Exploring generalizations of a result about cubic polynomials

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Abstract

For a real cubic function with three distinct roots, the tangent at the mean of any two of them passes through the third. This elegant and elementary result seems to have first surfaced as an investigation for students studying the International Baccalaureate. There are number of ways for generalizing this result; in this article we look at two of them: higher degree polynomials, and polynomials over the complex numbers, and the quaternions. Thus we show how the result can be defined in considerable generality over general domains. This may be considered as a case study in the use of a CAS to generalize a simple result.

1 A simple result about cubic polynomials

The following result is known:

> Let $p(x)$ be a cubic polynomial. For any two roots x_1 and x_2 of p, the tangent to p at $(x_1 + x_2)/2$ crosses the x-axis at the other root $x = x_3$.

Figure 1 shows this. Note that although the figure shows the result for three real and unequal roots, the result is in fact more general. One full treatment is given by Jean-Jacques Dahan [4], who credits the problem to a mathematical investigation as part of the International Baccalaureate program.

We first give a simple proof. Note that the x intercept of the tangent to a function $f(x)$ at $x = a$ is given by

$$a - \frac{f(a)}{f'(a)}$$

as we know from Newton's method. Also, if $p(x) = k(x - x_1)(x - x_2)(x - x_3)$, then

$$p'(x) = k\Big((x - x_2)(x - x_3) + (x - x_1)(x - x_3) + (x - x_1)(x - x_2)\Big)$$

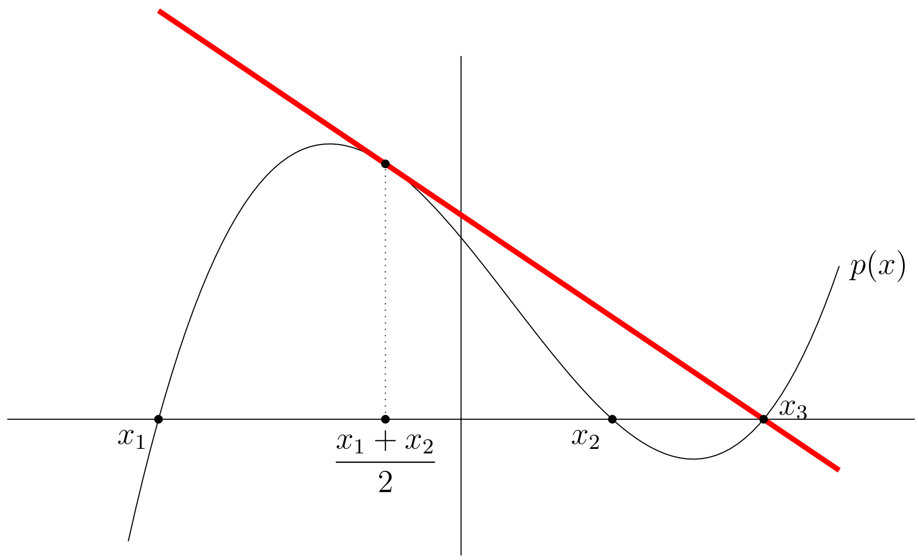

Figure 1: A cubic with a tangent

and so

$$\frac{p'(x)}{p(x)} = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \frac{1}{x - x_3}.$$

Consider a tangent at $(k, p(k)$ which goes through x_3. We have

$$k - \frac{f(k)}{f'k} = x_3$$

or

$$\begin{aligned} \frac{1}{k - x_3} &= \frac{f'(k)}{f(k)} \\ &= \frac{1}{k - x_1} + \frac{1}{k - x_2} + \frac{1}{k - x_3} \end{aligned}$$

This means that

$$\frac{1}{k - x_1} + \frac{1}{k - x_2} = 0$$

or that

$$k = \frac{x_1 + x_2}{2}.$$

Note that this can be also done using a Computer Algebra System; for ease of explanation we shall use Axiom [1], an open source package, which we first introduce.

2 Axiom

Axiom is the current open-source descendant of a powerful system called ScratchPad, which was developed at IBM in the late 1960s and 1970s. It seems to be little used, and in part this is because of a very old-fashioned interface. Another argument militating against it is the different forks—owing to some disagreements between the lead developers, there are three forks: Axiom, which is the original; FriCAS, which is possibly the most actively developed; and OpenAxiom.

