

# Homothetic centers of three circles and their three-dimensional applications

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## Abstract

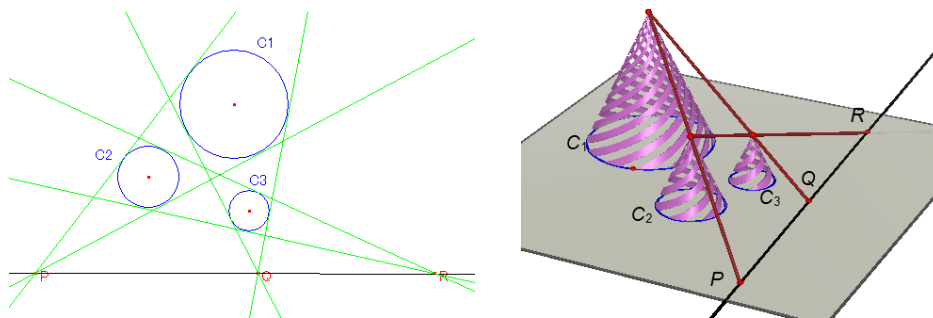
*In this paper, we recall the famous Monge's theorem of three circles. There are several proofs for the theorem. One of the proofs is that of using three similar right cones. Inspired by the proof, we propose a three-dimensional problem of the same angles of elevation: For three similar right cones on the ground, find the places from where three angles of elevation are equal to each other. There are at most two places. With dynamic geometry software, we can simply construct the solutions. In addition, the relation between two solutions is cleared.*

## 1. Introduction

Figure 1.1 shows the famous Monge's theorem of three circles and its three-dimensional intuitive proof:

### Theorem 1.1 (Monge's theorem of three circles)

*For any three circles in a plane, none of which is inside one of the others, the intersection points of each of the three pairs of external tangent lines are collinear.*



**Figure 1.1** The Monge's theorem (left) and its three-dimensional proof (right).

Regarding three circles on the plane as the top view of three similar right cones as in Figure 1.1(right), we can easily know the external homothetic centers by connecting the vertices of the cones with lines. These lines are on the plane passing through three vertices of cones, hence three external homothetic centers lie on the intersection of the plane and the base plane. This is the reason why three external homothetic centers are collinear. Here is a question. When you are on the line passing through three external homothetic centers, how do the three vertices of the cones look like from your eyes? You must answer that the three vertices look like collinear. Inspired by this problem, we will propose the next problem: For three similar right cones on the ground, find the places from where three angles of elevation are equal to each other. There are infinitely many

places from where three vertices look like collinear, however, there are at most two places from where three angles of elevation are equal to each other.

In section 2, we review an easy construction of homothetic centers of two circles. Using the homothetic centers, we will construct the points from where the angles of elevation are equal to each other for two cones in section 3. A relation between circle of Apollonius and angle of elevation is cleared.

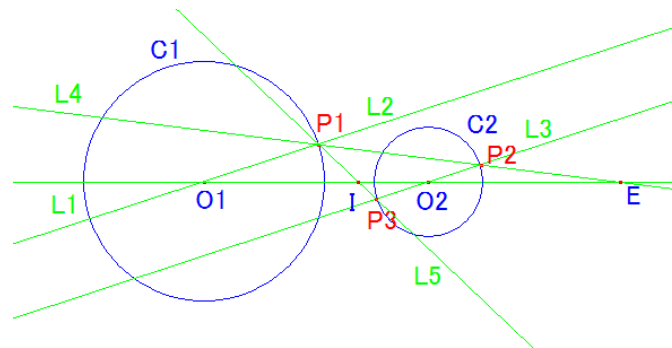
Finally, we construct the solution of the same elevation problem of three cones in section 4. The relation between two solutions is revealed. All figures in this paper are drawn with dynamic geometry software; Cabri II plus and Cabri 3D.

## 2. Homothetic centers of two circles

In this section, we review the simplest construction of external homothetic center and internal homothetic center of two circles shown as Figure 2.1.

