

# Two examples on discrepancy potentials of dynamic geometry software

Yip-Cheung Chan

[mathchan@cuhk.edu.hk](mailto:mathchan@cuhk.edu.hk)

Department of Curriculum and Instruction  
The Chinese University of Hong Kong  
Hong Kong

## Abstract

*It is unavoidable that a computer program has its constraints. In a mathematics pedagogical program such as dynamic geometry software (DGS), the constraints may lead to deviations of mathematical representations from its intended mathematical concepts. If it is used properly, the deviations can open up pedagogical opportunities and hence may be called as discrepancy potentials of the tool. In this paper, examples of developing mathematical discussions by using the discrepancy potentials in two DGS tasks are discussed. It aims at elaborating the idea of discrepancy potentials in the design principles for effective DGS-based tasks. This paper relates to the topic "Mathematics Education using Information Communication Technology".*

## 1. Introduction

Dynamic geometry software (DGS) is a useful tool for manipulation of mathematical objects. In the process of manipulation, experimental aspect and theoretical aspect of mathematics are interplayed ([1]). The interplay can have potentials to enhance understanding of mathematical concepts and statements. Therefore, an important pedagogy research agenda is to investigate design principles for effective DGS-based tasks which can optimize the potential of experimental-theoretical interplay. One of the features of a tool-based task is that the mathematical representation produced by the tool (no matter it is software tool or concrete tool) can be different from the mathematical object itself. If we interpret a tool-based task as an experimentation of mathematical representation for theorizing mathematical objects, we may regard this discrepancy as a kind of experimental-theoretical interplay in tool-based tasks. The notion of discrepancy potential which is initiated by [2] and further elaborated in [3] and [4] highlights the pedagogical potentials of the discrepancy in tool-based tasks. It can be described as follows:

The discrepancy potential of a tool is a pedagogical space generated by (i) feedback due to the nature of the tool or design of the task that possibly deviates from the intended mathematical concept or (ii) uncertainty created due to the nature of the tool or design of the task that requires the tool users to make decisions. ([3], p. 212)

As evidenced from a tool-based mathematical task, such kind of discrepancy has potential to initiate meaningful mathematics discussion which could lead to deep understanding in mathematical concepts ([2] and [4]). In this paper, this claim will be further elaborated through two cases of teaching tasks by using GeoGebra (a kind of DGS). It highlights the importance of incorporating mathematical discussions based on the discrepancy embedded in DGS.

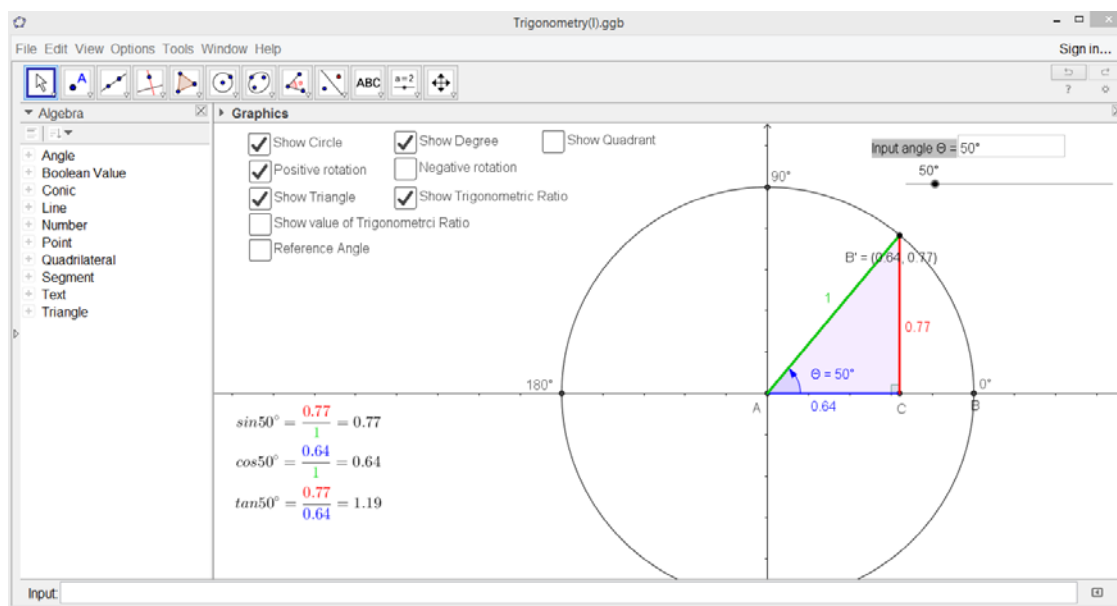
## 2. The Context

The two cases discussed in this paper are initially designed by two pre-service mathematics teachers Jane and Peter (pseudonyms) respectively. They were end-of-term assignments of a mathematics pedagogy course which focuses on integrating computer technologies in mathematics

