Graphing a quadrilateral using a single Cartesian equation

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Abstract

In this paper, we show that it is possible to graph an arbitrarily given quadrilateral (with known vertices) using only a single Cartesian equation. Crucially, we rely on matrix algebra; in particular, projective mappings which are commonly exploited in computer graphics but seldom mentioned in high school lessons or undergraduate matrix algebra courses. Our exploration is also helped by the use of a graphing calculator. Assuming no prior knowledge of matrices on the part of the reader, this paper introduces the necessary matrix-related machinery that our discussion requires.

1 Introduction

In the first author's high school experience, several functions and equations have been learnt in the mathematics lessons. One particular aspect that deeply attracted his attention is the *cartesian* connection between the shape of the graph and its equation. Our research odyssey began with some playful experimentations carried out on the advanced graphing function of the HP graphing calculator – an ICT tool widely used in Beijing mathematics classrooms. By keying the equation

$$|x| + |y| = 1, \tag{1}$$

one observes somewhat unexpectedly a diamond-shaped *square* (see Figure 1).

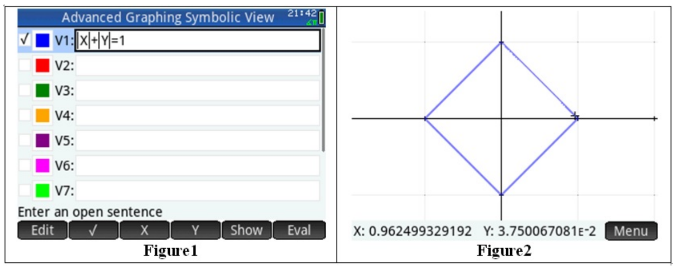

Figure 1: A diamond-shaped square: $|x| + |y| = 1$

The upshot here is that the four equations

$$\begin{aligned} x+y &= 1 \quad (Q_1 : x \geq 0,\ y \geq 0) && (2)\\ -x+y &= 1 \quad (Q_2 : x \leq 0,\ y \geq 0) && (3)\\ -x-y &= 1 \quad (Q_3 : x \leq 0,\ y \leq 0) && (4)\\ x-y &= 1 \quad (Q_4 : x \geq 0,\ y \leq 0) && (5) \end{aligned}$$

can in fact be replaced by an equivalent *single* Cartesian equation $|x|+|y|=1$. It is straightforward to verify that the above quadrilateral in Figure 1 is indeed a square whose side measures $\sqrt{2}$ units. Two research problems arise naturally from this observation. The first one is:

Problem 1 (Create textbook quadrilaterals) *Can we also create the usual* textbook quadrilaterals *(namely, rectangle, parallelogram, trapezium, and so on) encountered in typical high school geometry by using only a single Cartesian equation?*

To attack Problem 1, the first author experimented using a *trial-and-error* method with the advanced graphing function of the HP graphing calculator, and successfully produced some standard quadrilaterals. Later, a more systematic approach relying on 2×2 matrices is developed, as suggested by the second author, to produce the single Cartesian equations of these geometrical figures *without* trial-and-error. This approach is described in Section 2.

The second problem, which turns out to be much harder than the first, is:

Problem 2 (Create an arbitrarily given quadrilateral) *Given an arbitrary quadrilateral $PQRS$ with known coordinates $P=(x_0,y_0)$, $Q=(x_1,y_1)$, $R=(x_2,y_2)$ and $S=(x_3,y_3)$, obtain a single Cartesian equation whose graph is exactly the given quadrilateral $PQRS$.*

Section 3 records our collaborative efforts in answering Problem 2. Its solution requires the adaptation of the approach developed earlier; notably, an *upgrade* of the 2×2 matrix approach to a 3×3 one. We make essential use of projective mappings – a mathematical tool heavily used in the computer graphics for texture mapping and image warping, but seldom taught in high schools.

In tackling a novel problem, it is often the case that several mathematical concepts need to be invoked to give the desired solution. Alan Schoenfeld ([4]) terms the set of mathematical concepts or heuristics available to the problem solver as *problem solving resource.* We argue here that challenging mathematics problems can emerge from ordinary high school mathematics experience, and high school students can expand on his or her mathematical resource to tackle these novel problems – all in the spirit of mathematical exploration and technological experimentation, especially with the help of hand-held devices. In this paper, the reader is assumed to know about graphs of linear functions, elementary geometrical properties of quadrilaterals, and 2-dimensional vectors. Because we do not assume a prerequisite of matrix algebra, we take upon ourselves the task of developing the matrix machinery needed in this paper as we progress. As all the definitions and results about matrices presented here are standard (and of course, not new), we do not wish to burden the reader with extraneous references for these. Interested readers who are looking for a detailed treatment on matrices may wish to read [5]. For advanced exposition on projective geometry and its applications to computer graphics, the reader may consult [1, 3].

