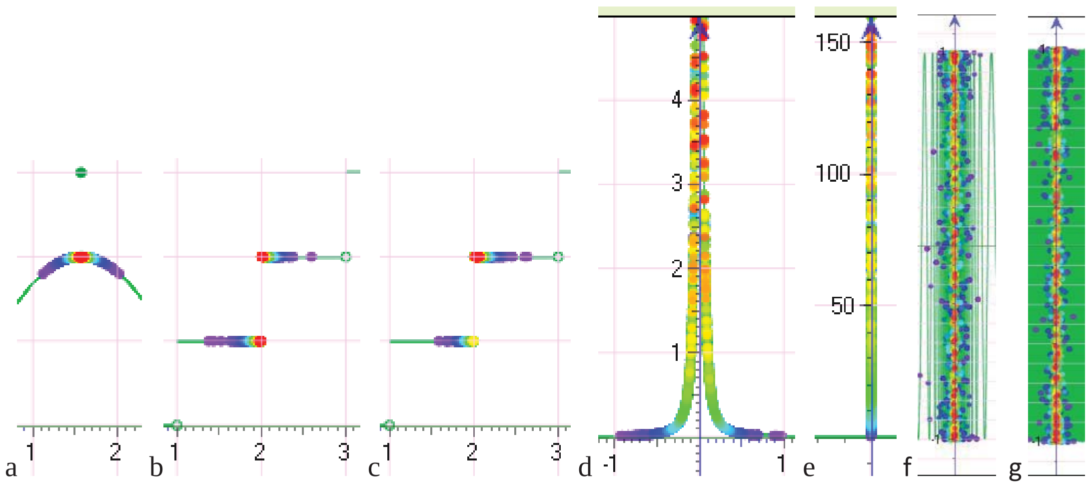

Figure 16

a) $y = \begin{cases} 3, x = \pi/2 \\ 2\sin x \end{cases}, \quad a = \dfrac{\pi}{2},$ b) $y = [x]$, $a = 2$, c) $y = [x]$, $a = 2.04$,

d), e) $y = \dfrac{1}{100x^2}$, $a = 0$ with different scales,

f), g) $y = \sin 1/x$, $a = 0$ with different scales.

- *Compare the two images d) and e) (f) and g)). What did the experimenter, to make the second image more "explanatory"? What is the value of* $\lim_{x \to 0} f(x)$ *if any?*

Cases f) and g) present a "strange" behavior of the random sequence in our model.

Scaling and repeated redrawing of random distribution (reconstruction of the sequence) *does not change the principle - the y-coordinate of red dots remain arranged randomly in the segment* [-1, 1].

It should be noted that the definition by *Heine* has a constructive nature. In fact, the value of L does not matter: finding that for all sequences $\{x_n\}$ converging to a - the sequences $\{f(x_n)\}$ ***converge to the same value*** P, we simply conclude that P is the limit of $f(x)$.

This fact converts the definition by *Heine* to *Sequential Criterion for Functional Limits:*

> Given a function $f(x)$ and an accumulation point a of Dom(f), the following two statements are equivalent:
>
> *a)* $\lim_{x \to a} f(x) = L$,
>
> *b)* **for** any sequence $\{x_n\}$ converging to a with terms $x_n \neq a$ for all $n \in \mathbf{N}$, the sequence $\{f(x_n)\}$ converges to L as $n \to \infty$.

The following *Divergence Criterion for functional limits* is just its corollary:

> $\lim_{x \to a} f(x)$ *does not exist* if and only if one of the following occur:
>
> *a)* there exist two converging to *a* sequences $\{x'_n\}$ and $\{x''_n\}$ in Dom(f) that $x'_n \neq a$ and $x''_n \neq a$ for all $n \in \mathbf{N}$ but $\lim_{n \to \infty} f(x'_n) \neq \lim_{n \to \infty} f(x''_n)$.
>
> *b)* there exists a converging to *a* sequence $\{x_n\}$ in Dom(f) that $x_n \neq a$ for all $n \in \mathbf{N}$ for which $\lim f(x_n)$ does not exist.