classrooms. The pre-service teachers were asked to design a self-contained teaching capsule of a mathematics topic with GeoGebra activities incorporated. In these two cases, there were some technical errors in the GeoGebra files. However, the present author (their course instructor) thinks that these technical errors can be used to enhance deep mathematical understanding. In the following sections, the designs and the technical errors of these two files will be described. Then, the author will elaborate how he would use these technical errors to initiate mathematical discussions. These two cases will illustrate the idea of discrepancy potentials in DGS.

### 3. The Case 1: Trigonometric functions

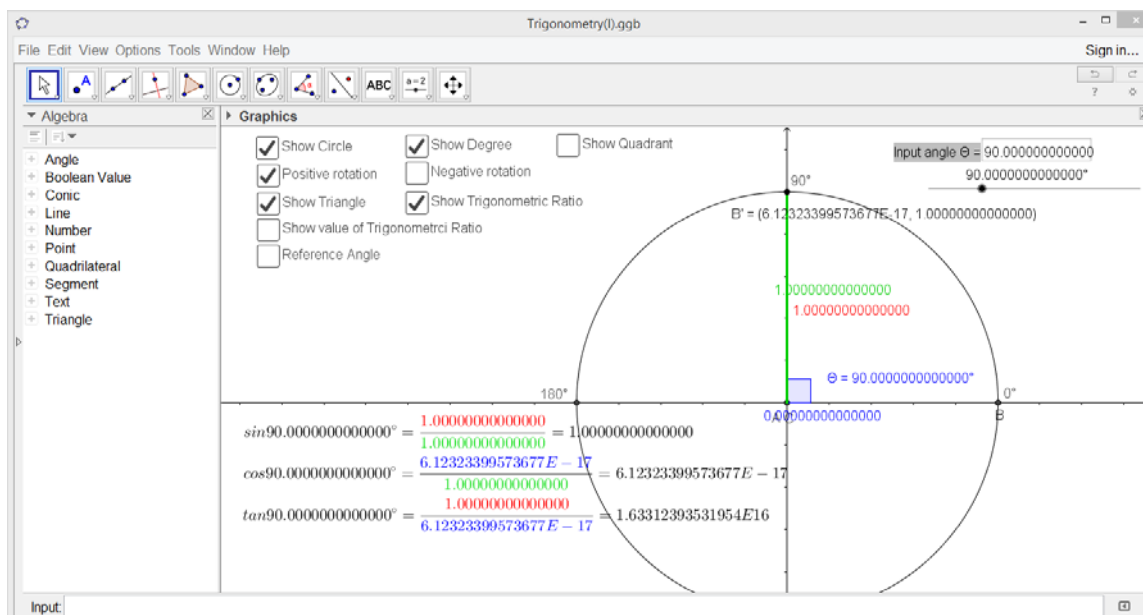
The topic of Jane's lesson design was trigonometric functions. She created a GeoGebra file to guide the students understand the definition of trigonometric ratios of angles in a rectangular plane and explore the signs of the trigonometric ratios and the reference angles. The GeoGebra file contains the standard unit circle representation of trigonometric functions. The input box and the slider enable the students to control the value of  $\angle\theta$ . As showed in figure 3.1,  $A = (0,0)$  is the center of the unit circle.  $B = (1,0)$  is the intersection point of  $x$ -axis and the circle.  $B'$  is the rotated point of Point  $B = (1,0)$  by  $\angle\theta$ . Point  $C$  is the perpendicular foot of  $B'$  on the  $x$ -axis. The auxiliary triangle  $B'CA$  defines the trigonometric functions of  $\angle\theta$ :  $\sin\theta = b = y(B')$  (that is, the  $y$ -coordinate of  $B'$ );  $\cos\theta = a = x(C)$  (that is, the  $x$ -coordinate of  $C$ ); and  $\tan\theta = \frac{b}{a}$ .