Axiom's great strength is its *strong typing*. Every object in Axiom has a type, which may be considered to be a computational domain which defines what operations are possible. Basic types include integers, matrices, polynomials, fields, and many others. Types can also be nested, so that for example you could define an object to be a matrix over the field of polynomials modulo 17.

Here are a few examples of Axiom, first factoring a polynomial over different number fields.

```
(1) -> p := x^6+x^4+x^2+1
```

$$x^6 + x^4 + x^2 + 1$$

Type: `Polynomial(Integer)`

```
(2) -> factor(p)
```

$$(x^2 + 1)(x^4 + 1)$$

Type: `Factored(Polynomial(Integer))`

```
(3) -> factor(p::POLY COMPLEX FRAC INT)
```

$$(x - \%i)(x + \%i)(x^2 - \%i)(x^2 + \%i)$$

Type: `Factored(Polynomial(Complex(Fraction(Integer))))`

Furthermore, we can adjoin a single element to the base field by giving a minimal polynomial. For example, suppose a is a root of $x^2+2=0$, and let $\mathbb{Q}[a]$ be the extension field of $\mathbb{Q}$ obtained by adjoining a. We can factorize over this field, too:

```
(4) -> aa:=rootOf(aa^2+2)
```

$$aa$$

Type: `AlgebraicNumber`

```
(5) -> factor(p,[aa])
```

$$(x^2 - aa\,x - 1)(x^2 + 1)(x^2 + aa\,x - 1)$$

Type: `Factored(Polynomial(AlgebraicNumber))`

We can in fact apply the same technique as above to the complex numbers, by extending the field $\mathbb{Q}$ by i and using `zeroOf` instead of `rootOf` to obtain a result using radicals:

```
(6) -> )cl pr aa
(7) -> aa:=zeroOf(aa^2+2)
```

$$\sqrt{-2}$$

Type: `AlgebraicNumber`

```
(8) -> bb:=zeroOf(bb^2+1)
```

$$\sqrt{-1}$$

Type: `AlgebraicNumber`

```
(9) -> factor(p,[aa,bb])
```

$$\left(x + \frac{-\sqrt{-2}\sqrt{-1} - \sqrt{-2}}{2}\right)\left(x + \frac{-\sqrt{-2}\sqrt{-1} + \sqrt{-2}}{2}\right)\left(x - \sqrt{-1}\right)\left(x + \sqrt{-1}\right)$$
$$\left(x + \frac{\sqrt{-2}\sqrt{-1} - \sqrt{-2}}{2}\right)\left(x + \frac{\sqrt{-2}\sqrt{-1} + \sqrt{-2}}{2}\right)$$

Type: `Factored(Polynomial(AlgebraicNumber))`

Axiom supports the full Risch algorithm for symbolic integration of elementary functions (the only open source system to do so), and can solve all of the integrals given by Charlwood [2], for example:

```
(10) -> integrate(asin(sqrt(x+1)-sqrt(x)),x)
```

$$\frac{(3\sqrt{x+1}+\sqrt{x})\sqrt{2\sqrt{x}\,\sqrt{x+1}-2x}+(8x+3)\operatorname{asin}(\sqrt{x+1}-\sqrt{x})}{8}$$

Type: `Union(Expression(Integer),...)`

Finally, a bit of linear algebra: a matrix and the Cayley-Hamilton theorem, starting with a little command to produce random numbers:

```
(11) -> r() == randnum(10)+5
```

Type: `Void`

```
(12) -> M := matrix [[r() for i in 1..3] for j in 1..3]
```

$$\begin{pmatrix} 10 & 5 & 9 \\ 5 & 7 & 5 \\ 6 & 7 & 13 \end{pmatrix}$$

Type: `Matrix(Integer)`

```
(13) -> cp := characteristicPolynomial(M,x)
```

$$-x^3+30x^2-177x+322$$

Type: `Polynomial(Integer)`

```
(14) -> eval(cp, x = M)
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Type: `Polynomial(SquareMatrix(3,Integer))`

Axiom applied to cubic polynomials

We start off by defining a generic cubic polynomial function and its derivative:

```
(15) -> p(x)==k*(x-a)*(x-b)*(x-c)
```

Type: `Void`

Without loss of generality, take a to be the root, and define the mean of the others to be s:

