See Graphs. Find Equations. Myth or Reality?

Wei-Chi YANG
wyang@radford.edu
Department of Mathematics
and Statistics
Radford University
Radford, VA 24142
USA

Abstract

In this paper, we will discuss challenges and discoveries the author has encountered and explored while using technological tools in teaching and research since the inaugural ATCM in 1995. We know that the graphing capabilities of a computer algebra system (CAS) and the use of graphing calculators in the generation of plots has revolutionized the teaching, learning, and research of mathematics. We shall use examples to demonstrate generated scattered plots for specific tasks when dynamic geometry system (DGS) is used. The question has become, however, if we, as educators, can inspire students to see a scattered plot, produced from a specific task, and have them readily identify the equation to the corresponding scattered plot? Finding appropriate equations can be an interesting and challenging enough task in two dimensions. It becomes even more of a challenge if we ask students to find equations for what is seen in three dimensions, real-life scenarios. If learners can visualize what is seen with the help of a technological tool, we hope students will be inspired to investigate problems further. We will also reflect on what Professor Wu, Wen-tsun *(see [13])* had envisioned in his plenary speech at the first ATCM in 1995 and will explore his predictions and visions as they apply today.

1 Introduction

In his plenary speech at the first ATCM of 1995 given by Professor Wu, Wen-tsun, (see [13]), he envisioned *elementary differential calculus with geometric applications be taught in high schools.* One may interpret this as a vision of calculus being made more accessible to high school students with the hope of having students being exposed to more real-life applications. How could this be accomplished? Teaching the procedures of graphing function f in a traditional calculus setting by using its first and second derivatives in a traditional, pencil and paper manner was often a difficult task. Students often became frustrated when making simple algebraic mistakes on the first and second derivatives. As a result, students could not graph f and were left with little time to explore (or even reach) the real-life optimization problems. In the early 1990's with the advent graphical tools, many educators phased out algebraic manipulation by hand and placed more emphasis on conceptual understandings with the assistance of the new, readily available graphical tools. One typical example involves being given a set of graphs which represent f, f' and f'', where students need to identify proper graphs for f, f' and f'' respectively. Another example is stated as follows: *The graph of the derivative function for a function f is given* in Figure 1. Assuming the x-intercepts for the following graph are respectively at $x = 1, 3$, and 5.

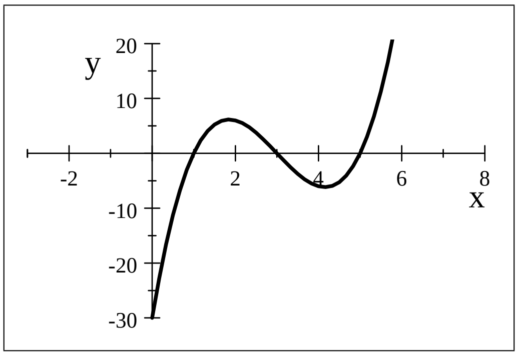

Figure 1.

Students are asked to (a) explain why f has an inflection point at the place where f' has a local maximum or local minimum, and (b) sketch the graph for a possible f. The purpose of both examples is for students to understand the relationship among f, f' and f'' conceptually. In 2002, thanks to the device that integrated DGS with CAS such as [1], the author were able to demonstrate one intuitive way of visualizing why $(\sin x)' = \cos x$ and $\sin^2 x + \cos^2 = 1$ to students by the following steps:

1. We first draw the graph of $y = \sin x$ (See Figure 2 in blue),
2. we construct a tangent line at a point A on the curve of $y = \sin x$ (see Figure 2 in black),
3. we animate the point A along $y = \sin x$ in a domain of x,
4. we collect the coordinates for A while A moves along the curve $y = \sin x$ (see first two columns from Table 1),
5. we collect the slopes of the tangent lines at A (see the last column from Table 1),
6. we drag x component of A and the Slope of the tangent line into the graph (highlighted in two bold columns from Table 1 to get Figure 2.
7. In Figure 3, we see the original blue curve of $y = \sin x$ and the red scattered plot by construction, what is this graph? [We know this should be that of $y = \cos x$.]
8. If we use the second column ($y = \sin x$) and third column ($y = \cos x$) from Table 1 and drag them into the picture of the Figure 4, we see a graph which is close to a circle. Do you know the equation of the circle? It should be the circle of $x^2 + y^2 = 1$ because we are sketching a parametric curve of $x = \sin t$ and $y = \cos t$ by using these two respective

columns from Table 1.