**Figure 3.1** visual representation of the definitions of trigonometric functions by coordinates

This dynamic figure provides a good visual representation of the definitions of trigonometric functions by coordinates. However, if we set  $\angle\theta = 90^\circ$  (by inputting the value to the input box or dragging the slider to this value), we would find that  $\sin\theta = 1$  and  $\cos\theta = 0$  which are correctly displayed; but  $\tan\theta$  displays a very large number (instead of the symbol  $\infty$ ). This phenomenon is caused by the rounding-off error of the software. If we set rounding as 15 significant figures, we will see that  $\cos\theta$  is a very small number (instead of 0) and hence  $\tan\theta$  is a very large number

(Figure 3.2). At this moment, we would be able to understand the source of the mistake, namely, point  $B'$  is rotated by an angle approximately equal to  $90^\circ$  (but not exact). This kind of error is frequently found in computer software. We may call it a “discrepancy potential”. The term “potential” implies that the discrepancy may enhance students’ conceptual understanding if the teacher can make use of it to initiate meaningful mathematics discussions.



**Figure 3.2** The representation of trigonometric functions at 90 degrees

In the following, we will propose how such kind of mathematics discussions can be arranged. First, with the rounding set to the default value (2 decimal places) and  $\theta$  set to  $90^\circ$ , the teacher can

point out that it shows  $\tan 90^\circ = \frac{1}{0} = 16331239353195370$  and ask the students whether they think

that it is correct. The teacher may remind the students that  $\tan \theta = \frac{\sin \theta}{\cos \theta}$  and then prompt the

students to think about the result when 1 is divided by 0. The latter problem is known to be a difficult question for school students. The teacher may ask them to multiply 0 on both sides. They should be able to realize the absurdity. Yet, the teacher can further discuss with the students the product of a very large number multiplied by 0. It may lead to discussion on the intuitive meaning of  $\infty \times 0$ . After that, the teacher may discuss with the students why a large number is displayed for  $\tan 90^\circ$ . Then, the rounding may be set to a more precise value, say 15 significant figures. As such,  $x(C)$  becomes a very small number (instead of zero) and  $\tan 90^\circ$  becomes

$\frac{1}{\text{small number}} = \text{large number}$ . (See figure 3.2 above.) The teacher may further discuss with the

students the product of a small number  $\times$  a large number, which would provide an intuitive idea of the notion of indeterminate in pre-calculus. Finally, the teacher may reset the rounding to 10 decimal places and reset the slider of  $\theta$  as interval from  $89^\circ$  to  $91^\circ$  and increment as  $0.00001^\circ$ . Then, the teacher may drag the slider of  $\theta$  slowly and ask the students to observe how the values of the trigonometric ratios,  $y(B')$  and  $x(C)$  change. It is expected that the students would realize the

sign of  $\tan \theta$  changes at  $90^\circ$ , and there is a rapid increase on the magnitude of  $\tan \theta$  when  $\theta$  is moved towards  $90^\circ$  and then a rapid decrease on the magnitude of  $\tan \theta$  when  $\theta$  is moved away  $90^\circ$  (Figure 3.3). Thus, the behaviour of tangent function near  $\theta = 90^\circ$  can be discussed. It provides a dynamic illustrative example to the concept of limit of a function. The above discussion extends the learning objective from the definitions of trigonometric functions to the behaviours of the functions, and hence broadens the learning area from trigonometry to calculus. This extension seems to be impossible without the mistake opportunity provided by the software.