```
(16) -> s:=(b+c)/2
```

$$\frac{1}{2}c+\frac{1}{2}b$$

Type: `Polynomial(Fraction(Integer))`

Now define the derivative of p (using Axiom's operator `D` for the derivative), and apply Newton's iteration to it at s:

```
(17) -> function(D(p(x),x),pd,x)
```

$$pd$$

Type: `Symbol`

```
(18) -> y:=s-p(s)/pd(s)
```

$$a$$

Type: `Fraction(Polynomial(Fraction(Integer)))`

3 Higher order polynomials

We now consider a generalization to higher order polynomials. We first observe that for an n-th order polynomial $p(x)$, with roots (which for the moment we shall assume to be real and distinct) $x_1, x_2, \ldots, x_n$, that

$$\frac{p'(x)}{p(x)} = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \cdots + \frac{1}{x - x_n}.$$

By applying the above reasoning, if we have a tangent to the curve $y = p(x)$ at $(a, p(a))$ which pass through a root x_m on the x-axis, then

$$\sum_{\substack{i=1 \\ i \neq m}}^{n} \frac{1}{a - x_i} = 0.$$

However, this expression is equal to

$$\frac{p'(a)}{p(a)} - \frac{1}{a - x_m}.$$

Alternatively, if

$$q(x) = \frac{p(x)}{x - x_m}$$

then

$$\frac{q'(x)}{q(x)} = \frac{p'(x)}{p(x)} - \frac{1}{x - x_m}.$$

Putting all this together, we have the following generalization of the initial result about cubics:

> Suppose $p(x)$ is an n-the degree polynomial, with n distinct real roots. Then a tangent through $(a, p(a))$ will pass through a root x_m on the x-axis if and only if $q'(a) = 0$, where $q(x) = p(x)/(x - x_m)$.

To see this in relation to cubics, suppose $p(x) = A(x - x_1)(x - x_2)(x - x_3)$. Let $x_m = x_3$. Then $q(x) = A(x - x_1)(x - x_2)$, and $q'(x) = A(2x - (x_1 + x_2))$. So if $q'(a) = 0$ then $a = (x_1 + x_2)/2$.

As an example of a higher order polynomial, suppose $p(x) = (x + 2)(x + 1)(x - 1)(x - 3)$, and let $x_m = 3$. Then $q(x) = (x + 2)(x + 1)(x - 1)$ and so $q'(x) = 3x^2 + 4x - 1$. The roots of $q'(x)$ are