2 Textbook quadrilaterals

In this section, we outline a general approach that allows us to construct single Cartesian equations required to produce the desired textbook quadrilaterals.

2.1 Rhombus

We begin with a (non-square) *rhombus* since Figure 1 is a special rhombus.

2.1.1 Identifying the transformation matrix

We identify that one way to maps the square onto a (non-square) rhombus is by stretching it along the x-axis by a factor of k ($k > 0$, for convenience). Let the existing (respectively, new) coordinates system to Oxy (respectively, OXY). Then stretching is given by two equations:

$$X = kx, \qquad Y = y \tag{6}$$

Before we proceed, we need some definitions.

Definition 3 (Matrix, (column/row) vector and matrix pre-multiplication) *An $m \times n$* matrix *is just a rectangular array $\mathbf{A}$ of real numbers a_{ij} with m rows and n columns, and is denoted by*

$$\mathbf{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Matrix addition *can be defined for any two $m \times n$ matrices $\mathbf{A} = (a_{ij})_{m\times n}$ and $\mathbf{B} = (b_{ij})_{m\times n}$ by $\mathbf{A} + \mathbf{B} = (a_{ij})_{m\times n} + (b_{ij})_{m\times n} := (a_{ij} + b_{ij})_{m\times n}$.* Matrix multiplication *is defined for any $m \times p$ matrix $\mathbf{A}$ and $p \times n$ matrix $\mathbf{B}$ by $\mathbf{AB} := (\sum_{k=1}^{p} a_{ik}b_{kj})_{m\times n}$. We call the $m \times 1$ matrices* column vectors *and $1 \times n$ matrices* row vectors*. For real number α and (row/column) vector $\mathbf{a} = (a_i)$, scalar multiplication of α to $\mathbf{a}$ is defined componentwise, i.e., $\alpha\mathbf{a} = \alpha(a_i) := (\alpha a_i)$.*

Example 4 *One common instance of matrix multiplication in this paper is the pre-multiplication of a 2×2 matrix $\mathbf{A}$ to a 2×1 vector $\mathbf{u}$ to yield another 2×1 vector $\mathbf{v}$ as shown below:*

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\mathbf{u}} := \underbrace{\begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}}_{\mathbf{v}}.$$

In this section, we use only 2×2 matrices and 2×1 vectors.

Compressing the two equations (6) into one requires matrix multiplication:

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{7}$$

Notably, any pre-multiplication by a fixed matrix induces a *linear transformation* on the set, $\mathbf{R}^2$, of 2-dimensional vectors. More precisely, for any 2×2 matrix $\mathbf{A}$, 2×1 vectors $\mathbf{p}$ and $\mathbf{q}$, and real numbers α and β, it holds that $\mathbf{A}(\alpha\mathbf{p} + \beta\mathbf{q}) = \alpha\mathbf{Ap} + \beta\mathbf{Aq}$. In other words,

Theorem 5 *Matrix pre-multiplication preserves vector addition and scalar multiplication.*

Specializing to $\mathbf{p} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}0\\1\end{pmatrix}$ allows us to recover the first column, $\mathbf{c}_1$, and second column, $\mathbf{c}_2$, of 2×2 matrix $\mathbf{A}$ by computing $\mathbf{c}_1$ as $\mathbf{Ap}$ and $\mathbf{c}_2$ as $\mathbf{Aq}$. This observation allows us to form the matrix, $\mathbf{A}_T$, that represents a given linear transformation, T, by writing down as column vectors $T\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}$. All in all, we have:

Theorem 6 (Equivalence of matrix pre-multiplication and linear transformation) *Every* 2×2 *matrix* $\mathbf{A}$ *induces a linear transformation* $T_{\mathbf{A}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *given by matrix pre-multiplication by* $\mathbf{A}$. *Conversely, every linear transformation,* T, *on* $\mathbb{R}^2$ *is induced by a matrix pre-multiplication in the above manner by a unique* 2×2 *matrix* $\mathbf{A}_T = \begin{pmatrix} T\begin{pmatrix}1\\0\end{pmatrix} & T\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}$.