### Construction 2.1 (Homothetic centers of two circles)

0. Input:  $C1$  and  $C2$ : two circles centered at  $O1$  and  $O2$ , respectively.
1.  $L1$ : line passing through  $O1$  and  $O2$ .
2.  $L2$ : any line passing through  $O1$ .
4.  $L3$ : line parallel to  $L2$  passing through  $O2$ .
5.  $P1$ : one of intersections of  $C1$  and  $L2$ .
6.  $P2$  and  $P3$ : intersections of  $C2$  and  $L3$  labeled as in Figure 2.1.
7.  $L4$ : line passing through  $P1$  and  $P2$ .
8.  $L5$ : line passing through  $P1$  and  $P3$ .
9. Output:  $E$  is the external homothetic center which is the intersection of  $L4$  and  $L1$ .  
 $I$  is the internal homothetic center which is the intersection of  $L5$  and  $L1$ .



**Figure 2.1** Construction of the homothetic centers of two circles.

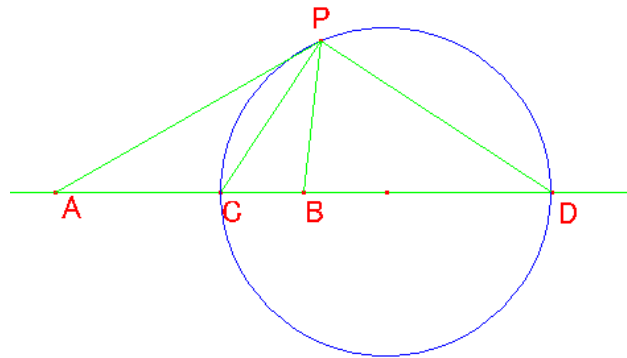
With this construction, we can easily confirm the Monge's theorem of three circles shown as in Figure 1.1. Now, let us regard Figure 2.1 as the top view of two similar right cones on the ground. If you are at point  $E$ , then two cones look like completely overlapped. If you are at point  $I$ , then the angle of elevation of one cone is equal to that of another cone. In fact, the set of points from where the angles of elevation are equal is the circle with diameter  $EI$ .

### 3. Angle of elevation problem of two cones

In this section, we will investigate the angle of elevation problem of two cones. To do this, we have to go back to the famous Apollonius theorem shown as in Figure 3.1:

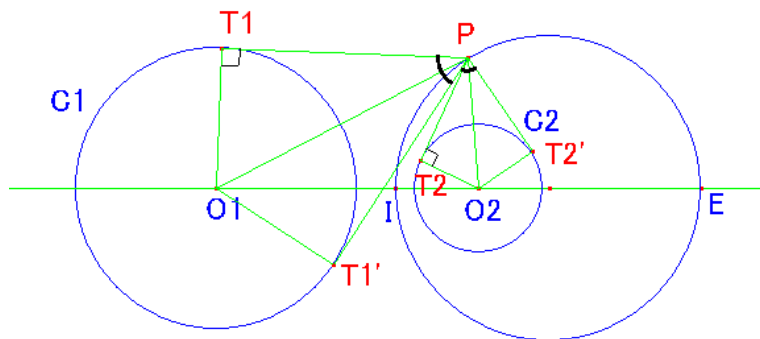
**Theorem 3.1** (The circle of Apollonius ([3] p.28)

*Let  $A, B$  be two fixed points on a line. The locus of a point  $P$  which moves so that the ratio of its distances from  $A$  and  $B$  is constant is the circle with diameter  $CD$  where  $C$  is the internal division and  $D$  is the external division with the ratio.*



**Figure 3.1** Circle of Apollonius.

Using Theorem 3.1, we can solve the next problem: For two circles on a plane, find the points from where the viewing angle of one circle is the same as that of another circle. Here viewing angle of the circle  $C$  from a point  $P$  is defined as the angle subtended by two tangent lines from  $P$  to the circle  $C$  shown as in Figure 3.2.