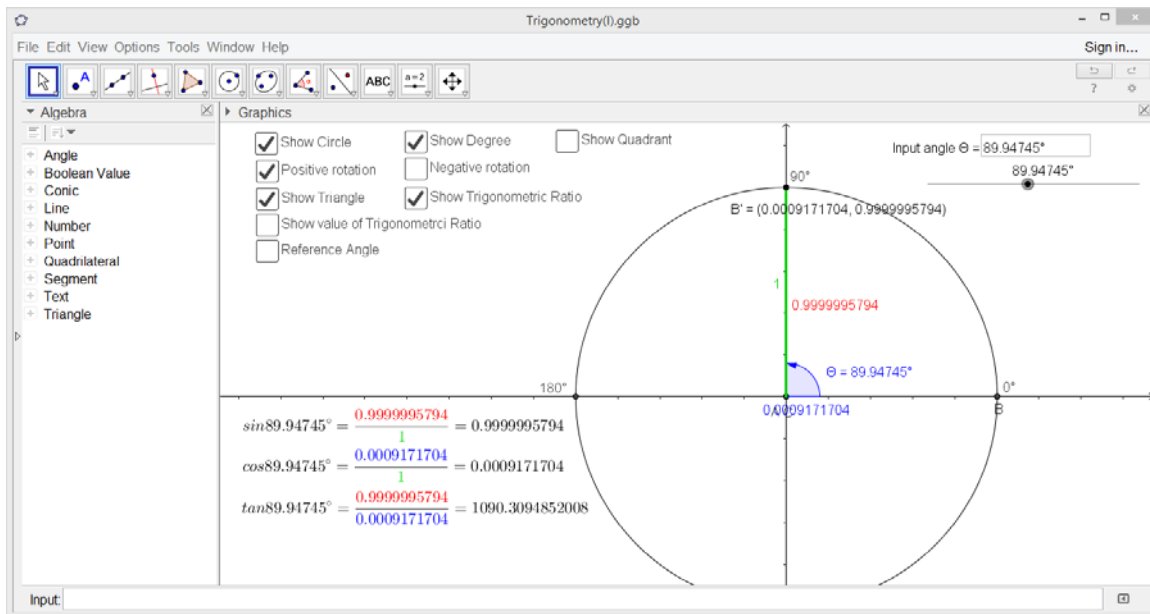
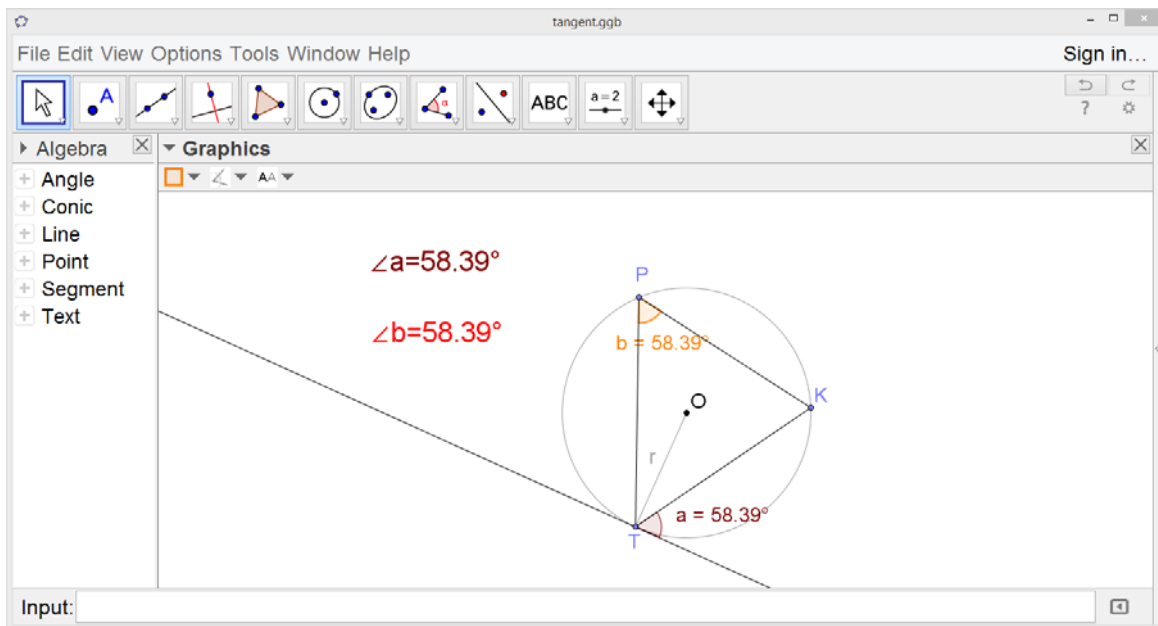