Example 7 *To graph a rhombus, we apply the horizontal stretch (with factor k) on the diamond-shaped square, and this transformation maps* $\begin{pmatrix}1\\0\end{pmatrix}$ *to* $\begin{pmatrix}k\\0\end{pmatrix}$ *and leaves* $\begin{pmatrix}0\\1\end{pmatrix}$ *unchanged. By Theorem 6, the horizontal k-stretch is represented by* $\mathbf{S}_{k,\rightarrow}\begin{pmatrix}k & 0\\0 & 1\end{pmatrix}$, *and so* $\begin{pmatrix}X\\Y\end{pmatrix} = \mathbf{S}_{k,\rightarrow}\begin{pmatrix}x\\y\end{pmatrix}$.

2.1.2 Obtaining the inverse matrix

Since we wish to obtain a new Cartesian equation (in terms of X and Y, the new coordinates system) out of the existing one $|x|+|y| = 1$ which is in terms of x and y (the existing coordinates system), we need to write $\begin{pmatrix}x\\y\end{pmatrix}$ in terms of X and Y. To do this, we need the following:

Definition 8 (Inverse of a square matrix) *An* inverse *of an square matrix, i.e.,* $n \times n$ *matrix,* $\mathbf{A}$, *if it exists, is the matrix* $\mathbf{A}^{-1}$ *such that* $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, *where* $\mathbf{I}_n$ *is the identity matrix defined by* $I_n := (\delta_{ij})_{n\times n}$, *and where* $\delta_{ij} := \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise.}\end{cases}$ *A square matrix* $\mathbf{A}$ *is said to be* invertible *if its inverse,* $\mathbf{A}^{-1}$, *exists.*

Theorem 9 (Inverse of an invertible matrix) *An inverse of a matrix* $\mathbf{A} = \begin{pmatrix}a & b\\c & d\end{pmatrix}$, *if it exists, turns out to be unique and is given by* $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -b\\-c & a\end{pmatrix}$.

Example 10 *Thus,* $\mathbf{S}_{k,\rightarrow}^{-1} = \begin{pmatrix}\frac{1}{k} & 0\\0 & 1\end{pmatrix}$ *so that* $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\frac{1}{k} & 0\\0 & 1\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix} = \begin{pmatrix}\frac{1}{k}X\\Y\end{pmatrix}$.

2.1.3 Obtaining the desired equation

Guided by the calculations done in Example 10, we can now substitute $x = \frac{1}{k}X$ and $y = Y$ into the existing equation (1) to obtain obtain $\left|\frac{1}{k}X\right| + |Y| = 1$. Because the graph is drawn using the same labels x and y for the principal axes, we replace X with x (respectively, Y with y) yielding

$$\left|\frac{1}{k}x\right| + |y| = 1. \tag{8}$$

Taking $k = 2$ for illustration sake, we generate a rhombus (see Figure 2).

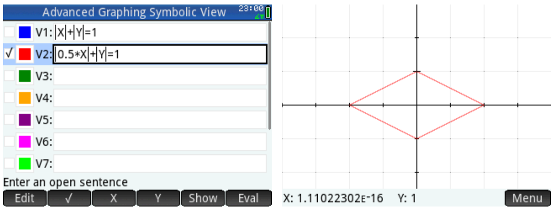

Figure 2: A rhombus: $|0.5x| + |y| = 1$

2.2 Parallelograms and rectangles

We now demonstrate the versatility of the approach outlined in Section 2.1 in generating the desired geometrical figures of a parallelogram, an upright square and a rectangle.

2.2.1 Parallelogram

We now make some modification to obtain a parallelogram (non-rhombus) by performing a special linear transformation called a *shear* which 'slants' horizontally the rhombus in Figure 2 by a shear factor of λ. Such a horizontal shear leaves $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ unchanged but maps $\mathbf{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} \lambda \\ 1 \end{pmatrix}$. Since shear is a linear transformation, it follows that the required *shear matrix* is $H_{\lambda,\rightarrow} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$, whose inverse equals to $H^{-1}_{\lambda,\rightarrow} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}$. Thus, we have $x = X - \lambda Y$, $y = Y$. Substituting into Equation (8), writing $\mu := \frac{1}{k}$ and replacing X by x (respectively, Y by y) yields

$$|\mu(-\lambda x + y)| + |y| = 1 \tag{9}$$

For illustration, setting $\mu = 0.5$ and $\lambda = 1$ yields a (non-rhombus) parallelogram (see Figure 3).

