Locus, Parametric Equations and Innovative Use of Technological Tools

Wei-Chi YANG
wyang@radford.edu
Department of Mathematics
and Statistics
Radford University
Radford, VA 24142
USA

Abstract

In this paper, we discuss two problems found from Chinese college entrance exam practice problems [8]. We see how original problems in 2D, stated in an exam static and somewhat uninspired setting, can be extended to other interesting cases in 2D and more challenging corresponding problems in 3D for students to explore with the help of a Dynamic Geometry Software (DGS) and a Computer Algebra System (CAS). We use a DGS to construct the locus or locus surface geometrically, and use a CAS to verify our locus or locus surface analytically. We shall see with the innovative use of technological tools, mathematics can be made more fun, accessible, challenging and applicable to broader group of students. Finally, we attempt to make these problems relevant to real-life applications, we invite readers to investigate how these problems can be interpreted differently.

1 Introduction

Ever since the document of *Innovation on Mathematics Curriculum and Textbooks in China* was released in 2006, technological tools have been adopted for explorations in many high schools in China. However, because college entrance examinations still play crucial components for students' future success, students and parents wonder how activities involving exploration could help students improve their exam grades. They are concerned that the nature of the assessment methods that students face in many countries does not reward exploration. However, we cannot ignore the fact that innovation and understanding do not come from drills or rote-type learning, but from exploration. The author believes that we should recognize the importance of stimulating the discussion of mathematics and its applications through timely use of technological tools (see *[7]* or *[9]*). In this paper, we present two problems that were found in college entrance exam practice problems from China (*[8]*). To make these problems more accessible, interesting and challenging at times, we propose using a DGS to construct the potential curve for a locus geometrically. While students may be able to solve locus equations by hand when the problems are simpler, they will soon discover that finding the algebraic equation for a locus by hand is virtually impossible when problems become more complicated. Consequently, they see the need of a DGS for construction purposes and a CAS to validate whether the algebraic equation for the locus matches with the plot that was obtained from the DGS. We shall see that the problems discussed in 2D can be extended to respective 3D scenarios when students have knowledge of multivariable calculus. The locus problems can be linked to real-life problems. The author states some possibilities and invites readers to imagine more applications on their own.

1.1 Original and Extended Scenarios

The original statement of the problem is stated as follows: *Given a unit circle centered at* $(0, 0)$ *and a fixed point at* $A = (2, 0)$. *Let* Q *be a moving point on the unit circle* C. *Find the locus* M *which is the intersection between the angle bisector* QOA *and line segment* QA. This is an easy exercise to verify that the locus M will be a circle, which we leave it as an exercise to the readers. It is natural to imagine when DGS and CAS tools are available for students to explore in a class, then quickly they may ask 'what if' scenarios. For example, we consider the following case:

Example 1 *Given an ellipse* C *of* $[x(t), y(t)] = [a\cos(t), b\sin(t)]$ *and a fixed point* $A = (p, q)$. *Let* Q *be a moving point on the ellipse. Find the locus* M *which is the intersection between the bisector* QOA *and line segment* QA.

Students may use their favorite DGS to construct the trace the locus M without too much trouble, which we use GInMA (*[4]*) below to demonstrate one possibility of the locus in red color of Figure 1 below.

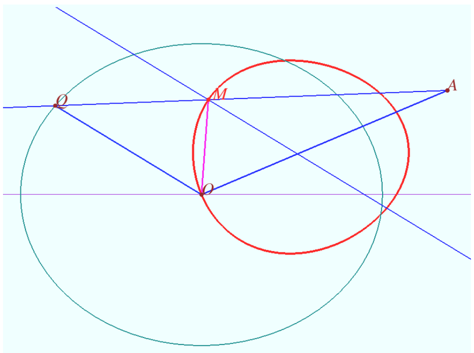

Figure 1 Locus, bisection and an ellipse

We note that a DGS allows users to drag the moving point Q and see how the corresponding locus M moves accordingly. Similarly, we can also make the point A to be movable and see how the locus changes accordingly (See Figure 2(a) and 2(b)). Being able to visualize and manipulate a dynamic graph constructed from a DGS will allow students to comprehend the original question quickly and make additional observations from *what if* scenarios. The next step students can do is to see if they can apply their math knowledge to derive the equation analytically for the locus and verify if what they see from DGS earlier is reasonable. We shall see below that the locus can be found with a little help from geometry and familiarity with parametric equations. First, we construct a line passing through M and is parallel to OQ, and label the intersection between such line and OA as B. It follows from the Angle Bisector Theorem that $\frac{MA}{MQ} = \frac{OA}{OQ}$. Suppose we denote $\frac{OA}{OQ}$ by $k(t)$, which is not constant but depends on t in this case. We see that triangle OBM is an isosceles with $MB = OB$. Therefore, we have

$$\frac{MA}{MQ} = \frac{AB}{OB} = \frac{AB}{MB} = \frac{OA}{OQ} = k(t). \tag{1}$$