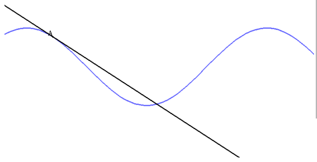

Figure 2. Tangent line at a point of $y = \sin x$

x	y	Slope
−5	0.958924	0.283674
−4.74359	0.999513	0.031208
−4.48718	0.974747	−0.22330
−4.23077	0.886246	−0.46320
−3.97436	0.739795	−0.67282
−3.71795	0.544972	−0.83845
−3.46154	0.314515	−0.94925
−3.20513	0.063493	−0.99798
−2.94872	−0.19168	−0.98146
−2.69231	−0.43432	−0.90075
−2.43590	−0.64856	−0.76115
−2.17949	−0.82040	−0.57178
−1.92308	−0.93859	−0.34503
−1.66667	−0.99541	−0.09571
−1.41026	−0.98714	0.159864
−1.15385	−0.91433	0.404985
−0.89744	−0.78173	0.623626
−0.64103	−0.59802	0.801490
−0.38462	−0.37520	0.926943
−0.12821	−0.12785	0.991793
0.128205	0.127854	0.991793
0.384615	0.375203	0.926943
0.641026	0.598018	0.801490
0.897436	0.781730	0.623626
1.153846	0.914328	0.404985
1.410256	0.987141	0.159864
1.666667	0.995408	−0.09571
1.923077	0.938588	−0.34503
2.179487	0.820397	−0.57178
2.435897	0.648563	−0.76115
2.692308	0.434322	−0.90075
2.948718	0.191681	−0.98146
3.205128	−0.06349	−0.99798
3.461538	−0.31452	−0.94925
3.717949	−0.54497	−0.83845
3.974359	−0.73980	−0.67282
4.230769	−0.88625	−0.46320
4.487179	−0.97475	−0.22330
4.743590	−0.99951	0.031208
5	−0.95892	0.283674

Table 1. Data points for $y = \cos x$

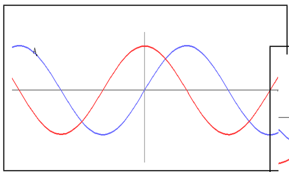

Figure 3. Blue is $y = \sin x$ and Red is $y = \cos x$

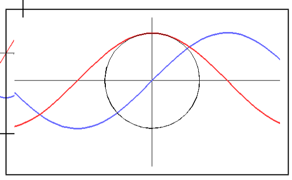

Figure 4. $\sin^2 x + \cos^2 = 1$

The next example was encountered in 2002. The author was surprised, while exploring [1], by how to visualize the scattered plot for the derivative of an implicit function and asked students how the equation of the plot came about. We start with an arbitrary ellipse of $0.5109x^2 + 1.456y^2 - 0.3584xy + 0.1247x - 0.09944y - 0.7033 = 0$,which is shown in black of Figure 5. By collecting the slopes of the tangent lines at various point D on the ellipse, we are able to visualize the scattered red plot that is shown in red in Figure 5. Since we can see the plot, we may pose the following questions:

1. Can you identify what the red curve is? [By the way of construction, the red curve is the scattered plot for the derivative of the given ellipse.]

2. How do you verify the red curve represents the derivative of the ellipse equation? [First we solve the equation of the ellipse in terms of y, which yields in two branches, let's say $y_1(x)$ and $y_2(x)$. Next we find $\frac{d}{dx}(y_1(x))$ and $\frac{d}{dx}(y_2(x))$. Finally, we sketch $y = \frac{d}{dx}(y_1(x))$ and $y = \frac{d}{dx}(y_2(x))$.

3. Why do you see the vertical lines shown in red below? [We note that for each of $y_1(x)$ and $y_2(x)$, we have two vertical tangents.]

With some exploration using available technological tool(s), we are able to make our teaching methods and contents interesting and make students ponder and understand content both visually and mathematically.