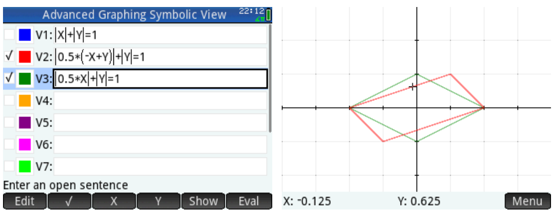

Figure 3: A parallelogram: $|0.5(-x + y)| + |y| = 1$

2.2.2 Rectangle

To create an upright rectangle, one needs to make the ‘tilted’ square in Figure 1 upright. This requires one to rotate it $\theta = \frac{\pi}{4}$ radians (i.e., 45°) counterclockwise about the origin. In general, a rotation of θ radians counterclockwise maps $\mathbf{p} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$, and hence is represented by the *rotation matrix* $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. When $\theta = \frac{\pi}{4}$, we have: $R_{\frac{\pi}{4}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, whose inverse is given by $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Using the usual procedure, we obtain the equation

$$\left|\frac{x+y}{\sqrt{2}}\right| + \left|\frac{x-y}{\sqrt{2}}\right| = 1 \tag{10}$$

which produces the desired upright square (see Figure 4).

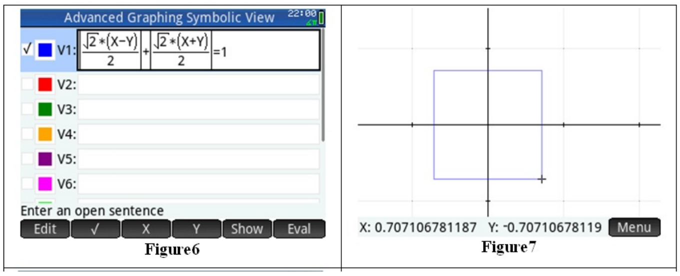

Figure 4: An upright square

To obtain an rectangle from an upright square, we merely need to stretch the square by a certain positive scaling factor, say $k > 0$, horizontally. By our development in Section 2.1.2, we arrive at the equations

$$\left|\frac{\frac{1}{k}x+y}{\sqrt{2}}\right| + \left|\frac{\frac{1}{k}x-y}{\sqrt{2}}\right| = 1 \tag{11}$$

For the sake of illustration, take $k = \frac{1}{2}$, we obtained a rectangle (see Figure 5). Notice that the horizontal stretch of the square in Figure 4 by a factor of $\frac{1}{2}$ indeed results in a rectangle whose length is $\frac{1}{2\sqrt{2}}$, and whose breadth is unchanged, i.e., $\sqrt{2}$. This feature is confirmed in Figure 5.

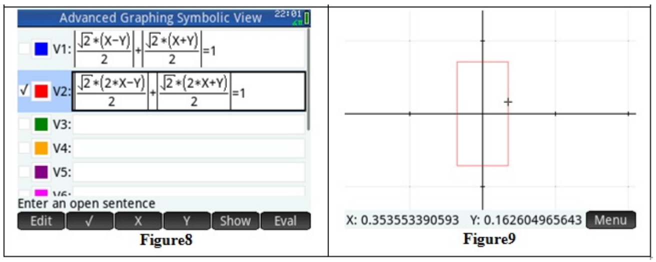

Figure 5: An upright rectangle

3 Graphing an arbitrarily given quadrilateral

Being able to produce the graphs of some textbook quadrilaterals using only one Cartesian equation is somewhat unsatisfying mathematically. What one wishes to achieve is to be able to use a single Cartesian equation to graph *any arbitrarily given* quadrilateral. In this section, we specifically attend to this problem, i.e., Problem 2.

Recall from Section 1 that we can graph a diamond shaped square, which lies in the domain space, using a simple Cartesian equation $|u| + |v| = 1$. Our plan is to solve Problem 2 by adapting the general approach outlined in Section 2 to this current context via the steps below:

1. Find the transformation that maps this square to the given quadrilateral.
2. Code this transformation as a matrix and obtain its inverse.
3. Use this inverse of the matrix to obtain the equations connecting the new and existing coordinates systems, and then substitute these equations into the existing equation $|u| + |v| = 1$ to obtain the desired equation.

It turns out that a ready solution to this mapping sub-problem (1) is available from the sub-discipline of computer science: *image warping*. Our presentation below is fashioned after [2].