Since $\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA} = \overrightarrow{OB} + k(t)\overrightarrow{OB}$, we see

$$\overrightarrow{OB} = \frac{1}{k(t)+1}\overrightarrow{OA} \tag{2}$$

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \overrightarrow{OA} - \frac{1}{k(t)+1}\overrightarrow{OA} \tag{3}$$

$$= \frac{k(t)}{k(t)+1}\overrightarrow{OA}. \text{ Thus, we obtain} \tag{4}$$

$$\overrightarrow{BM} = \frac{k(t)}{k(t)+1}\overrightarrow{OQ} \text{ and} \tag{5}$$

$$\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM} = \frac{1}{k(t)+1}\overrightarrow{OA} + \frac{k(t)}{k(t)+1}\overrightarrow{OQ} \tag{6}$$

$$= \frac{1}{\frac{OA}{OQ}+1}\overrightarrow{OA} + \frac{\frac{OA}{OQ}}{\frac{OA}{OQ}+1}\overrightarrow{OQ} \tag{7}$$

$$= \frac{OQ}{OA+OQ}\overrightarrow{OA} + \frac{OA}{OA+OQ}\overrightarrow{OQ}. \tag{8}$$

In addition, because $OQ = \sqrt{a^2\cos^2 t + b^2 \sin t}$ and $OA = \sqrt{p^2+q^2}$, the parametric equation for the locus M can be seen directly from Eq. (8) above. With the help of a CAS, which we use MAPLE (*[6]*) here, we plot the locus M together with the original ellipse when $a = 2, b = 1, p = 2$ and $q = 0$ in Figure 2(c):

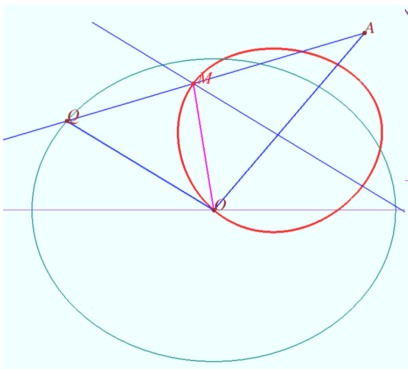

Figure 2(a) Point A is outside the ellipse

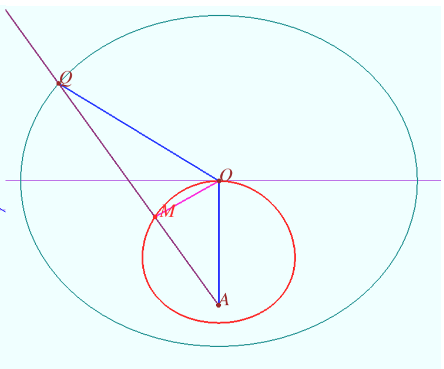

Figure 2(b) Point A is inside the ellipse

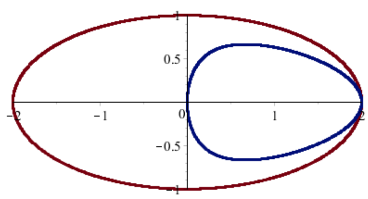

Figure 2(c) Plot generated by MAPLE

We pose another scenario when we replace the ellipse by a cardioid as follows:

Exercise 2 *Given a cardioid C of the form $[x(t), y(t)] = [(1-\cos(t))\cos(t)+1, (1-\cos(t))\sin(t)]$ and a fixed point $A = (p, q)$. Let Q be a moving point on the cardioid. Find the locus M which is the intersection between the bisector QOA and line segment QA.*

As we have mentioned earlier, we encourage students to use a DGS to explore their possible locus before validating analytically with their favorite CAS to see if the locus seen from a DGS matches that of the CAS. We use *[4]* to draw the locus when we vary the point A in Figures 3(a) and 3(b) below. We also use MAPLE *[6]* to plot the locus analytically when $p = 3$ and $q = 2$ in Figure 3(c).

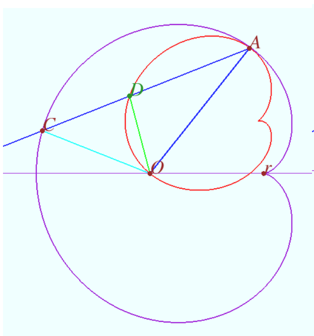

Figure 3(a) Locus and a cardioid

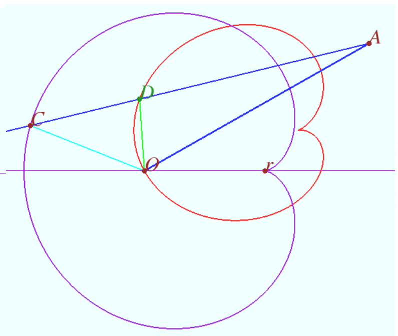

Figure 3(b) Point A is outside the cardioid

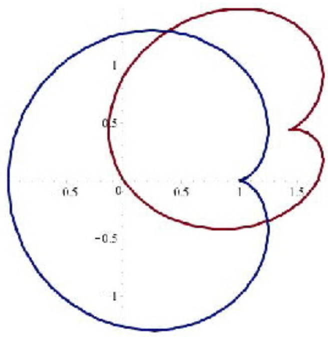

Figure 3(c) Locus generated by MAPLE

1.1.1 Possible Real Life Applications in 2D

1. Consider the Figures 2(a) or 3(b), an allied aircraft Q is moving along the shape of a given curve C, which could be an ellipse or a cardioid. The allied aircraft carrier is set up at the point A (outside curve C), which communicates with a command center $O = (0, 0)$. The enemy decides to move along (roughly) at the intersection between the angle bisector QOA and QA to avoid being targeted. Find the possible route for the enemy.