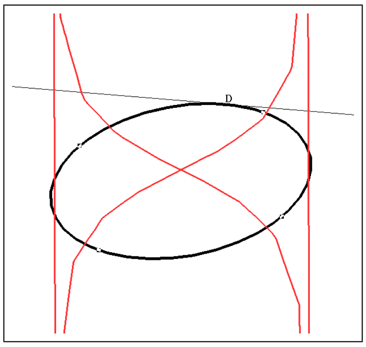

Figure 5. Derivative of an ellipse

We further remark that the concept of uniform continuity (one size of δ that fits all given $\epsilon > 0$) is important for learning advanced mathematics such as in Riemann integration theory. The author realizes that using graphical capability from a CAS can assist students to capture key concepts behind uniform continuity. Moreover, visualization using graphs can motivate the concept of uniform convergence for a series of functions, simulate a function that is nowhere differentiable but continuous functions, and the Gibbs phenomenon from a Fourier series (see *[15]* or *[16]*). In Section 2, the author shares how a discrete probability problem can be turned into a continuous calculus optimization problem. In Section 3, we use examples to show how animations can make students be more interested in mathematics and how animation problems can be both interesting and challenging. In Section 4, we use technological tools to demonstrate how a challenging university entrance examination (practice problems) from China can be made accessible and interesting. Moreover, those entrance practice problems can be used for further exploration when technological tools are available. In Section 5, we mention some real life problems that are encountered but not would have initially thought would relate to the world of mathematics.

2 Seeing is just beginning

We show in the following example how we can turn a discrete probability problem into an analytical problem (see *[19]*), and use the graphical and computational capability within a CAS to solve this problem.

Example 1 *Suppose we have n white balls and n black balls which we are going to place in two urns A and B (in any way we please), as long as at least one ball is placed into each urn. After this has been done, a second person walks into the room and selects one ball at random. Our problem is to maximize the probability that this person draws a white ball..*

We suppose that the distribution of the balls in the urns A and B is as described in the following table:

	A	B
Number of White Balls	x	$n-x$
Number of Black Balls	y	$n-y$

If $P(x, y, n)$ is the probability that a single ball drawn at random will be white then

$$P(x,y,n) = \frac{1}{2}\left(\frac{x}{x+y} + \frac{n-x}{2n-x-y}\right).$$

From now on we shall assume that $n = 50$. We begin our educational guesses of where the maximum for the function by looking at the Table 2, which shows the numerical values of $P(x, y, 50)$, and the graph of $z = P(x, y, 50)$ as shown in Figure 6.

$P(0,1,50) = .25253$	$P(1,0,50) = .74747$
$P(1,1,50) = .5$	$P(2,1,50) = .58076$
$P(1,2,50) = .41924$	$P(25,25,50) = .5$
$P(50,1,50) = .4902$	$P(1,50,50) = .5098$
$P(50,0,50) = .4902$	$P(50,49,50) = .25253$
$P(49,50,50) = .74747$	$P(49,49,50) = .5$

Table 2. Table of $P(x, y, 50)$

To solve the problem analytically we need to find the maximum value of the expression $P(x, y, 50)$ as the point (x, y) varies through the rectangle $[0, 50] \times [0, 50]$ from which the points $(0, 0)$ and $(50, 50)$ have been removed.

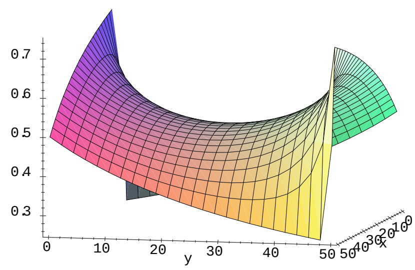

Figure 6. Plot for a probability function

From the looks of this surface it seems unlikely that the maximum value of z will be achieved at a critical point. The maximum appears to be at the left or right extremities of the figure. If we solve the following equations

$$\begin{aligned}\frac{\partial}{\partial x}P(x,y,50) &= 0 \text{ and}\\ \frac{\partial}{\partial y}P(x,y,50) &= 0,\end{aligned}$$

then we obtain

$$\{y = 25, x = 25\}.$$

As we have already seen, the maximum value of z does not occur at the point $(25, 25)$ .We conclude that the expression $P(x, y, 50)$ takes a maximum value of .74747 at the point $(0, 1)$ and again at the point $(49, 50)$. This means that we can maximize the probability that a white ball will be selected by placing no white ball and just one black ball in urn A and all the other balls in urn B. Alternatively we can place no white ball and just one black ball in urn B and all the other balls in urn A. We refer readers to explore other possibilities at *[19].*

3 Animations Make Mathematics Fun to Explore

In Wu's plenary speech at ATCM 1995 (see *[13]*), he emphasized that *Algebra and Geometry should work in congruence with teaching.* One may interpret his remarks as follows: If we integrate CAS with DGS in the evolving technological era, we can discover many interesting problems. From 2012-2015, the author had numerous opportunities to discuss how mathematics can be made more interesting with animations and with the developer of geometric software V. Shelomovskii and his team (see *[4]*).