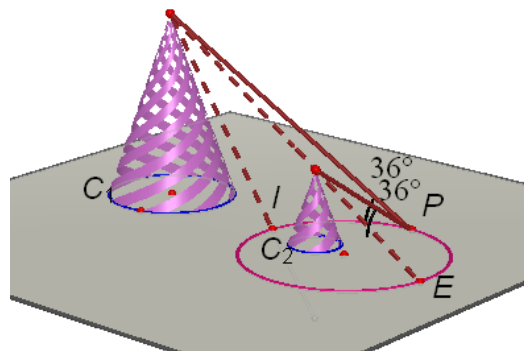
**Figure 3.2** Circle of the same viewing angle.

**Proposition 3.1** (Viewing angle of two circles)

Let  $C_1$  and  $C_2$  be two circles in a plane, none of which is inside of the other. Let  $E$  and  $I$  be the external and internal homothetic centers of the two circles, respectively. Then for any point  $P$  on the circle with diameter  $EI$ , the viewing angle of  $C_1$  from  $P$  is the same as the viewing angle of  $C_2$  from  $P$ .

**Proof.** Let  $P(T_1)$  and  $P(T_2)$  be tangent lines from  $T$  to  $C_1$  and  $C_2$ , respectively as in Figure 3.2. It is enough to show that  $\angle(T_1)P(O_1) = \angle(T_2)P(O_2)$ . Let  $r_1$  and  $r_2$  be the radii of circles  $C_1$  and  $C_2$ , respectively. Then  $I$  is the internal division of  $O_1$  and  $O_2$  such that  $(O_1)I : (O_2)I = r_1 : r_2$ . And  $E$  is the external division  $O_1$  and  $O_2$  such that  $(O_1)E : (O_2)E = r_1 : r_2$ . Then, the circle with diameter  $EI$  is the Apollonius circle of  $O_1$  and  $O_2$  with the ratio  $r_1 : r_2$ . Then,  $P(O_1) : P(O_2) = r_1 : r_2$ . Therefore, triangles  $\triangle(T_1)P(O_1)$  and  $\triangle(T_2)P(O_2)$  are similar to each other. In particular,  $\angle(T_1)P(O_1) = \angle(T_2)P(O_2)$ . ■

Regarding Figure 3.2 as the top view of two similar cones on the ground, we directly clear the angle of elevation of two cones, since the viewing angle is directly proportional to the angle of elevation. Figure 3.3 shows the places from where the angle of elevation of one cone is the same as that of another cone.



**Figure 3.3** Angle of elevation problem of two cones.

#### 4. Angle of elevation problem of three cones

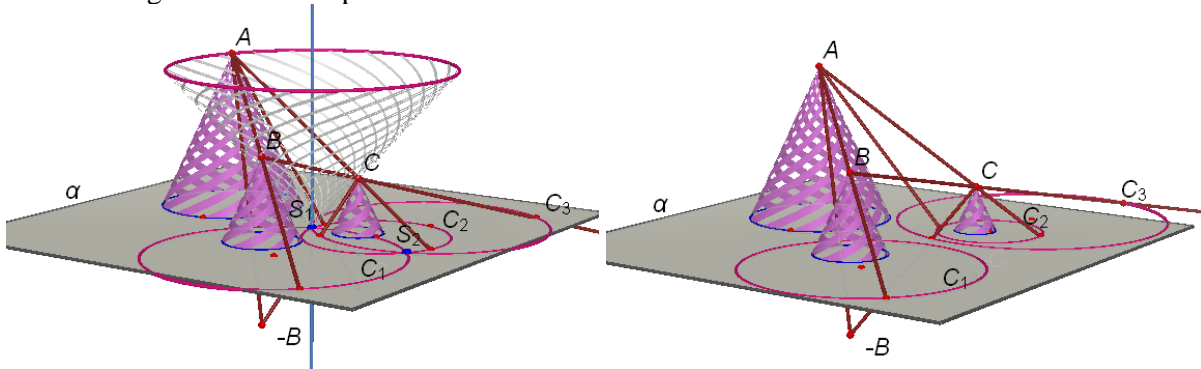
With Proposition 3.1, we can find out the solution of angle of elevation problem of three cones. The solution is the intersections of three circles shown as in Figure 4.1 (left).