2. A game is described as follows: We refer to Figure 3(d). A light source Q, pointing at a point O, is moving along a curve C (could be an ellipse or cardioid as described in Example 1 or Exercise 2). The reflected light ray is always kept at the direction of $\overrightarrow{OA}$, where A is a fixed point. It is known that the target M is always staying at the intersections between the line segment of AQ and the normals of mirror sticks (the mirror sticks are shown in line segments L or $L'$when point Q is moved to Q' in Figure 3(d)).

The game is for you to maneuver the mirror sticks so you can hit the target M.

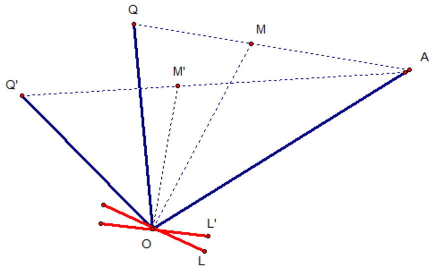

Figure 3(d) A game and reflections

3. Use your imagination to interpret your real-life scenarios.

1.2 Extensions to 3D scenarios

Here we describe a possibility for students to explore once they have knowledge of parametric equations for surfaces. We shall see that the bisection theorem used in 2D is still valid in the 3D explorations. Specifically, we explore the scenarios when we replace the ellipse and cardioid by an ellipsoid and a cardioid surface respectively. We state these two scenarios as follows:

Example 3 *Given an ellipsoid S of $x(\theta,\varphi) = a\sin\varphi\cos\theta, y(\theta,\varphi) = b\sin\varphi\sin\theta$ and $z(\theta,\varphi) = c\cos\varphi$.and a fixed point $A = (p,q,r)$. We pick a moving point Q on the ellipsoid S. Find the locus M which is the intersection between the bisector QOA and line segment QA.*

We show a scenario of the locus of M in Figure 4(a) below using *[4]*. We note that Eq.(8) can be extended to find the parametric equation for the locus surface in 3D as follows:

$$\begin{bmatrix} X(\theta,\varphi) \\ Y(\theta,\varphi) \\ Z(\theta,\varphi) \end{bmatrix} = \frac{OQ}{OA+OQ}\begin{bmatrix} p \\ q \\ r \end{bmatrix} + \frac{OA}{OA+OQ}\begin{bmatrix} x(\theta,\varphi) \\ y(\theta,\varphi) \\ z(\theta,\varphi) \end{bmatrix} \quad (9)$$

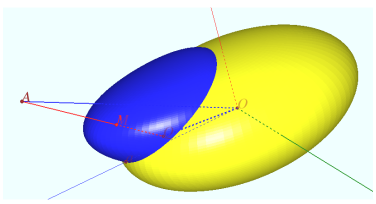

Figure 4(a) Locus surface and an ellipsoid

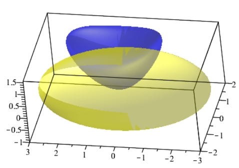

Figure 4(b) A non-convex locus generated by Maple

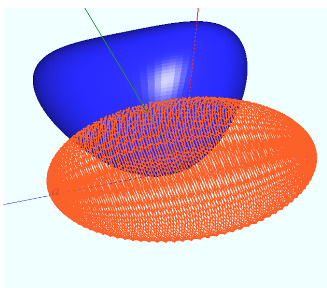

Figure 4(c) A non-convex locus from GInMA

Through various exploration by adjusting the shape of the ellipsoid (a, b, c) and the fixed point $A = (p, q, r)$, we found an interesting non-convex locus when $(a, b, c) = (3, 2, 1)$ and $(p, q, r) = (3, 1, 1)$, which are shown using *[6]* in Figure 4(b) and *[4]* in Figure 4(c) respectively. We pose another scenario when we replace the ellipsoid by a cardioid surface, which we leave it as an exercise as follows:

Exercise 4 *Given a cardioid surface S by rotating the 2D curve of $[x(t), y(t)]$, where $x(t) = a(1 - \cos t)\cos(t) + a, y(t) = a(1 - \cos t)\sin(t)$ and $t \in [0, 2\pi]$, around the $x - axis$. We let A be a fixed point and pick a moving point Q on the cardioid surface S. Use a DGS or CAS to find the locus M which is the intersection between the angle bisector QOA and the line segment QA.[Hint: We see that the cross sections of the surface S are circles parallel to $yz - plane$, whose centers are on the $x - axis$ with radius $y(t)$. If we let angle φ be the angle between the vector from center to the point on each cross section and the positive $y - axis$. Then the parametric surface becomes $[x(t), y(t)\cos\varphi, y(t)\sin\varphi]$, where $t \in [0, 2\pi]$ and $\varphi \in [0, \pi]$.]*