3.1 Homogeneous coordinates

To carry out our plan, we need these concepts:

Definition 11 (Projective space, homogeneous coordinates) *On the set of non-zero 3-dimensional vectors, $\mathbb{R}^3 - \{\mathbf{0}\}$, define an equivalence relation $\sim$ as follows:*

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2) \iff \exists\lambda \neq 0.\ (x_1, y_1, z_1) = \lambda(x_2, y_2, z_2).$$

The equivalence class induced by $\sim$ which contains the point (x, y, z) is denoted by $[x, y, z]$, and each such triple (in square brackets) is referred to as homogeneous coordinates*. The collection of all equivalence classes induced by $\sim$ is called the* projective space*, and is denoted by $\mathbb{P}$.*

Each 2-dimensional vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ is represented by the homogeneous vector $\begin{bmatrix} xw \\ yw \\ w \end{bmatrix}$, where w is an arbitrary non-zero real number – often chosen to be 1 so that the actual 2-dimensional vector may be recovered from its homogeneous representation conveniently without a need for division by w. Notice that the projective space is a superset of $\mathbb{R}^3$ in the sense that it includes points of the form $[x', y', 0]$, where $x',\ y' \neq 0$; these extra points are called the *points at infinity* and are meant to represent the directions in the Euclidean space. As mentioned in the introduction, when we avail ourselves of the tools of projective geometry we upgrade the 2-dimensional vectors to 3-dimensional vectors and, likewise, 2×2 matrices to 3×3 matrices.

3.2 Projective mappings

We now to turn to the projective analogue of linear transformations.

Definition 12 (Projective mappings) *A* projective mapping $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ *is one defined explicitly by:*

$$x = \frac{au + bv + c}{gu + hv + i}, \qquad y = \frac{du + ev + f}{gu + hv + i}, \tag{12}$$

where $a, b, c, d, e, f, g, h, i$ are real numbers.

Remark 13 *1. For easier manipulation, one use the homogeneous matrix form given by*

$$\begin{bmatrix} x' \\ y' \\ w \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} u' \\ v' \\ q \end{bmatrix}, \tag{13}$$

where $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x'/w \\ y'/w \end{pmatrix}$ for $w \neq 0$, and $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u'/q \\ v'/q \end{pmatrix}$ for $q \neq 0$.

We denote the source vector $\begin{bmatrix} u' \\ v' \\ q \end{bmatrix}$ by $\mathbf{v}_s$ and the target vector $\begin{bmatrix} x' \\ y' \\ w \end{bmatrix}$ by $\mathbf{v}_t$. The projective mapping, ψ, is thus represented by the matrix $\mathbf{M}_{st} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. In this way, ψ can be seen to have been 'lifted' to a mapping $T_{\mathbf{M}_{st}} : \mathbb{P} \longrightarrow \mathbb{P}$.

2. *Because we are working with homogeneous coordinates, projective mappings are homogeneous and so for any $\lambda \neq 0$, matrices $\mathbf{A}$ and $\lambda\mathbf{A}$ can be regarded as equivalent projective mappings. Although there are 9 coefficients in the matrix representation of a projective mapping, we have in actuality 8 degrees of freedom. For this reason, we may assume without loss of generality that $i = 1$.*

3. *Composition of projective mappings is given by the matrix multiplication of their representing matrices.*

Projective mappings share a lot of pleasant properties of affine mappings (i.e., these are analogues of linear transformations suited to the context of affine spaces – loosely speaking, vector spaces which are translated away from the origin). One very important property, similar to Theorem 5, is:

Theorem 14 *Projective mappings preserve lines at all orientations.*

Remark 15 *Since a quadrilateral is determined by its vertices, the above theorem ensures that if a projective mapping does exist between a given square and a given quadrilateral that maps each vertex of the square to the corresponding vertex of the quadrilateral (in order), then the mapping will also map the edges of the square to the corresponding ones of the quadrilateral.*

3.3 Solution of problem

Having identified the salient transformation as the projective mapping and having coded it in matrix form (in homogeneous form), we now find the inverse of the representing matrix $\mathbf{M}_{st}$. Using the adjoint-determinant form, the inverse of $\mathbf{M}_{st}$ is given by

$$\mathbf{M}_{st}^{-1} := \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix} = \begin{bmatrix} ei - fh & ch - bi & bf - ce \\ fg - di & ai - cg & cd - af \\ dh - eg & bg - ah & ae - bd \end{bmatrix}. \tag{14}$$