Figure 3.3 Zooming the behaviour of tangent function near 90 degrees

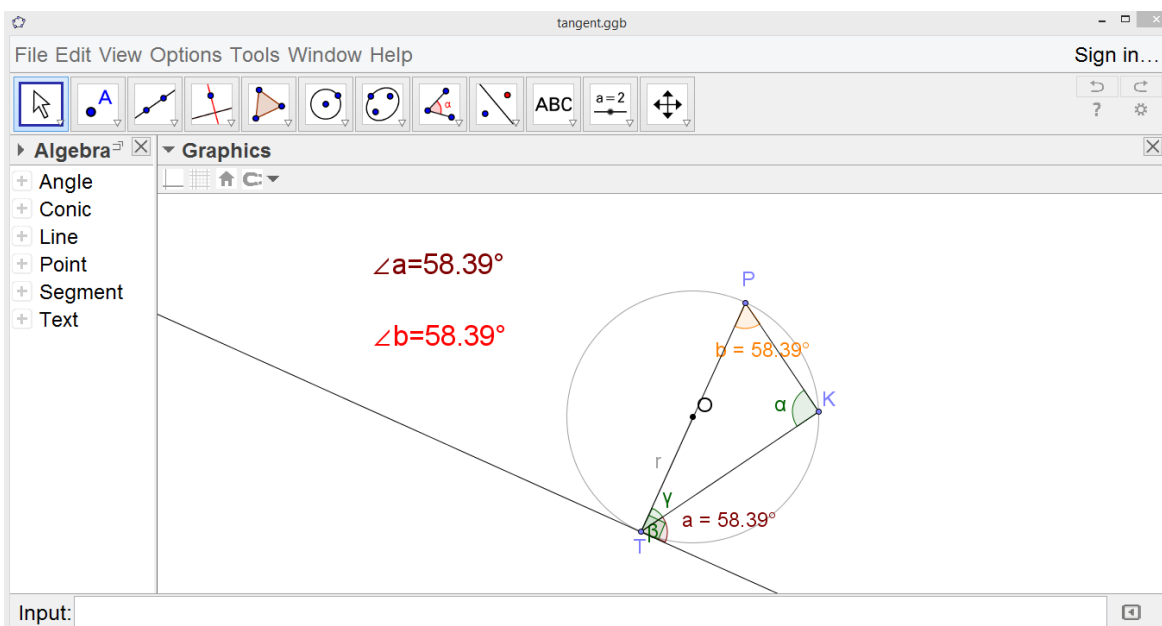
#### 4. Case 2: Tangent to a circle

There are many properties related to tangent to a circle. Some of them are not easy to be understood. One of the properties is that a tangent-chord angle of a circle equals to an angle in the alternate segment (usually abbreviated as: theorem of angle in alternate segment). Peter constructed a GeoGebra figure to illustrate this property (figure 4.1). While dragging point  $P$  along the circle,  $\angle a$  (tangent-chord angle) keeps equal to  $\angle b$  (angle in the alternate segment). However, problem arises when point  $P$  crosses point  $K$ . In that case, the orientation of the triangle is changed and hence  $\angle a$  is no longer equal to  $\angle b$ . Indeed, this phenomenon which is a discrepancy potential highlights a common misconception of the students. For a tangent to the circle and a chord, there are two tangent-chord angles in which they are complement to each other (that is, their sum equals to  $180^\circ$ ). Students may have difficulty in identifying which tangent-chord angle corresponding to which angle in the alternate segment. The discrepancy due to the orientation change of the triangle can enhance students' abilities in identifying the correct angle pair and deeper their understanding on the orientation of a geometric figure which is an important but often overlooked issue.