We use *[6]* to sketch the locus surface (in blue) and the original cardioid surface as follows with $A = (1, 2, 3)$ and $a = 1$ as follows:

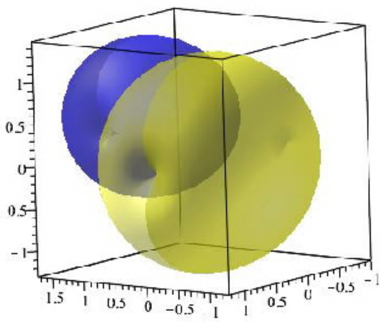

Figure 5 Locus surface
generated by MAPLE

1.2.1 Possible Real Life Applications in 3D

1. An allied aircraft Q is moving along the shape of a given ellipsoid or a cardioid surface. The allied aircraft carrier is set up at the point A, which communicates with a command center O at the center of the ellipsoid or cardioid surface. The enemy's aircraft decides to move along at the intersection between the angle bisector QOA and QA to avoid being hit. Find the possible route for the enemy.

2. A game is described as follows: A light source Q, pointing at a point $O = (0, 0, 0)$, is moving along a surface S (it could be an ellipsoid or a cardioid surface as described in Example 3 or Exercise 4). The reflected light ray is always kept at the direction of $\overrightarrow{OA}$, where A is a fixed point in the space. It is known that the target M is always staying at

the intersections between the line segment of AQ and the normals of mirror planes when Q is moving along S. The game is for you to maneuver the mirror planes so that the target M can be hit.

3. Use your imagination to interpret your real-life scenarios.

2 Exploration 2

We next show how the following original locus problem, originated from a practice problem for college entrance in China, actually serves one way of constructing an ellipse from two given circles.

2.1 Original and Extended Statements

Example 5 *We are given two concentric circles centered at $O = (0, 0)$ with radii of 1 and 2 respectively. We are given a moving point A on the unit circle. We construct the line OA to intersect at a point B on the outer circle. We then construct the line l_1 passing through B and is parallel to $y - axis$. Finally, we construct the line l_2 passing through the point A and is parallel to $x - axis$. Find the locus for the point P that is the intersection between l_1 and l_2.*

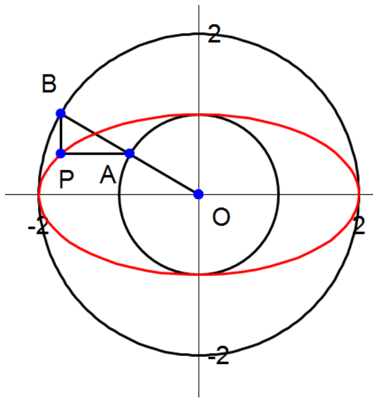

Figure 6 Generating an ellipse from two concentric circles

Although finding the locus is quite elementary with simple knowledge from trigonometry, this problem actually serves a good purpose of understanding how the parametric equation for an ellipse is being formed. If we assume the radii of inner and outer circles to be b and a respectively. It is quite simple from *[3]* (Figures 7(a) and 7(b)) that the locus of the desired ellipse will be of the form of $[a \cos t, b \sin t]$.It is interesting to note from *[3] that 'For extreme accuracy it's probably the best method. It's convenient for use on a drafting board with T-square* *and triangles'.*

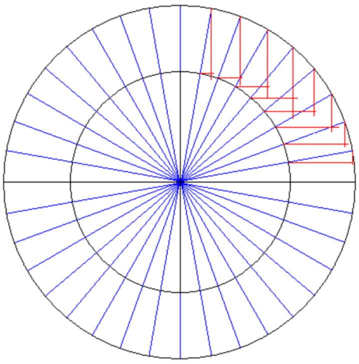

Figure 7(a) Constructions of an ellipse

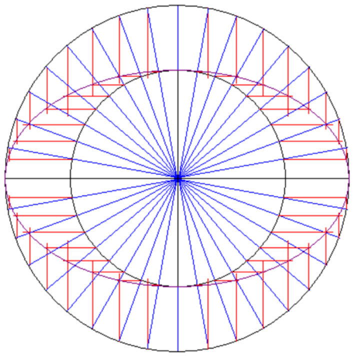

Figure 7(b) Locus derived from the construction

It is natural to ask now what the locus would be if we replace the outer circle by an ellipse.

Example 6 *We are given a circle C with radius r_0 and centered at $O = (0,0)$, and an ellipse of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, that is outside the given circle. Let A be a moving point on the circle. Suppose we construct the line OA to intersect at a point B on the ellipse. We construct the line l_1 passing through B and is parallel to $y-axis$. Next we construct the line l_2 passing through the point A and is parallel to $x-axis$. (a) Find the locus for the point P that is the intersection between l_1 and l_2. (b) Find the point B which yields the maximum area for the triangle APB.*