$$a_1 = \frac{-\sqrt{7} - 2}{3}, \quad a_2 \frac{\sqrt{7} - 2}{3}.$$

Figure 2 shows the tangents through each $(a_i, p(a_i))$.

Now let's try with one of the "inner roots", say $x = 1$. In this case $q(x) = x^3 - 7x - 6$ with roots

$$a_1 = -\sqrt{\frac{7}{3}}, \quad a_2 = \sqrt{\frac{7}{3}}.$$

and the result is shown in figure 3. It is clear that for an n-th degree polynomial, with n distinct real roots, there will be $n - 2$ tangents to its curve passing through any given root.

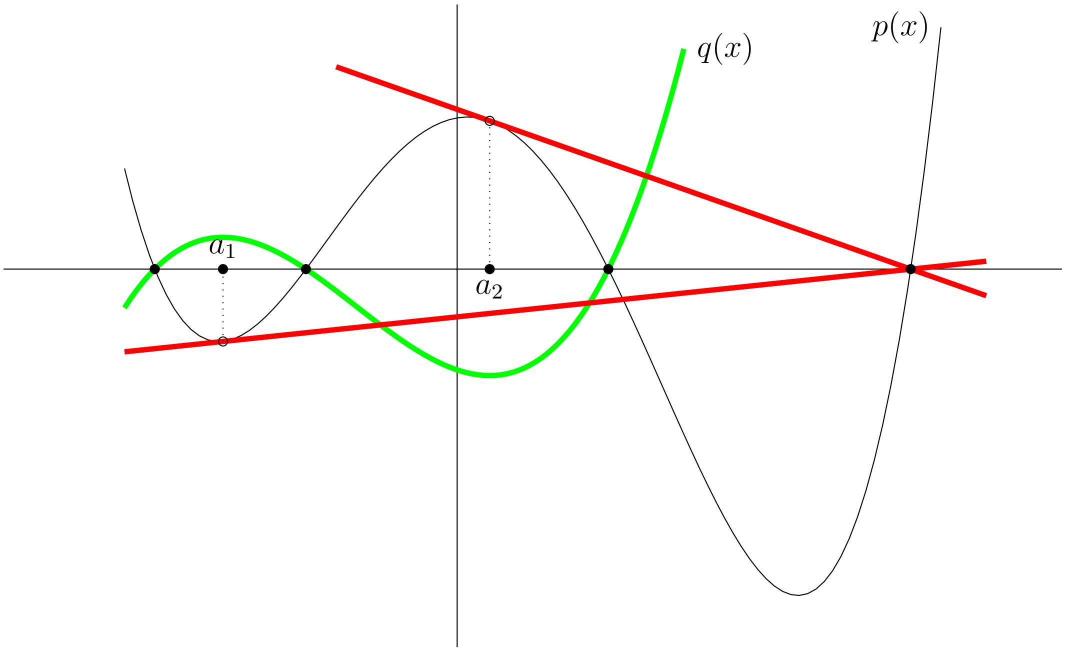

Figure 2: Tangents to a quartic passing through a root

Polynomials over the field of complex numbers

We now show that the previous results can be extended to polynomials defined on the complex numbers. Note that there is in fact nothing in the previous discussion which requires our polynomials to be defined over the real numbers only. We might not be able to draw the curve of a polynomial $p : \mathbb{C} \to \mathbb{C}$, but the same operations as above can be applied.

For example, suppose

$$p(z) = (z - 2 + 3i)(z - 1 - i)(z + 2 - i)(z + 1 + 2i)$$

and choose $r = 2 - 3i$. Then

$$\begin{aligned} q_r(z) &= (z - 1 - i)(z + 2 - i)(z + 1 + 2i) \\ &= z^3 + 2z^2 + (2 - i)z - 1 - 7i \end{aligned}$$

and so

$$q'(z) = 3z^2 + 4z^2 + 2 - i.$$

We find that the roots of q' are

$$\begin{aligned} a_1 &= \frac{\sqrt{-2 + 3i} - 2}{3} \approx -0.368008 + 0.55805i \\ a_2 &= \frac{-\sqrt{-2 + 3i} - 2}{3} \approx -0.965326 - 0.55805i \end{aligned}$$

and so using a_1, we find that

$$a_1 + \frac{p(a_1)}{p'(a_1)} = 2 - 3i.$$

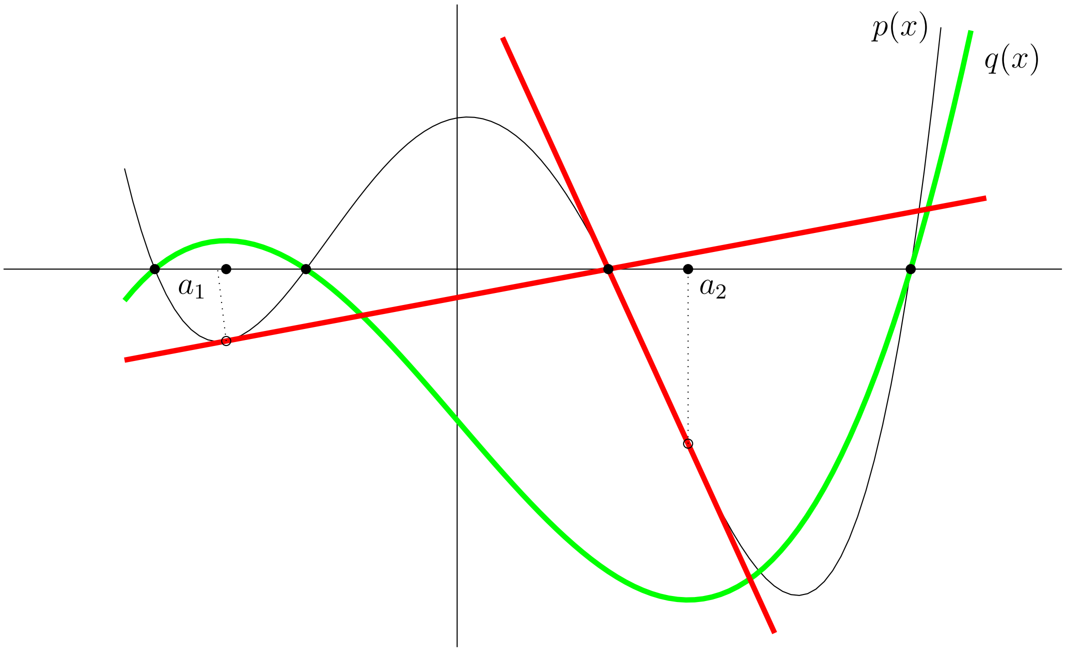

Figure 3: Tangents to a quartic passing through another root

This is best shown using Axiom, with univariate polynomials over the field of complex numbers:

```
(19) -> p:UP(z,Complex FRAC INT)
```

Type: Void

```
(20) -> p:=(z-2+3*%i)*(z-1-%i)*(z+2-%i)*(z+1+2*%i)
```

$$z^4+3iz^3+(-2+5i)z^2+(-2+i)z+23+11i$$

Type: UnivariatePolynomial(z,Complex(Fraction(Integer)))

```
(21) -> r:=2-3*%i
```

$$2-3i$$

Type: Complex(Integer)

```
(22) -> q:UP(z,Complex FRAC INT):=p/(z-r)
```

$$z^3+2z^2+(2-i)z-1-7i$$

Type: UnivariatePolynomial(z,Complex(Fraction(Integer)))

```
(23) -> pd:=D(p,1)
```

$$4z^3+9iz^2+(-4+10i)z-2+i$$

Type: UnivariatePolynomial(z,Complex(Fraction(Integer)))

```
(24) -> qd:=D(q,1)
```

$$3z^2+4z+2-i$$

Type: UnivariatePolynomial(z,Complex(Fraction(Integer)))

```
(25) -> qs:=radicalSolve(qd=0,z)
```

$$\left[z=-\frac{1}{2}\sqrt{-\frac{8}{9}+\frac{4}{3}i}-\frac{2}{3}, z=\frac{1}{2}\sqrt{-\frac{8}{9}+\frac{4}{3}i}-\frac{2}{3}\right]$$

Type: List(Equation(Expression(Complex(Fraction(Integer)))))

```
(26) -> a1:=rhs(qs.1)
```

$$-\frac{1}{2}\sqrt{-\frac{8}{9}+\frac{4}{3}i}-\frac{2}{3}$$

```
                                        Type: Expression(Complex(Fraction(Integer)))
(27) -> a1-subst(p,z=a1)/subst(pd,z=a1)
```

$2 - 3i$

```
                                        Type: Expression(Complex(Fraction(Integer)))
```

and we see that the result is the root we chose at the start. We are now in a position to state a general result for polynomials with distinct roots:

> Let $p(z) : \mathbb{C} \to \mathbb{C}$ be an n-th order polynomial, with n distinct roots. For a chosen root r, define $q_r(z) = p(z)/(z - r)$. Then for any root s of the derivative $q_r'(z)$, we have
>
> $$s - \frac{p(s)}{p'(s)} = r.$$

The proof is the same as the previous proof given for real polynomials.

Polynomials over quaternions

We now show that the result can be generalized even further, to polynomials defined over the ring of quaternions, as long as the roots commute by multiplication—recall that if q and r are quaternions, then in general $qr \neq rq$. We define the quaternions to be objects which have the form

$$a + bi + cj + dk$$

where $a, b, c, d \in \mathbb{R}$, and

$$i^2 = j^2 = k^2 = ijk = -1.$$

Note that this last expression implies that $ij = -ji$, $ik = -ki$, $jk = -kj$, hence the non-commutativity of quaternion multiplication. The ring of quaternions is denoted $\mathbb{H}$. To read more about quaternions, a nice elementary view is provided by Tait [5] and a more modern view by Conway and Smith [3].

Suppose that $a, b, c \in \mathbb{H}$ are all multiplicatively commutative, so that $ab = ba, ac = ca, bc = cb$. This can be ensured if each of a, b, c is an integer power of a single quaternion, so that $a = u^m$, $b = u^n$, $c = u^k$, for some quaternion u and integers m, n, k. Alternatively, we could fix real numbers u, v and w, and define

$$a = a_1 + m(ui + vj + wk), \quad b = b_1 + n(ui + vj + wk), \quad c = c_1 + k(ui + vj + wk)$$

where a_1, b_1, c_1 are any real numbers.

Consider the polynomial

$$p(x) = (x - a)(x - b)(x - c)$$

where $a, b, c \in \mathbb{H}$ all commute, and $p : \mathbb{H} \to \mathbb{H}$. Define

$$p'(x) = 3x^2 - 2(a + b + x)x + (ab + bc + ca).$$

If $s = (b+c)/2$ then

$$s - p(s)\left(p'(s)\right)^{-1} = a.$$

We can check this again in Axiom, starting by clearing all currently defined variables, and then defining three quaternions a, b, c:

```
(28) -> )cl pr all
                                                                Type: Void
(29) -> u:QUAT FRAC INT:=quatern(2,1,-2,-1)
```
$$2 + i - 2j - k$$
```
                                          Type: Quaternion(Fraction(Integer))
(30) -> a:=3+2*(u-real(u))
```
$$3 + 2i - 4j - 2k$$
```
                                          Type: Quaternion(Fraction(Integer))
(31) -> b:=-2-3*(u-real(u))
```
$$-2 - 3i + 6j + 3k$$
```
                                          Type: Quaternion(Fraction(Integer))
(32) -> c:=1-5*(u-real(u))
```
$$1 - 5i + 10j + 5k$$
```
                                          Type: Quaternion(Fraction(Integer))
(33) -> p(x)==(x-a)*(x-b)*(x-c)
                                                                Type: Void
```

Now we can defined the derivative in terms of the coefficients of $p(x)$:

```
(34) -> cs:=coefficients(p(x));
                                    Type: List(Quaternion(Fraction(Integer)))
(35) -> pd(x)==3*x^2+2*cs.2*x+cs.3
                                                                Type: Void
```

Now we can choose any root; without loss of generality choose $r = a$, and so $s = (b+c)/2$:

```
(36) -> s:=1/2*(b+c)
```
$$-\frac{1}{2} - 4i + 8j + 4k$$
```
                                          Type: Quaternion(Fraction(Integer))
(37) -> y:=s-p(s)*(pd(s))^(-1)
```
$$1 - 5i + 10j + 5k$$
```
                                          Type: Quaternion(Fraction(Integer))
(38) -> test(y=a)
```
true
```
                                                             Type: Boolean
```

Although the above example is for cubics, it could be extended to higher powered polynomials; the main difficulty being computing the roots of polynomials over the quaternions. However, roots of quaternions can be computed by a method very similar to that of complex numbers, and also using de Moivre's theorem.

4 Polynomials with repeated roots

We now consider the case where not all roots are distinct. Note that if r is a repeated root, then $p'(r) = 0$ and so the expression

$$r - \frac{p(r)}{p'(r)}$$

will be undefined.

For example, consider the polynomial

$$p(x) = (x-1)^2(x-2)(x-3)$$

and choose $r = 2$. Then $q(x) = (x-1)^2(x-3)$, and we find that the roots of $q'(x)$ are $7/3$ and 1. If $s = 7/3$ then

$$s - \frac{p(s)}{p'(s)} = r.$$

Clearly we need to avoid using a root a of $q_r'(x)$ which is a repeated root of $q(x)$ and hence of $p(x)$.

The result we want can be thus written as

> Let $p(z) : \mathbb{C} \to \mathbb{C}$ be an n-th order polynomial. For a chosen root r, define $q_r(z) = p(z)/(z-r)$. Let s be any root of $q_r'(z)$ which is *not* a repeated root of $q_r(z)$. Then we have
>
> $$s - \frac{p(s)}{p'(s)} = r.$$