Example 2 *We consider Figures 7(a)-7(c) below constructed by V.SHELOMOVSKII using [4].*

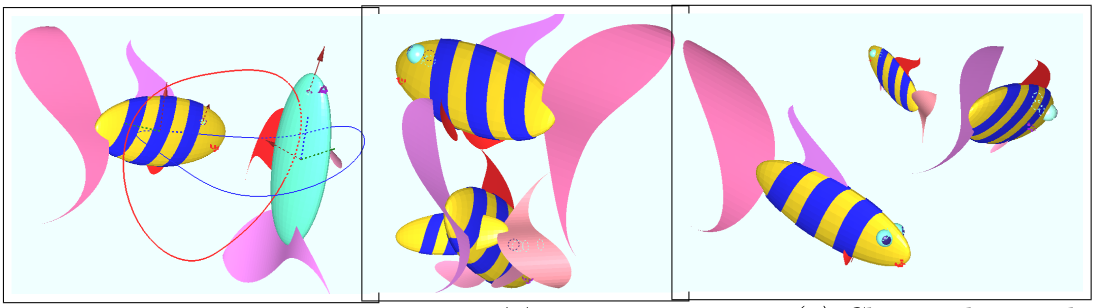

Figure 7(a) Fish with space curve

Figure 7(b) One fish bumps into the other

Figure 7(c) Change the speed of one fish

We ask the following questions to those students who have learned the concepts of space curve and the Frenet.Serret frame. Instead of asking how fish, in Figures 7(a)-7(c), can be generated, we pose the following general questions:

1. How can we make a 3D fish swim in a space? [We first select a space curve for a fish to follow. Next, we determine the unit tangent vector, unit normal vector and unit binormal vector at each given point on the space curve. We then pick a point on the fish in such a way that the fish is following the direction of the unit tangent vector at the given point on the space curve.]

2. Describe, in mathematics term, how two fish can swim without running into each other. [We may choose two non-intersecting space curves for two respective given fish to follow. Of curse two fish should not be too wide (see animation from *[6]*).]

3. What about three fish? Is it sufficient that we create three non-intersecting space curves, then three fish will not run into each other? [As we see from Figure 7(b) that one fish might run into the other (see animation from *[7]*). Thus we may make one fish swim slower or faster than the other two so they do not run into each other (see animation from *[8]*).]

4. How do we make sure fish not to swim upside down? [Since in real life scenario, fish do not swim upside down. We therefore carefully select the unit binormal vector at every point on the space curve to be pointed upward.]

This example shows that the integration of Geometry and Algebra is crucial for advancing knowledge in mathematical fields. Furthermore, the example demonstrates the basic skills needed in calculus (for both students and teachers) as applied to solving open-ended, exploratory activities which have become more prevalent in current mathematics curriculum.

4 Locus and Optimization Problems

The author was surprised to see many interesting locus problems on both university entrance exams and university pre-test practice problems. We consider the following problem which has been modified by the author from the original practice problem (see *[10]*).

Example 3 *We consider the following Figure 8. We are given a fixed circle shown in blue (or inner circle) that is centered at* $(0,0)$*. The point* C *is a fixed point that lies on the* $x-axis$*. The points* E *and* F *are on the inner circle so that the angle* $\measuredangle FCE$ *is kept as* 90*-degree. We form the rectangle* $GFCE$*, by using two perpendicular edges* CF *and* CE *to find the point* G*. (a) Find the locus of* G*. In addition, the author added the following follow-up questions for students to explore when technological tools are at hand. (b) Find the locus of* G *if* $\measuredangle FCE = 30, 60$ *degrees and etc. (c) Find the location of* G *so that the area of* $GFCE$ *yields a maximum while* $\measuredangle FCE$ *is kept as* $30, 60$ *or 90 degrees respectively.*