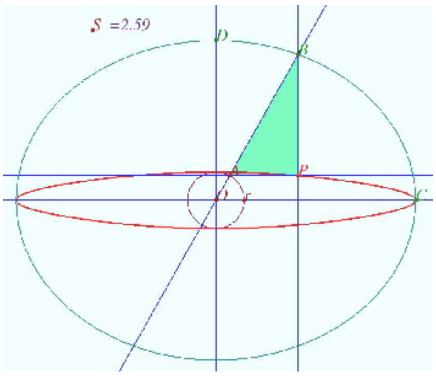

Figure 8 Generating the locus from a circle and an ellipse

We note that the part (a) of this problem can be solved by hand without too much work: We write $A = (A_x, A_y), B = (B_x, B_y)$, and let $OB = r, \measuredangle BOC = \theta$, then $B = (a\cos\theta, b\sin\theta)$. We see $OB^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2\cos^2\theta + b^2(1-\cos^2\theta) = b^2 + (a^2-b^2)\cos^2\theta$. Thus $r^2 = b^2 + (a^2-b^2)\frac{1+\cos 2\theta}{2}$, which leads to

$$OB = r = \frac{\sqrt{2}ab}{\sqrt{a^2+b^2-(a^2-b^2)\cos 2\theta}}. \tag{10}$$

If we write locus $P = (P_x, P_y)$, then $(P_x)^2 = \left(\frac{\sqrt{2}ab}{\sqrt{a^2+b^2-(a^2-b^2)\cos 2\theta}}\right)^2 \cos^2\theta = \dfrac{2a^2b^2\cos^2\theta}{a^2+b^2-(a^2-b^2)\cos 2\theta}$ and $(P_y)^2 = r_0^2 \sin^2\theta$. For part (b), the area of ABP is the absolute value of

$$\frac{1}{2}(AP)(BP) = \frac{1}{2}(P_x - A_x)(B_y - P_y) = \frac{1}{2}(r\cos\theta - r_0\cos\theta)(r\sin\theta - \sqrt{r_0}\sin\theta). \qquad (11)$$

Now we substitute r in Eq. (10) into the area of ABP and use a CAS to simplify $\frac{1}{2}(AP)(BP)$ to the following form:

$$\frac{1}{4}\sin 2\theta\left(\frac{\sqrt{2}ab}{\sqrt{a^2+b^2-(a^2-b^2)\cos 2\theta}} - r_0\right)^2. \qquad (12)$$

The locus corresponds to an ellipse is sketched in Figure 8 using *[4]*. If we use a CAS such as *[6]* with specific numeric values of $a = 5, b = 4$ and $r_0 = \frac{1}{\sqrt{2}}$, we find the maximum area of ABP to be 3.5631, which occurs when θ is about 0.655308 radians or 37.5464 degrees. In the following example, we investigate a similar locus problem but centers for two curves are at different locations, which we stated the problem as follows:

Example 7 *We are given a circle C^* centered at $O = (0,0)$ with radius r_0, and a cardioid which resembles the shape of $r = a(1-\cos\theta)$, where $\theta \in [0, 2\pi]$ enclosing the given circle C^* as shown in Figure 9(a). We are given a moving point A on the circle. Suppose we construct the line OA to intersect at a point B on the cardioid. We construct the line l_1 passing through B and is parallel to $y-axis$. Next we construct the line l_2 passing through the point A and is parallel to $x-axis$. (a) Find the locus for the point P that is the intersection between l_1 and l_2. (b) Find the point B on the cardioid which yields the maximum area for the triangle APB.*

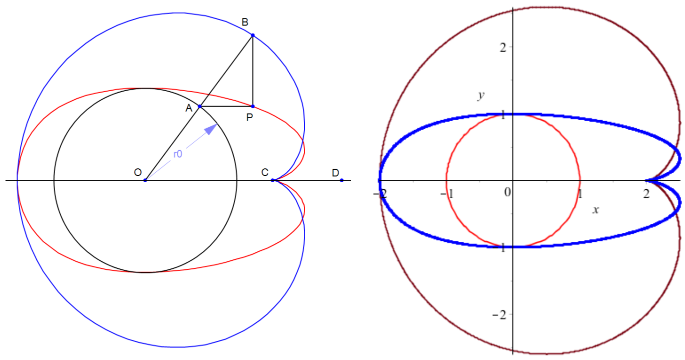

Figure 9(a) Locus generated by [2]   Figure 9(b) Locus generated by [6]

First, we notice in Figure 9(a), the cardioid, of the shape $r = a(1-\cos\theta)$, enclosing the circle, is centered at the point C and the circle C^* is centered at $(0,0)$. If we use $O = (0,0)$ as the center of the cardioid enclosing the circle, we may write the parametric equation $[x(\varphi), y(\varphi)]$ for such cardioid as $x(\theta) = a(1-\cos\theta)\cos(\theta) + OC$ and $y(\theta) = a(1-\cos\theta)\sin(\theta)$ with $OC > r_0$. Now, we let $\theta = \measuredangle BOC, \varphi = \measuredangle BCD$, $OB = R, OC = a$. We next will express R in terms of a and angle