**Construction 4.1** (Angle of elevation problem of three cones)

0. Input: three cones with vertices  $A$ ,  $B$ , and  $C$ .
1.  $-B$ ,  $-C$ : reflections of  $B$  and  $C$  in the base plane  $\alpha$ , respectively.
2.  $C_1$ : circle with diameter  $AB \cap \alpha$  and  $A(-B) \cap \alpha$ .
3.  $C_2$ : circle with diameter  $AC \cap \alpha$  and  $A(-C) \cap \alpha$ .
4.  $C_3$ : circle with diameter  $BC \cap \alpha$  and  $(-B)C \cap \alpha$ .
5. Output:  $S_1$ ,  $S_2$ : intersections of  $C_1$ ,  $C_2$ , and  $C_3$ .

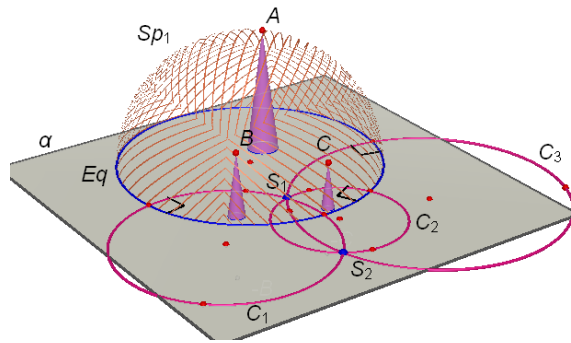
In fact, depending the configuration of the three cones, there are 2, 1, or 0 solutions as in Figure 4.1. If there is a solution, we can easily confirm that the angles of elevation are equal to each other with dynamic geometry software. To see this, construct an upside-down cone with vertex at the solution,

axis perpendicular to the base plane, and passing through vertex  $A$  shown as in Figure 4.1 (left). Then, we can see other vertices  $B$  and  $C$  are also on this upside-down cone. In this way, we can solve the angle of elevation problem of three cones.



**Figure 4.1** Angle of elevation problem of three cones: two solutions (left) and no solution (right).

In the following argument, we assume that there are two solutions  $S1$  and  $S2$  shown as in Figure 4.1 (left).



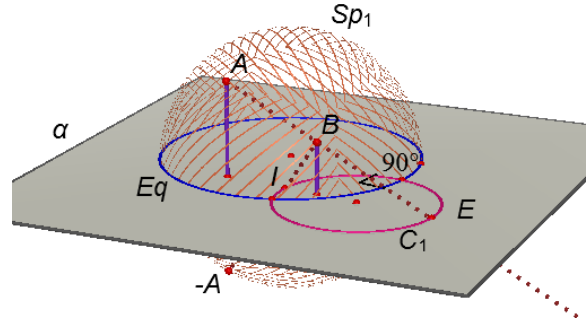
**Figure 4.2** Sphere passing through three vertices of cones and two solutions  $S1$  and  $S2$ .

Final study is to clear the relation between two solutions  $S1$  and  $S2$ . Here, let us consider the sphere  $Sp1$  passing through three vertices of cones  $A$ ,  $B$ , and  $C$  perpendicular to the base plane  $\alpha$ . Figure 4.2 shows that one of solutions  $S1$  is the inversion of another solution  $S2$  with respect to sphere  $Sp1$ . Let  $Eq$  be the equator which is the intersection of sphere  $Sp1$  and the base plane  $\alpha$ . With dynamic geometry software, we can check that three circles  $C1$ ,  $C2$ , and  $C3$  in Construction 4.1 intersect orthogonally with circle  $Eq$ . This means that circles  $C1$ ,  $C2$ , and  $C3$  are invariant under the inversion with respect to circle  $Eq$  on the base plane  $\alpha$ . Hence, we can show that  $S1$  is the inversion of  $S2$  with respect to sphere  $Sp1$ , if we prove that  $C1$  intersects orthogonally with  $Eq$ , because with the same argument, both  $C2$  and  $C3$  also intersect with  $Eq$  orthogonally. Therefore, in the following discussion, let us focus on the relation between circles  $Eq$  and  $C1$ .