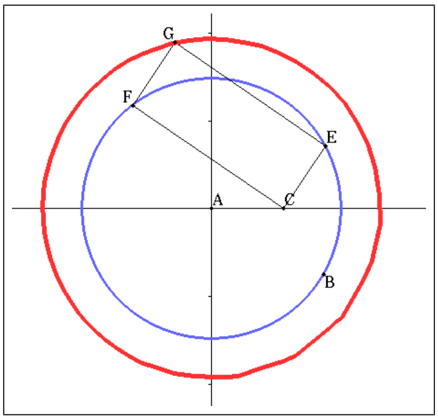

Figure 8. Locus in red when angle is 90 degree

We assume the equation of the circle is $x^2 + y^2 = 4$, the point $C = (1, 0)$, the angle $\measuredangle FCE = 90$. We leave other scenarios to readers as exercises. Indeed, it can be proved analytically that the locus of G is a circle that is centered at $(0, 0)$. In this example, we will not find the locus of G analytically but we show how technological tool can be used for making conjectures that hopefully will inspire more students be interested in mathematics. The red plot is generated by using the coordinate of the point G when the point F is animated along the circle. We use the techniques that we have used earlier when visualizing $(\sin x)' = \cos x$. We recall that G is constructed by using the equation of $\overrightarrow{CG} = \overrightarrow{CE} + \overrightarrow{CF}$.Therefore we can make the scatter plot for the locus of G that is shown in red curve of Figure 8. Similarly, we adopt the same technique to generate the scattered plot of the area function against the $x-component$ of F, which can be seen in red of Figure 9(a). The red also shows the location of F, which roughly corresponds to the maximum value of the area. The Figure 9(b) shows the location of F which roughly corresponds to the minimum area of the rectangle. It is natural to predict that we can motivate more students to be interested in mathematics and make mathematics more accessible if technological tools are used creatively to help them for visualization. Moreover, students can be inspired to do more challenging problems too.

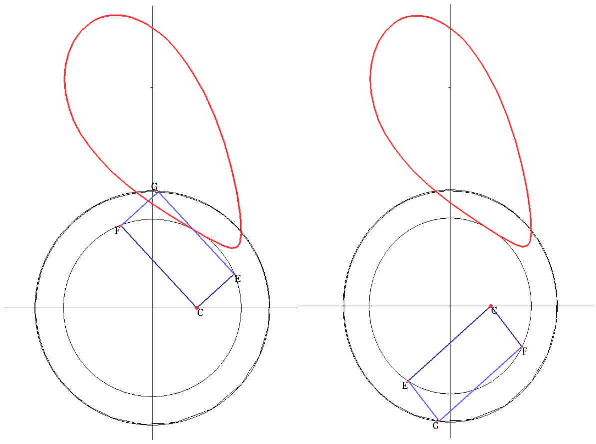

Figure 9(a). Scatter plot of the area function

Figure 9(b). Estimate the location of the minimum area

The preceding problem shed some lights on the importance of integrating the knowledge of both CAS and DGS for training students in current environment. The following Example 4 is a natural extension from Example 3. First, we use the dynamic software *[4]* to demonstrate how the problem can be made accessible to most students and how the problem can be generalized to more challenging ones in 2D and corresponding ones in 3D. In *[13]*, Professor Wu pointed out that *Geometry teaching should be applications-oriented.* We believe that when DGS and CAS are integrated, we can find more applications even from those problems found in university entrance exam practice problems in China.

Example 4 *Consider the following Figure 10, we are given the blue figure (or thick curve) of* $\frac{x^2}{2.24^2} + \frac{y^2}{1.15^2} = 1$ *and the fixed point* $D = (1, 0)$ *on the major axis. We are also given two fixed* *points E and F on the ellipse so that $\measuredangle FDE = 90$ degree. We form the rectangle $GFDE$ by using two perpendicular edges DE and DF. (a) Find the locus of G. (b) Find the location of E so that the area of $GFDE$ yields a maximum or minimum respectively.*

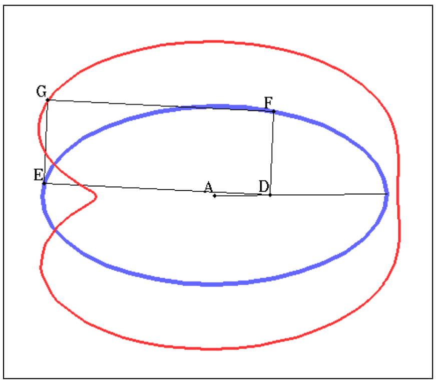

Figure 10. Locus and an ellipse

We first use a technological tool to simulate the plot of the locus, which can be seen in red color of Figure 10 or black color of Figures 11(a) and 11(b). The question here is, of course, **if we can see the scattered plot for the locus of G, are we able to find its equation?** Similarly, we can sketch the scattered plot for the area function against the $x-coordinate$ of E, which is shown in red of Figure 11(a) or 11(b) below. Instead of finding the locus of G analytically in this paper but we shall show analytically but we shall show how we find numerical solutions to where position E results in maximum and minimum area of $GFDE$ respectively. While exploring with *[1]*, we note the maximum area of the rectangle $DEGF$ (see Figure 11(a)) occurs when the $x-coord$ of E near -2.16. Analogously, the minimum area of the rectangle $DEGF$ (see Figure 11(b)) occurs when the $x-coord$ of E is close 2.24. We shall show analytically and numerically with Maple that the maximum and minimum area of

$DEGF$ occur at $x = -2.196627754$ and $x = 2.233594267$ respectively for point E.