φ. We write locus $P = (P_x, P_y), A = (A_x, A_y)$, and $B = (B_x, B_y)$, and note $P_y = A_y = r_0 \sin\theta$ and $P_x = B_x$. Note the original blue cardioid can be represented by $r = a(1 - \cos\varphi)$. We observe $B_x = R\cos\theta = a + r\cos\varphi$ and $B_y = R\sin\theta = r\sin\varphi$, which leads to

$$\begin{aligned} R^2 &= a^2 + 2ar\cos\varphi + r^2 \\ &= a^2 + 2a\left(a(1-\cos\varphi)\right)\cos\varphi + a^2(1-\cos\varphi)^2 \\ &= a^2\left(2 - \cos^2\varphi\right). \text{ This implies} \qquad (13) \\ R &= a\sqrt{2-\cos^2\varphi}. \qquad (14) \end{aligned}$$

In view of $P_x = B_x$, we see

$$\begin{aligned} \frac{P_x}{a} &= 1 + \frac{r}{a}\cos\varphi = 1 + (1-\cos\varphi)\cos\varphi = \sin^2\varphi + \cos\varphi, \\ P_x &= a\left(\sin^2\varphi + \cos\varphi\right). \qquad (15) \end{aligned}$$

Furthermore, we see

$$\begin{aligned} \frac{P_y}{r_0} &= \sin\theta = \frac{r}{R}\sin\varphi = \frac{r\sin\varphi}{a\sqrt{2-\cos^2\varphi}} = \frac{\sin\varphi\,(1-\cos\varphi)}{\sqrt{2-\cos^2\varphi}} \text{ and} \\ P_y &= r_0\left(\frac{\sin\varphi\,(1-\cos\varphi)}{\sqrt{2-\cos^2\varphi}}\right). \qquad (16) \end{aligned}$$

The parametric representative of locus P using angle φ is then $[P_x, P_y]$. We plot the locus $[P_x, P_y]$ together with cardioid and circle when $r_0 = 1$ and $OC = 2$ in Figure 9(b) with the help of *[6]*. If we make the substitution of $t = \tan\frac{\varphi}{2}$, then we can see that $t^4 + 4t^3\cot\theta - 4t^2 - 1 = 0$, which yields

$$\frac{P_x}{a} = \frac{2t^2}{1+t^2} \text{ and } \frac{P_y}{a} = \sin\theta. \qquad (17)$$

The Eq. (17) gives a representative for the locus P in terms of angle θ. The sketch of the locus corresponding to a cardioid is shown using *[2]* in Figure 9(a). We leave it as an exercise to find the maximum area for the triangle ABP.

2.1.1 Possible Real Life Interpretation in 2D

1. A sea rock is similar to the shape of half of a circle. An airplane is flying on the path of C (either a bigger circle, an ellipse or a cardioid that is enclosing the circle). The airplane decides to drop a basket tied to a vertical ladder intending to rescue people standing at a point A on the sea rock. But because of the tides, the sea rock may be covered by various water levels at times. The tides are assumed to be lines parallel to the sea level. We assume those people who need to be rescued from the sea rock may need to swim to the location where the basket is dropped. (a) Find the locus of the rescuing basket. (b) Furthermore, it is decided that the best place the airplane should drop the basket is at the point where the area of the triangle ABP reaches its maximum. Find the place where the airplane should drop the basket.

2. Use your imagination to interpret your real-life scenarios.

2.2 Extensions to 3D scenarios

In view of three cases we described in 2D. We naturally extend the corresponding scenarios to respective 3D. Specifically, we state these scenarios as follows:

Example 8 *We are given two concentric spheres centered at* $O = (0, 0, 0)$ *with radii of* a *and* b *(with* $a < b$*) respectively (see Figure 10 using [4] below. The blue is the unit sphere and the yellow is the sphere of radius* 2*. We are given a moving point* A *on the unit sphere. We extend the ray* OA *to intersect the outer sphere at a point* B*. We project point* B *onto the plane* E *(in purple), that is passing through* A *and is parallel to* xy *plane, at a point* P*. (In other words, the vector* AP *is perpendicular to the normal vector of the plane* E*.) (a) Find the locus for the point* P*. Find the point* B *that will yield the maximum area for the triangle* APB*.*

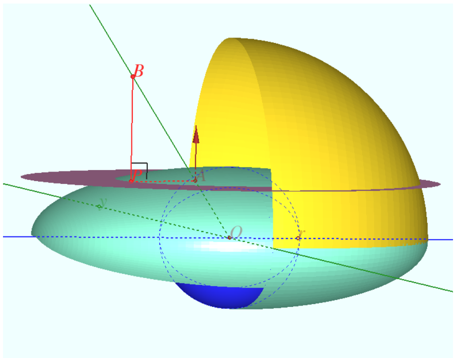

Figure 10 Generating an ellipsoid from two concentric spheres

We write $A = (A_x, A_y, A_z), B = (B_x, B_y, B_z)$, and the locus $P = (P_x, P_y, P_z)$. We use the spherical coordinate system by letting the angle φ to be the angle between OB and the positive $z - axis$, and the angle θ to be the angle between the projection of OB onto xy-plane and the positive $x - axis$. If we let $a = OA$ and $b = OB$, then we see that $B_z = b\cos\varphi$, $B_x = b\sin\varphi\cos\theta$ and $B_y = b\sin\varphi\sin\theta$. We note that $P_z = A_z = a\cos\varphi$, $P_x = B_x = b\sin\varphi\cos\theta$ and $P_y = B_y = b\sin\varphi\sin\theta$. It shows that the locus surface in this case is an ellipsoid of the form

$$\frac{P_x^2}{b^2} + \frac{P_y^2}{b^2} + \frac{P_z^2}{a^2} = 1. \tag{18}$$

We may interpret part (a), finding the locus surface, as one way of constructing an ellipsoid as stated in *[5]*. We construct the locus surface in green as seen in Figure 10 with the help of *[4]*. We leave it as an exercise to find the maximum area for the triangle ABP.