**Proposition 4.1** (Orthogonal intersection of  $Eq$  and  $C1$  (Figure 4.3))

Let  $SP1$  be the sphere passing through  $A$  and  $B$  perpendicular to the base plane  $\alpha$ . Let  $Eq$  is the intersection of  $SP1$  and the base plane  $\alpha$ . Let  $-A$  be the reflection of  $A$  in the base plane  $\alpha$ . Let

$E$  be the intersection of line  $AB$  and the base plane  $\alpha$ . Let  $I$  be the intersection of line  $(-A)B$  and the base plane  $\alpha$ . Let  $C_1$  be the circle on the base plane  $\alpha$  with diameter  $EI$ . Then,  $Eq$  and  $C_1$  intersect orthogonally.

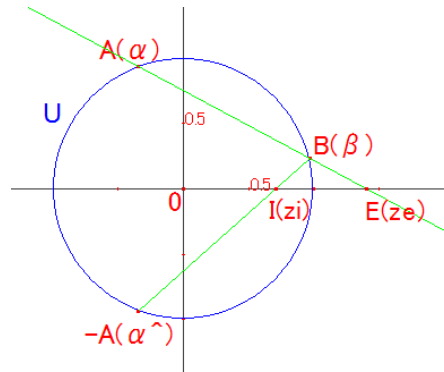


**Figure 4.3** Orthogonal intersection of  $Eq$  and  $C_1$ .

Before we prove Proposition 4.1, we prepare the following proposition.

**Proposition 4.2** (inverse relation between  $E$  and  $I$  (Figure 4.4))

Let  $U$  be the unit circle on a complex plane. Let  $A(\alpha)$  and  $B(\beta)$  be two points on  $U$ . Let  $-A(\bar{\alpha})$  be the conjugate of  $A(\alpha)$ . Let  $E(ze)$  be the intersection of  $AB$  and the real axis. Let  $I(zi)$  be the intersection of  $(-A)B$  and the real axis. Then,  $ze \cdot zi = 1$ , that is,  $E$  is the inversion of  $I$  with respect to  $U$ .



**Figure 4.4** Relation between  $E$  and  $I$ .

**Proof of Proposition 4.2.** The equation of line  $AB$  is  $\begin{vmatrix} z & \bar{z} & 1 \\ \alpha & \bar{\alpha} & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$ . Because  $ze$  satisfies the equation and  $ze = \bar{ze}$ ,

$$ze = \frac{\alpha\bar{\beta} - \bar{\alpha}\beta}{(\alpha - \bar{\alpha}) - (\beta - \bar{\beta})}.$$

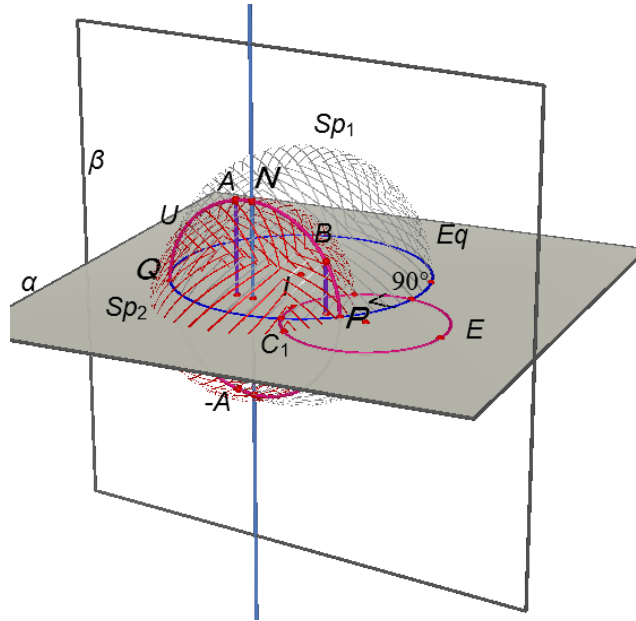
In the similar way, the equation of line  $(-A)B$  is  $\begin{vmatrix} z & \bar{z} & 1 \\ \bar{\alpha} & \alpha & 1 \\ \beta & \bar{\beta} & 1 \end{vmatrix} = 0$ . Because  $zi$  satisfies the equation and  $= \bar{z}$ ,