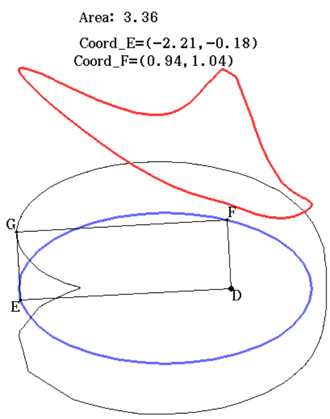

Figure 11(a). Scattered plot and the maximum area

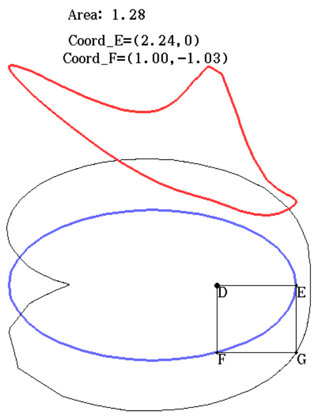

Figure 11(b) Scattered plot and the minimum area

We outline how we optimize the area function by Lagrange Method. We consider $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ when $a = 2.24$ and $b = 1.15$. We set $D = (1, 0)$, and note that $E = (x_1, y_1) = (a \cos \varphi_1, b \cos \varphi_1)$ and $F = (x_2, y_2) = (a \cos \varphi_2, b \cos \varphi_2)$ are two points on the ellipse.

1. Equation 1 ($Eq1$): We substitute line DE into the equation of ellipse to get $Eq1$.
2. Equation 2 ($Eq2$): We substitute line DF into the equation of ellipse to get $Eq2$.
3. Equation 3 ($Eq3$): If β is the angle between DE and DF,we have $\cos \beta = \frac{\overrightarrow{DE} \cdot \overrightarrow{DF}}{(\|\overrightarrow{DE}\|)(\|\overrightarrow{DF}\|)}$.
4. Objective function is the area function:

$$f(a, b, \varphi_1, \varphi_2) = \left(\sqrt{(a \cos \varphi_1 - 1)^2 + (b \sin \varphi_1)^2}\right)\left(\sqrt{(a \cos \varphi_2 - 1)^2 + (b \sin \varphi_2)^2}\right). \quad (1)$$

5. (a) Consider the Lagrange function

$$L(\varphi_1, \varphi_2, x_1, x_2, k_1, k_2, k_3) = f(a, b, \varphi_1, \varphi_2) + k_1 Eq1 + k_2 Eq2 + k_3 Eq3. \quad (2)$$

   Set $\nabla L(\varphi_1, \varphi_2, x_1, x_2, k_1, k_2, k_3) = 0$ and solve for $\varphi_1, \varphi_2, x_1, x_2, k_1, k_2, k_3$ with the help of Maple.

   (b) We remark that using Lagrange method finding the critical points only produces necessary solutions where extreme values might occurs, we need to check numerically and see which solutions produce the maximum area and minimum area respectively. After numerical verification with Maple, we find the maximum area occurs when $x_1 = -2.196627754$ and the minimum area happens when $x_2 = 2.233594267$.

Discussions:
We describe further possible exploration activities below:

1. Repeat the problem when we change the angle $\measuredangle FDE$ to other fixed angles such as 30 or 60 degrees.

2. Explore if this locus and extremum problem can be generated to corresponding one in 3D.

We remark that these open ended projects are excellent choices for adopting technological tools for exploring mathematics. Examinations alone cannot be the sole measurement of student success. It will be important to see how math curriculums can include the component of exploration with the help of technological tools and where real life exploration can be found. In a recent article (see *[12]*), it is stated that '*Taiwan plans a radical reform of its education system, one aiming to set it apart in East Asia by playing up creativity and student initiative instead of the rote of memorization curriculums that dominates classroom learning techniques in this part of the world.*' While many educators, researchers and parents would applaud this brave and bold initiative, how the government can really implement this agenda remains to be seen. Indeed, it is not enough to "say the right things", but knowing how to develop implementable and sustainable strategies to carry out the strategy. We outline necessary knowledge for teachers so technological tools can be integrated in a math curriculum to motivate more students be interested in the STEM (Science, Technology, Engineering and Mathematics) area.