Discussions:

1. We notice that it is not possible to construct an ellipsoid of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with $a \neq b \neq c$ by using only two spheres.

2. The immediate question one may ask is if it is possible to construct an ellipsoid *geometrically* of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ with $a \neq b \neq c$.

3. One may also ask why the parametric equation for an ellipsoid of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ can be expressed as $[x(\theta,\ \varphi),y(\theta,\ \varphi),z(\theta,\ \varphi)]=[a\cos\theta\sin\varphi,b\sin\theta\sin\varphi,c\cos\varphi]$,where $\theta\in[0,2\pi]$ and $\varphi\in[0,\pi]$.

Now it is natural to replace the outer sphere by an ellipsoid and see how the locus surface may vary. In particular, we consider the following:

Exercise 9 *We are given a sphere centered at* $O=(0,0,0)$ *with radii of* r_0*, and an ellipsoid that is centered at* $(0,0,0)$ *and outside the given sphere. The blue is the sphere and the yellow is the ellipsoid). We are given the moving point* A *on the sphere. We extend the ray* OA *to intersect the outer ellipsoid at a point* B*. We project point* B *onto the plane* E *(in purple), that is passing through* A *and is parallel to* xy *plane, at a point* P*. (In other words, the vector* AP *is perpendicular to the normal vector of the plane* E*.) (a) Find the locus for the point* P *(b) Find the point* B *which yields the maximum area for the triangle* APB*.*

We write the parametric equation for the ellipsoid as $[a\cos\theta\sin\varphi,b\sin\theta\sin\varphi,c\cos\varphi]$.We let φ be the angle between $\overrightarrow{AB}$ and positive $z-axis$ and θ be the angle between the projection of OB onto the xy-plane and the positive $x-axis$. Then the locus $[P_x,P_y,P_z]$ can be written as follows:

$$P_x = B_x = x(\theta,\varphi), \tag{19}$$
$$P_y = B_y = y(\theta,\varphi) \text{ and} \tag{20}$$
$$P_z = A_z = r_0\cos\varphi. \tag{21}$$

We leave it as an exercise to find the maximum area for the triangle ABP.

We explore how we can replace the outer ellipsoid by a cardioid surface below:

Example 10 *We are given a sphere centered at* $O=(0,0)$ *with radius of* r_0*, and the cardioid surface* S*, by rotating* $[x(t),y(t)]=[a(1-\cos t)\cos t+a,a(1-\cos t)\sin t]$ *around the* $x-axis$*. Let* A *be a moving point on the sphere. We extend the ray* OA *to intersect the outer cardioid surface at a point* B*. We project point* B *onto the plane* E*, that is passing through* A *and is parallel to* xy *plane, at a point* P*. (a) Find the locus for the point* P *(b) Find the point* B *which yields the maximum area for the triangle* APB*.*

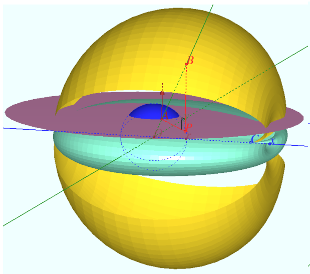

Figure 11(a) A sphere, cardioidal surface and locus

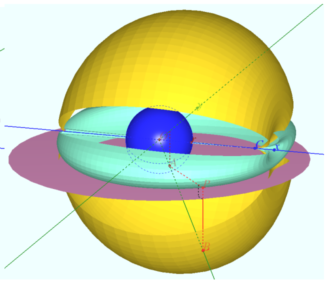

Figure 11(b) Locus surface when point A varies

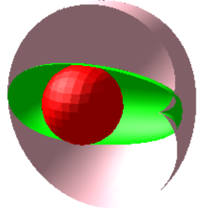

Figure 11(c) Locus generated by MAPLE

As mentioned in Exercise 4, the cardioid surface can be written as $[x(t), y(t)\cos\varphi, y(t)\sin\varphi]$,where $t \in [0, 2\pi]$ and $\varphi \in [0, \pi]$. As we have seen in Example 7 that the locus for $[x(t), y(t)]$ is $[x^*(t), y^*(t)] = \left[a\left(\sin^2 t + \cos t\right), r_0\left(\frac{\sin t(1-\cos t)}{\sqrt{2-\cos^2 t}}\right)\right]$. Due to symmetry, the locus surface for the cardioid surface is $[x^*(t), y^*(t)\cos\varphi, y^*(t)\sin\varphi]$. We plot various views of cardioid surfaces together with spheres and respective locus surfaces in Figures 11(a) and (b) using *[4]. We verify the locus surface analytically with [6] when* $a = 2$ and $r_0 = 1$ in Figure 11(c).