Explicitly, the inverse projective mapping is thus given by

$$u = \frac{Ax + By + C}{Gx + Hy + I}, \qquad v = \frac{Dx + Ex + F}{Gx + Hy + I} \tag{15}$$

We are now ready to solve the second research problem. The graph of the following single Cartesian equation $|u| + |v| = 1$ is exactly the square determined by $(u_0, v_0) = (0, 1), (u_1, v_1) = (1, 0), (u_2, v_2) = (0, -1)$ and $(u_3, v_3) = (-1, 0)$ (see Figure 1). Substituting Equations (15) into $|u| + |v| = 1$, the desired single Cartesian equation which graphs $PQRS$ is then given by:

$$\left| \frac{Ax + By + C}{Gx + Hy + I} \right| + \left| \frac{Dx + Ey + F}{Gx + Hy + I} \right| = 1, \tag{16}$$

where A, B, C, D, E, F, G and I are to be determined. By Remark 13(2), we may assume without loss of generality that $I = 1$. The projective mapping defined by Equations (15) perform the following assignments:

$$u_n = \frac{Ax_n + By_n + C}{Gx_n + Hy_n + 1}, \qquad v_n = \frac{Dx_n + Ey_n + F}{Gx_n + Hy_n + 1}. \tag{17}$$

for $n = 0, 1, 2, 3$. These equations are equivalent to

$$x_nA + y_nB + C - x_nu_nG - y_nu_nH = u_n, \qquad x_nD + y_nE + F - x_nv_nG - y_nv_nH = v_n. \tag{18}$$

These equations form a system of 8 linear equations in A, B, C, D, E, F, G and H below:

$$\begin{pmatrix} x_0 & y_0 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_1 & y_1 & 1 & 0 & 0 & 0 & -x_1 & -y_1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ x_3 & y_3 & 1 & 0 & 0 & 0 & x_3 & y_2 \\ 0 & 0 & 0 & x_0 & y_0 & 1 & -x_0 & -y_0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 & x_2 & y_2 \\ 0 & 0 & 0 & x_3 & y_3 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \tag{19}$$

We now demonstrate the solution of the above linear system using the following example.

Example 16 *Let us solve Problem 2 when specialized to*

$$P = (x_0, y_0) = (-2, 1), Q = (x_1, y_1) = (2, 2), R = (x_2, y_2) = (1, -3), S = (x_3, y_3) = (-2, 2).$$

Solving Equation 19 in this case yields:

$$\begin{array}{lll} A = \frac{2}{7}, & B = \frac{3}{14}, & C = \frac{5}{14} \\ D = -\frac{57}{224}, & E = \frac{57}{224}, & F = 0 \\ G = \frac{31}{224}, & H = \frac{9}{224}, & I = 1 \end{array}$$

Keying the equation

$$\left|\frac{\frac{2}{7}x+\frac{3}{14}y+\frac{5}{14}}{\frac{31}{224}x+\frac{9}{224}y+1}\right|+\left|\frac{-\frac{57}{224}x+\frac{57}{224}y}{\frac{31}{224}x+\frac{9}{224}y+1}\right|=1 \tag{20}$$

into the advanced graphing function of HP *calculator, we obtain the desired quadrilateral PQRS:*

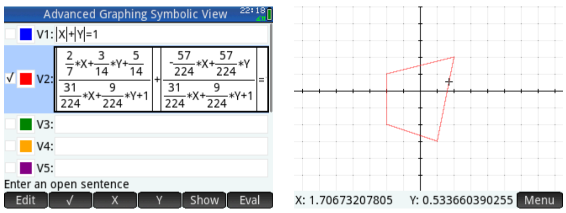

Figure 6: Quadrilateral $PQRS$

Remark 17 *Tongue-in-cheek: There is absolutely no way Equation (20) can be obtained by trial-and-error!*

4 Conclusion

We have employed matrices as convenient tool to equationally describe the change of bases, i.e., the coordinate systems Oxy to OXY, and obtain the desired single Cartesian equation whose graph is the given quadrilateral. Our exploration is greatly helped by a graphing calculator. A general solution to the linear system (19) has been worked out but, for a lack of space, is omitted. For future work, we want to consider the problem of graphing general conics from the unit circle $x^2 + y^2 = 1$ using only a single cartesian equation, and make connections between that and the current one we solved.

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