1. Use DGS or similar technological tools to simulate animations in two dimensions.

2. Encourage students to make conjectures through their observations from step 1.

3. Encourage students to verify their results using a DGS or CAS for 2D case.

4. Extend students observations to a 3D scenarios with technologies if possible.

5. Prove the corresponding results for 3D cases analytically using a DGS, CAS or appropriate tools.

6. Extend our results to finite dimensions or beyond if possible.

5 Explore Real Life Problems

In *[13]*, Professor Wu pointed out that *some of the geometry problems to be solved are those problems arising from actual life.* While investigating the general inverses in 2-D, 3-D and their applications (see *[14]*), the author was led to the concept of caustic curve with the help of Phillip Todd see a video clip in (*[11]*). Caustic curve is the envelope of light rays reflected (or refracted) by a curved surface or object, see Figure 12(a) and the video clip at *[11].* In the differential geometry of curves, the evolute of a curve is the locus of all its centers of curvature. Equivalently, it is the envelope of the normals to a curve. Caustic curve is the evolute of the orthotomic curve. Caustic is also the locus of all its centers of curvature of orthotomic curve or the envelope of the orthotomic normals. However, the caustic surfaces have not been discussed much in literatures. Author created the Figure 12(b) and the movie clip can be found at *[17]).* The caustic surface was created when a light is reflecting through a transparent cup. In addition, after seeing the light and water show at Marina Bay (see Figure 12(c) or see movie clips at *[18]*) when author was visiting Singapore in January of 2014, author asked himself if the water and light show has anything to do with the concept of caustic surface? If not, how so? In addition, would it be cheaper to create similar effects by using the concept of caustic surface such as the one demonstrated in Figure 12(b)? Finally, since we can see such a water and light show, are we able to use mathematics equations to simulate the effect?

Figure 12(a). Caustic curve in 2D

Figure 12(b). Caustic surface in 3D

Figure 12(c). Is this related to caustic surface?

The next real-life example shows that we would have hoped that the designers of the Vdara Hotel in Las Vegas had some knowledge about the effect of caustic surface. According to *[2]*, '*When the Vdara hotel first opened in Las Vegas on December 1, 2009, visitors relaxing by the pool would experience intense periods of heat at certain times of the day, at certain times of the year. This intense heat was caused by the reflection of solar radiation from the curved, reflective surface on the South-facing side of the hotel. This model shows how a caustic surface is generated in the pool area around the time and date the problems were first reported.*' (See Figures 13(a),(b) and (c).)

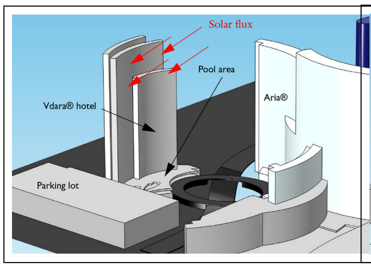

Figure 13(a). Vicinity of Vdara Hotel

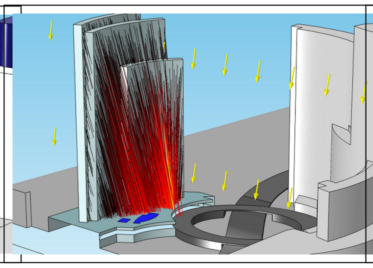

Figure 13(b). Vectors of solar radiation

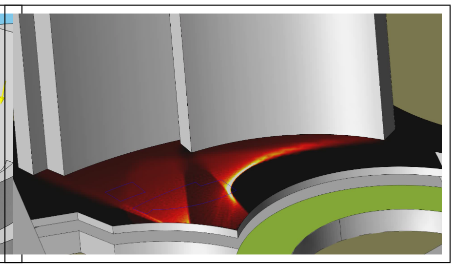

Figure 13(c) High-intensity shaded in red

In the same article, it states that '*the animation (created by Ruud Börger) in [2] shows the intensity of the rays projected onto the pool area between 11:30 am and 1 pm in September 2010. As you can see, the high-intensity region starts initially on the side of the pool area closest to the parking lot and then sweeps over the pool complex towards the road.*' From this example, it is natural to conjecture that technological tools will lead the general public to gain more common knowledge which cannot be acquired otherwise. We, as educators, can predict that future students should have broader knowledge on all subjects and their respective applications.