2.2.1 Possible Real Life Interpretation in 3D

1. A sea rock is similar to the shape of half of a small sphere. An airplane is flying on a path of C, which lies on the surface of an ellipsoid or a cardioid surface. We assume the ellipsoidal or cardioid surface is enclosing the sphere. The airplane decides to drop a basket which tied to a vertical ladder to rescue people who are stuck in the sea rock. We assume those people who need to be rescued from the sea rock may need to swim to the location where the basket is dropped. But because of the tides, sea rock will be covered by various of water level at times. The tides are planes that pass through a moving point A on the sea rock and are parallel to the sea level). (a) Find the locus of the rescuing basket. (b) Furthermore, it is decided that the best place the airplane should drop the basket is at the point when the area of the triangle ABP reaches its maximum. Find the place where the airplane should drop the basket.

2. Use your imagination to interpret your real-life scenarios.

Discussions:

We remark that some of these open ended projects in 2D and 3D are excellent choices for students to explore. Examinations alone should not be the sole measurement of a student's success. It will be important to see how a math curriculum includes proper components of exploration with the help of technological tools, where real life applications can be found. In an article (see *[1]*), it is stated that 'Taiwan plans a radical reform of its education system, one aiming to set it apart in East Asia by playing up creativity and student initiative instead of the rote memorization that dominates classroom learning in this part of the world.' While many educators, researchers and parents would applaud this brave and bold initiative. However, how do the government really implement this agenda remains to be seen. It is not how to say the right thing but how to develop strategies to see it through. We outline necessary knowledge for a teacher so technological tools can be integrated in a math curriculum to motivate more students be interested in the STEM (Science, Technology, Engineering and Mathematics) area.

1. Use a DGS to simulate animations in two dimensions.

2. Encourage students to make conjectures through their observations from step 1.

3. Encourage students to verify their results using a CAS for 2D case.

4. Extend students' observations to a 3D scenarios if possible.

5. Prove corresponding results for 3D cases analytically using a CAS if possible.

6. Extend results to finite dimensions or beyond if possible.

3 Conclusions

We turned two static college entrance exam practice problems into interesting exploratory types of problems both in 2D and 3D respectively. We notice that the required mathematical knowledge of those extended 3D problems can be even accessible to high school students if they are familiar with parametric equations of 3D surfaces. However, we do see the need of developing more 3D DGS for visualizing purpose. Allowing users to drag and view figures from different perspectives definitely assists us when we attempt to set up complex algebraic equations.

It is common sense that teaching to the test can never promote creative thinking skills. Furthermore, it will loose potential students who might pursue mathematics related fields in the future. We know that, addressing the importance and timely adoption of technological tools in teaching, learning and research can never be wrong. Therefore, we encourage ATCM communities to continue creating innovative examples by adopting technological tools for teaching and research and to influence their colleagues and communities and the decision makes in their respective countries. We should consider selecting those examples that can be explored from middle to high schools, university levels, or even beyond; and they should be STEM related and link mathematics to real-world applications if possible. Access to technological tools has motivated us to rethink how mathematics should be presented, to make it more interesting, and to reveal its role as a cross disciplinary subject. There is no doubt that these technological tools have helped learners to discover mathematics and to become aware of its applications.

4 Acknowledgements

The author would like to thank Vladimir SHELOMOVSKII for creating several inspiring GInMA dynamic figures. The author also would like to thank Qiuxia Li for editing several examples and providing numerous inspiring discussions about the level of high school math content knowledge from her teaching experiences in China.

References

[1] An article from the Christian Science Monitor at http://www.csmonitor.com/World/Asia-Pacific/2015/0823/Creative-demand-Taiwan-says-radical-school-reform-will-set-it-apart

[2] Geometry Expression, see http://www.geometryexpressions.com/.

[3] Dutch, S, Constructing Ellipses, https://www.uwgb.edu/dutchs/MATHALGO/Ellipses.HTM.

[4] Geometry in Mathematical Arts (GInMA): *A Dynamic Geometry System*, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.

[5] Sir William R. Hamilton, On the Construction of the Ellipsoid by two Sliding Spheres, Proceedings of the Royal Irish Academy, vol. 4 (1850), p. 341–342. https://www.emis.de/classics/Hamilton/Sliding.pdf.

[6] Maple: A product of Maplesoft, see http://maplesoft.com/.

[7] McAndrew, A., Yang, W.-C. (2016). Locus and Optimization Problems in Lower and Higher. The Electronic Journal of Mathematics and Technology, 10(2), 69-83. https://php.radford.edu/~ejmt/deliveryBoy.php?paper=eJMT_v10n2p1.

[8] Shi GI Gin Bang 2016, Strategies for high school complete review, ISBN 978-7-5634-5289-701, published by YanBian University Publishing company.

[9] Yang, W.-C. See Graphs. Find Equations. Myth or Reality? (pp. page 25-38). Proceedings of the 20th ATCM, the electronic copy can be found at this URL: http://atcm.mathandtech.org/EP2015/invited/2.pdf, ISBN:978-0-9821164-9-4 (hard copy), ISSN 1940-4204 (online version), Mathematics and Technology LLC.