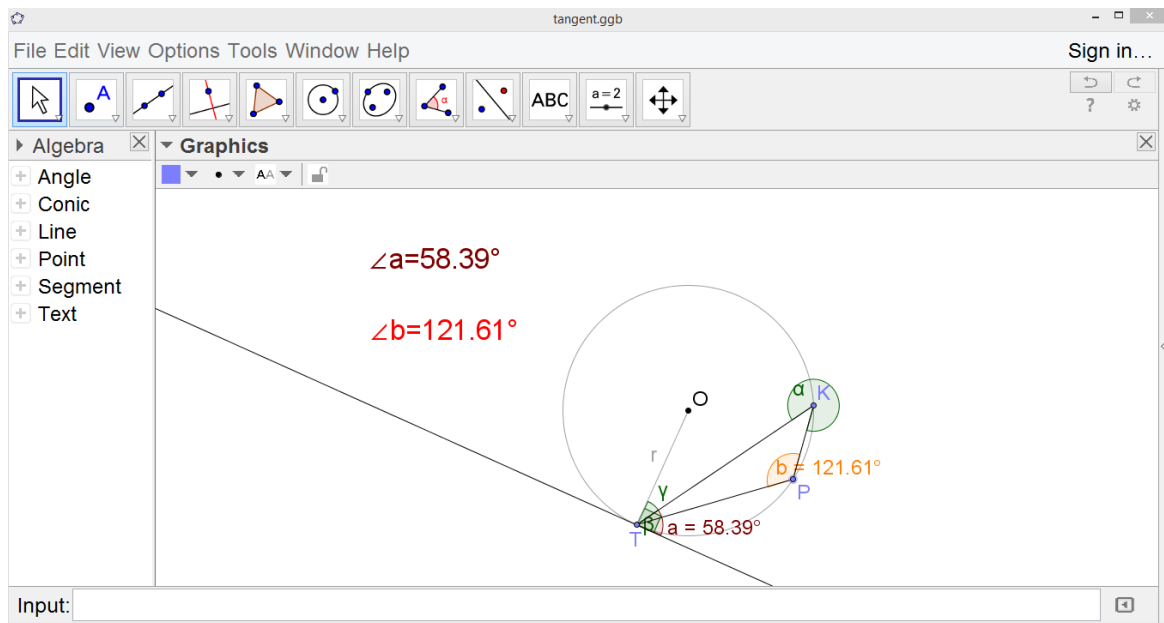
**Figure 4.1** theorem of angle in alternate segment

In the following, we will propose how such kind of mathematics discussions can be arranged. First, the teacher can drag  $P$  along the circle and ask the students to put attention on the values of  $\angle a$  and  $\angle b$ . While point  $P$  moves along the circle,  $\angle TPK (= \angle b)$  changes whereas its value keeps unchanged. (This is the property of angles in the same segment of a circle.) When  $P$  reaches a position where  $PT$  passes through the center  $O$  (that is,  $PT$  is a diameter),  $\angle a$  becomes  $90^\circ$ , and hence  $\angle b + \angle \gamma = 90^\circ$ . It is also noted that  $\angle a + \angle \gamma = \angle \beta = 90^\circ$  (figure 4.2). It gives a proof for the theorem of angle in alternate segment.



**Figure 4.2** Angle in alternate segment for a right-angled triangle

After explaining the proof of the theorem, the teacher can continue to drag  $P$  along the circle to make the discrepancy visible. After point  $P$  crosses point  $K$ , the teacher can point out that  $\angle a \neq \angle b$  and asks the students why it happens (Figure 4.3). The teacher can ask the students why the above proof is not applicable. By observing the diagram, it is expected that the students can realize that  $\angle a$  becomes a reflexive angle (that is, it is greater than  $180^\circ$ ) and hence the orientation of  $\triangle TPK$  is changed from clockwise to anti-clockwise. Thus,  $\angle TPK (= \angle b)$  does not correspond to the tangent-chord angle  $\angle a$  any more. The teacher can further ask the students to find a relationship between the values of  $\angle a$  and  $\angle b$ . They should be easily find that  $\angle a + \angle b = 180^\circ$  and thus  $\angle b$  should correspond to the complementary angle of  $\angle a$ . It is hope that this explorative activity with GeoGebra can help students to recognize the correct angle pair for the theorem of angle in alternate segment through investigating the orientation of the triangle.



**Figure 4.3** Change of orientation when point  $P$  crosses point  $K$

## 5. Conclusion

Computer program such as DGS has its own design constraints. The constraints may lead to deviations from the intended mathematical concepts. These deviations have potentials to provide pedagogical opportunities and hence can be called as discrepancy potentials. In this paper, the discrepancy potentials of two DGS files are discussed. The discrepancies are due to rounding off errors (Example 1) and change of orientation (Example 2), which are originated from the design natures of the software. The discussion above illustrated how these discrepancies can be used for deepen our mathematical understanding and clear-up misconceptions.

## References

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