6 Conclusions

We have heard from teachers and educators globally that students today are less motivated and competent in basic algebraic manipulation skills than their counterparts from previous years. Indeed it is a difficult task to balance curriculums that emphasize the exploration of mathematics while at the same time, require rote algebraic manipulation skills. On the one hand, it is common sense that if a curriculum is laden with too much examination/testing or is based solely on "teaching to the test", such curriculums do little for the promotion of creative thinking skills. Furthermore, curriculums stand to lose potential students who might pursue math related fields in the future. On the other hand, thanks to the rapid advancement of technological tools, we have started seeing students exploring readily available knowledge through the internet and other technologies. Students may not possess in-depth knowledge of a field, but they are able to integrate various aspects of knowledge, possibly with the help of technologies, to explore and complete viable projects. We have seen in many ATCM papers where technologies have helped teachers, students, and researchers integrate knowledge from a variety of fields of study. We believe by integrating knowledge from Geometry and Algebra or DGS with CAS, students will be able make their own conjectures effectively and verify their observations analytically. We know that by addressing the importance and timely adoption of technological tools in teaching, learning, and research, we will never be wrong. Therefore, we encourage ATCM communities to continue creating innovative examples by adopting technological tools for teaching and research and equally as important, influencing your colleagues, communities, and decision makers in your respective countries to do the same. Selecting examples that can be explored from middle and high school, university level, and beyond, problems should be STEM related by linking mathematics to real-world applications. If we can see a plot through exploration with technological tools, we should encourage students to find the algebraic equations representing graphs even if students may not have an in-depth knowledge of a particular topic. After all, should students not be able to find a particular algebraic equation, it at least provides motivation to such students to acquire higher mathematics knowledge in order to do so. Technologies have made us re-evaluate how to make mathematics an interesting and cross disciplinary subject. Through the advancement of technological tools, there is no doubt that learners will be able to discover more mathematics and mathematical applications in their lives.

References

[1] ClassPad Manager for ClassPad II Series, 90 days trial download, see http://classpad-manager-for-classpad-ii-series.software.informer.com/

[2] B. Cunningham, An Analysis of Caustic Surface Generation at the Vdara Hotel, December 12, http://www.comsol.com/blogs/an-analysis-of-caustic-surface-generation-at-the-vdara-hotel/.

[3] Geometry Expression, see http://www.geometryexpressions.com/.

[4] Geometry in Mathematical Arts (GInMA): *A Dynamic Geometry System*, see http://deoma-cmd.ru/en/Products/Geometry/GInMA.aspx.

[5] GInMA animation: *An animation of two fish.* see http://www.radford.edu/wyang/Tsukuba/fish.MOV.

[6] GInMA animation: *An animation of two fish swimming with two respective curves*, https://mathandtech.org/Yang/Fish/two_fish.mp4.

[7] GInMA animation: *An animation of three fish swimming but one fish ran into the other*, https://mathandtech.org/Yang/Fish/three_fish_flaws.mp4.

[8] GInMA animation: *An animation of three fish swimming with no problem by adjusting the speed,* https://mathandtech.org/Yang/Fish/three_fish_okay.mp4.

[9] Maple: A product of Maplesoft, see http://maplesoft.com/.

[10] Shi GI Gin Bang 2016, Strategies for high school complete review, ISBN 978-7-5634-5289-701, page 647, published by YanBian University Publishing company.

[11] Todd, P., Mathematics of Light Caustics at YouTube*:* https://www.youtube.com/watch?v=UBSu7NJiI3c.

[12] An article from the Christian Science Monitor at http://www.csmonitor.com/World/Asia-Pacific/2015/0823/Creative-demand-Taiwan-says-radical-school-reform-will-set-it-apart.

[13] Wu,W.-t., Geometry Problem-Solving And Its Contemporary Significance, Proceedings of the First Asian Technology Conference in Mathematics, pp 67-72.

[14] Yang, W.-C. & Lo, M.-L. *General Inverses in 2-D, 3-D, applications inspired by Technology*, the Electronic Journal of Mathematics and Technology (eJMT), Volume 3, Number 1, ISSN 1933-2823, Mathematics and Technology LLC.

[15] Yang, W.-C. (2011), Technology Has Shaped Up Mathematics Communities, Proceedings of the 16th ATCM, the electronic copy can be found at this URL: http://atcm.mathandtech.org/EP2011/invited_papers/3272011_19335.pdf.

[16] Yang, W.-C. (1995), Examples in Advanced Calculus and Scientific Workplace, Proceedings of the 8th International Collegiate Technology Conference in Mathematics (ICTCM8) [Electronic version]. Retrieved from: http://archives.math.utk.edu/ICTCM/EP-8/C88/pdf/paper.pdf.

[17] Yang, W.-C. (2014), Created water shows at Mariana Bay, Singapore in 2014, mathandtech.org/Yang/Caustic/3D_caustic.mov.

[18] Yang, W.-C. (2014), Created water shows at Mariana Bay, Singapore in 2014, mathandtech.org/Yang/Caustic/

[19] Yang,W.-C. and Lewin, J. Exploring Mathematics with Scientific Notebook