For example, suppose

$$p(x) = (x-1)^3(x-2)^3.$$

Choose $r = 1$, so that $q(x) = (x-1)^2(x-2)^3$. Then

$$q'(x) = (x-2)^2(x-1)(5x-7)$$

and so the only root we can choose for s is $7/5$.

In general, suppose that

$$p(x) = (x-a)^m(x-b)^n f(x)$$

where $f(x)$ is a polynomial none of whose roots are equal to a or b. Pick $r = a$, so that

$$q_r(x) = (x-a)^{m-1}(x-b)^n f(x)$$

and

$$q_r'(x) = (x-a)^{m-2}(x-b)^{n-1}t(x)$$

where

$$t(x) = (m-1)(x-b)f(x) + n(x-a)f(x) + (x-a)(x-b)f'(x).$$

Note that

$$t(a) = (m-1)(a-b)f(a), \quad t(b) = n(b-a)f(b)$$

both of which are non-zero since $a \neq b$ and neither a nor b are roots of $f(x)$. This means that if $t(x)$ has a real root, it will be different from a and b.

For example, take $p(x) = (x-1)(x-2)^3$ and $r = 1$. Since $q_r(x) = (x-2)^3$ then its only root is repeated, and hence there is no tangent which passes through the x-axis at $x = r$, except for the tangent $y = 0$. However, if $r = 2$ so that $q_r(x) = (x-1)(x-2)^2$, the roots of $q_r'(x)$ are $x = 4/3$ and $x = 2$. The only non-repeated root is $x = 4/3$, which gives us a tangent, as figure 4 shows.

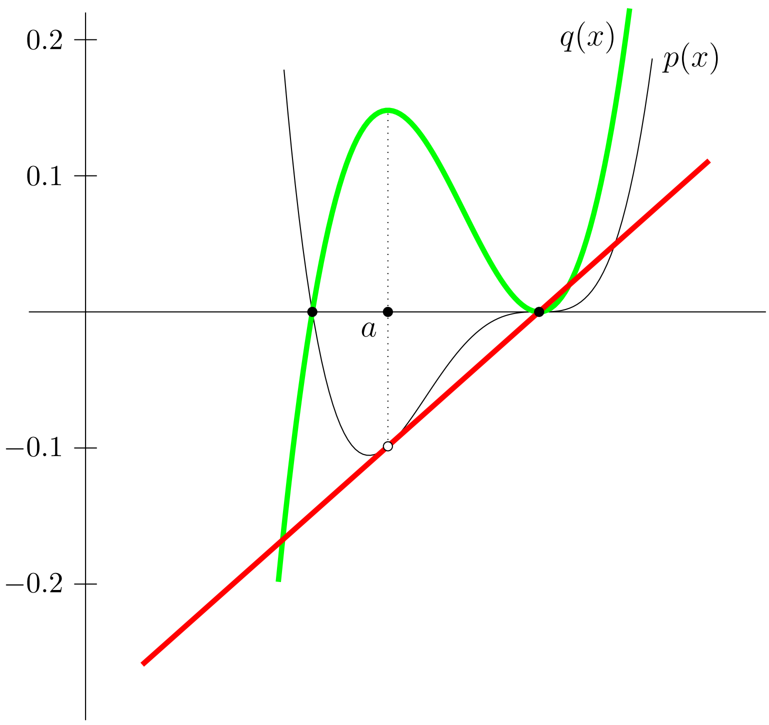

Figure 4: A case with repeated roots

From the above discussion, we know that if $p(x) = (x-a)(x-b)^n$ then there is no tangent line to $p(x)$ through the x-axis at $x = a$ (aside from the x-axis itself). However, if $r = b$ then there will be exactly one tangent at

$$s = \frac{a(n-1)+b}{n} = a + \frac{b-a}{n}.$$

If $p(x) = (x-a)^n(x-b)^m$, with each $m, n \geq 2$, then for $r = a$, say, the roots of $q_r'(x)$ can be found to be

$$\frac{b(n-a)+am}{n+m-1}, \quad a, \quad b.$$

Since a and b correspond to repeated roots of $p(x)$, the only tangent which passes through the root at $x = a$ is at

$$s = \frac{b(n-a) + am}{n+m-1}.$$

Note that this corresponds to the previous discussion of the function $p(x) = (x-a)^n(x-b)^m f(x)$, here with $f(x) = 1$.

5 Conclusions

We have seen how a very simple result, easily within the reach of secondary school students, can be made more interesting by generalizing it over extended number systems. This can be made easier with the use of a computer algebra system, and in fact these generalizations make for a elegant computational exercise. Exploring these results, with an appropriate computer system—or hand held calculator with enough power—would make for a possible extension or outreach activity.

As well as the mathematics itself, this shows the power of generalization in mathematics. Students (and their teachers) may wish to explore further, either along the lines discussed here, or in the previous article by Dahan [4].

References

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