$$zi = \frac{\alpha\beta - \bar{\alpha}\bar{\beta}}{(\alpha - \bar{\alpha}) + (\beta - \bar{\beta})}.$$

$$\text{Then, } ze \cdot zi = \frac{\alpha\bar{\beta} - \bar{\alpha}\beta}{(\alpha - \bar{\alpha}) - (\beta - \bar{\beta})} \cdot \frac{\alpha\beta - \bar{\alpha}\bar{\beta}}{(\alpha - \bar{\alpha}) + (\beta - \bar{\beta})} = \frac{\alpha^2 - \beta^2 - \bar{\beta}^2 + \bar{\alpha}^2}{(\alpha - \bar{\alpha})^2 - (\beta - \bar{\beta})^2} = 1,$$

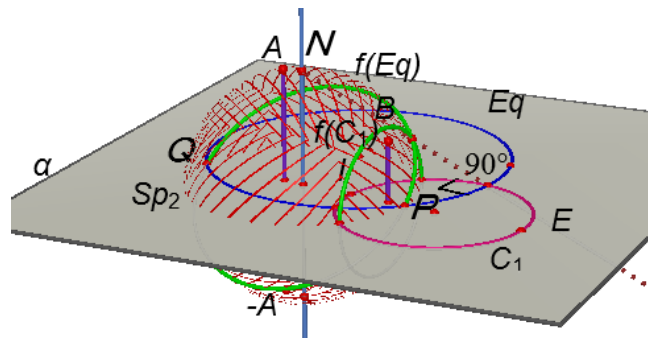
where we use  $|\alpha| = |\beta| = 1$ . This completes the proof. ■

**Proof of Proposition 4.1.** Let  $\beta$  be the plane passing through  $A, B$ , and  $-A$ . Let  $U$  be the circle given as the intersection of  $\beta$  and  $Sp_1$  as in Figure 4.5. We have already seen that  $I$  is the inverse of  $E$  with respect to  $U$ .



**Figure 4.5** Setting for the proof of Proposition 4.1.

Let  $P$  and  $Q$  be two intersections of  $U$  and the base plane  $\alpha$ . Let  $Sp_2$  be a sphere with diameter  $PQ$ . Let  $N$  be the top of sphere  $Sp_2$ . Now let us consider the stereographic projection  $f$  ([1] p.93 , [2] p.260, [4] p.74) from the base plane  $\alpha$  to sphere  $Sp_2$  with respect to  $N$  shown as in Figure 4.6.



**Figure 4.6** Stereographic projection of  $C_1$  and  $Eq$ .

Stereographic projection has two strong properties: conformal and circle-to-circle correspondence. Regarding  $P$  and  $Q$  as the poles of sphere  $Sp_2$ ,  $f(Eq)$  is a longitude of sphere  $Sp_2$ , because  $P$  and  $Q$  are on  $Eq$ . On the other hand,  $f(C_1)$  is a latitude of sphere  $Sp_2$ , because  $I$  is the inversion of  $E$  with respect  $Sp_2$  including  $U$ . In general, latitude  $f(C_1)$  and longitude  $f(Eq)$  intersect orthogonally, therefore,  $C_1$  and  $Eq$  which are inverse images by the stereographic projection, also intersect orthogonally. This completes the proof. ■

In consequence, the solutions  $S_1$  and  $S_2$  are coincident, if and only if,  $S_1$  is on the sphere passing through  $A, B, C, -A, -B,$  and  $-C$ .

### References

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