- *Justify by means of Fig.16 inexistance of limit in the exploration tasks d)-g).*

Our suspicions about the behavior of function $y = \sin 1/x$, when x approaches to $a = 0$ can not be easily veryfied by the *a*) version of *Divergence Criterion* with model M4. Its single random sequience does not fit the task of construction of two different concrete sequences.

Fortunately, *VisuMatica* allows to highlight extremums. They are looking expressive in the context of our exploration (Fig.18 a),b)). Colored points accent the extremums: points of maximum are shown in blue, while points of minimum – in orange.

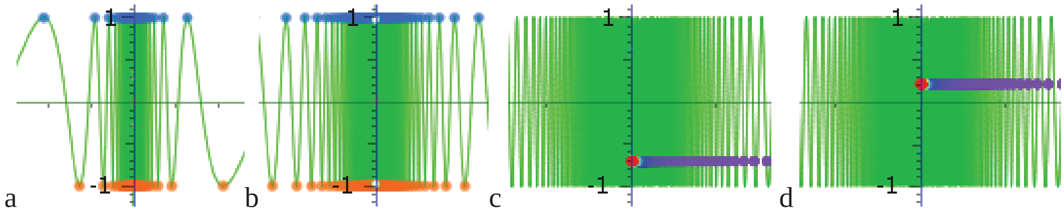

Figure 18

Images a) and b) in Fig.18 present the exploration by changing the x-axis window's boundaries. It becomes clear that the two sets of extremum points *include sequences with two different limits* -1 *and* 1: the blue points converge to (0, 1), and the orange ones are approaching to (0, -1). In accordance with *Divergence Criterion a*) *function* $f(x)$ *diverges at* 0.

- *Find expression that define sequence marked in blue (orange) and converges to* 1(-1).

Pay students attention to the presence of other sequences $\{(x_n, f(x_n))\}$ with $\{f(x_n)\}$ *converging to every possible limit in interval* [-1, 1]. *For example, any line* $y = b$, $b \in [-1, 1]$ *crosses the graph of the function at an infinite number of points that can be represented as an infinite constant sequence* $\{f(x_n) = b\}$, *converging to* b.

- *Are there any other converging sequen*ces $\{f(x_n)\}$?

The following model will help students to handle similar tasks, related to the issue of *Convergence-Divergence Criterion*:

Fig.18 c), d) shows the outputs of model **M5** that "*solves*" the last task and consisits of:

- function $f_1(x)$, defined as "y=sin(1/x)",
- sequence $s_1(n)$, defined as "x(n)=1/(2*n*pi-pi/2+**b**)" – set to invisible,
- sequence $s_2(n)$, defined as "y(n)=f1(s1(n))",
- set boundaries of variable ***b*** in the list box to interval "[0, 2*pi]".

The sequence is multicolored in the same manner as in M4 (see Fig.15).

- Explore the model - play with the ***b*** parameter and zoom in if necessary.
- Redefine s_1 to approach to the accumulation point from the other side.

In general, a diverging at some acccumulation point ***a*** function *f(x)* can define converging sequences $f(x_n)$ with finite $\lim_{n\to+\infty} f(x_n)$, while $\lim_{n\to+\infty} x_n = a$, so called ***partial limits***.

Playing with the **b** *parameter students discover the fact that the partial limits are filling in the whole interval* [-1,1].

Conclusions

In result of the study:

1. *Was proved the* ***importance*** *and shown the* ***function*** *and* ***place*** *of* ***counterexamples*** *in mathematics studies.*
2. *Was confirmed the fundamental role of the mental actions of* ***recognition*** *and of* ***drawing conclusions from the fact of belonging/not belonging to the concept*** *in mastering the concept as well as* ***in the development of students logical thinking****.*
3. *Was created an accurate and* ***optimal typology of relevant tasks****.*
4. *Was provided a general algorithm of construction of such systems of types.*
5. *Presented* ***general ideas of building relevant software*** *were realized in Microsoft Excel© VBA application and the author's noncommercial program VisuMatica, which ease the creation of even complex models by the students themselves.*
6. *Experimental results show the effectiveness of the proposed approach.*

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