Visualization of quaternions with Clifford parallelism

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Abstract: *In this paper, we try to visualize multiplication of unit quaternions with dynamic geometry software Cabri 3D. The three dimensional sphere S^3 is identified with the set of unit quaternions. Multiplication of unit quaternions is deeply related with Clifford parallelism, a special isometry in S^3.*

1. Introduction

The quaternions H are a number system that extends the complex numbers. They were first described by William R. Hamilton in 1843 ([1] p.186, [3]). The identities

$$i^2 = j^2 = k^2 = ijk = -1,$$

where i, j and k are basis elements of H, determine all the possible products of i, j and k:

$$ij = k, \quad ji = -k, \quad jk = i, \quad kj = -i, \quad ki = j, \quad ik = -j.$$

For example, if $a = 1 + j$ and $b = 2j + 3k$, then, $ab = (1 + j)(2j + 3k) = -2 + 3i + 2j + 3k$. We are very used to the complex numbers C and we have a simple geometric image of the multiplication of complex numbers with the complex plane. How about the multiplication of quaternions? This is the motivation of this research. Fortunately, a unit quaternion a with norm one ($\|a\| = 1$) is regarded as a point in S^3. In addition, we have the stereographic projection from S^3 to R^3. Therefore, we can visualize three quaternions a, b, and ab as three points in R^3. What is the geometric relation among them? Clifford parallelism plays an important role for the understanding of the multiplication of quaternions.

In section 2, we review the stereographic projection. With this projection, we try to construct Clifford parallel in Section 3. Finally, we deal with the multiplication of unit quaternions in Section 4.

2. Stereographic projection of the three-sphere

The stereographic projection of the two dimensional sphere S^2 ([1, page 260], [2, page 74]) is a very important map in mathematics. It maps a sphere minus one point (the north pole N) to the plane containing the equator by projecting along lines through N as in Figure 2.1.

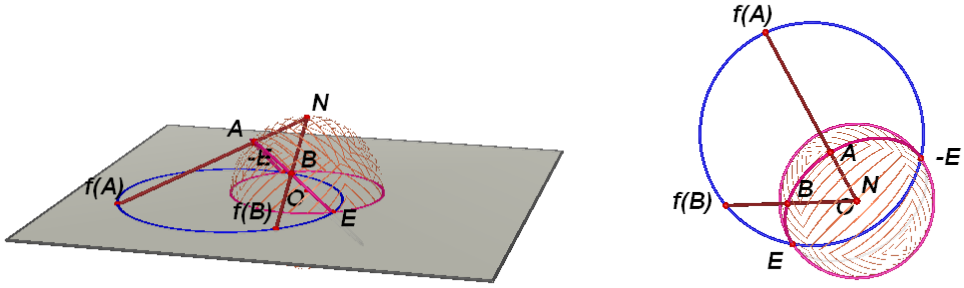

Figure 2.1 The stereographic projection of S^2 (left) and its top view (right).

This projection preserves angles, and maps a circle on the sphere to a circle on the plane. In particular, a great circle on the sphere is projected to a circle passing through two antipodal points (E and $-E$ in Figure 2.1) on the equator. This projection enables us to draw spherical objects on a plane. In this sense, the unit circle is regarded as the equator, and a circle passing through two antipodal points on the unit circle is regarded as a great circle.

In the same way, the stereographic projection from the north pole of the three-dimensional sphere S^3 onto the three-dimensional Euclidean space R^3 enables us to draw spherical objects in R^3. In this sense, the unit sphere is regarded as the equator (geodesic plane) of S^3, and a circle passing through two antipodal points on the unit sphere is regarded as a great circle. Here, let us review how to construct the geodesic passing through arbitrary fixed two points A and B.

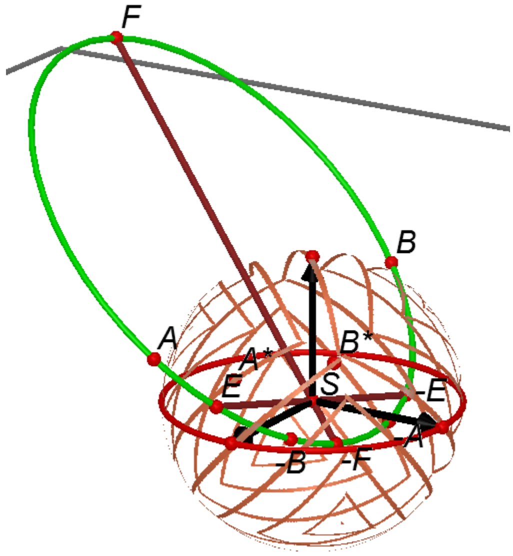

Figure 2.2 Stereographic projection of S^3.

Construction 2.1 (great circle passing through A and B)
0. Input: Any points A and B.
1. A^*, B^*: inversion of A and B with respect to the unit sphere (the equator).
2. $-A, -B$: point symmetry of A^* and B^* with respect to the origin S (the south pole).
3. Output: circle $A(-B)(-A)B$ is the great circle passing through A and B.

Note that in this model, distance is measured by radians. $A(-A) = B(-B) = E(-E) = \pi$. In Figure 2.2, let F be the farthest point on the great circle from the origin S in Euclidean sense. Then, $FE = E(-F) = (-F)(-E) = (-E)F = \pi/2$. In the stereographic projection, angles are preserved, however, distances are not preserved, therefore, we have to pay attention to measuring distances. Let $a = s + a_1 i + a_2 j + a_3 k$ be a unit quaternion, then $s^2 + a_1^2 + a_2^2 + a_3^2 = 1$. This equation implies that $(s, a_1, a_2, a_3) \in S^3$. Hence, we can see unit quaternions in R^3 as in Figure 2.3. For $a = s + a_1 i + a_2 j + a_3 k$:

$$(x, y, z) = \frac{(a_1, a_2, a_3)}{1 + s}, \quad (s, a_1, a_2, a_3) = \frac{1}{1 + r^2}(1 - r^2, 2x, 2y, 2z),$$

where $r^2 = x^2 + y^2 + z^2$. Note that in this paper, x-axis (i-axis) directs upwards shown as in Figure 2.3. For example, $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$, $1 = (0,0,0)$, -1 =(point at infinity), and $0.5(1 + i + j + k) = (1/3,1/3,1/3)$.

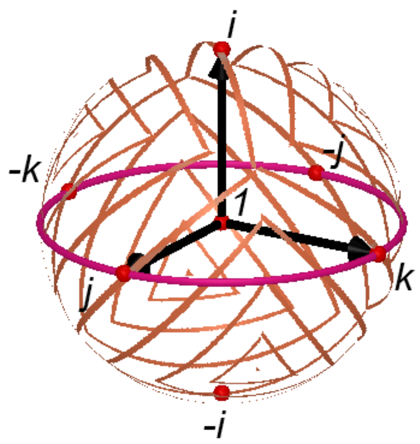

Figure 2.3 Visualization of bases of unit quaternions.

3. Clifford parallelism

For the visualization of the multiplication of unit quaternions, let us review Clifford parallelism ([2] p.298). Figure 3.1 is an image of the multiplication of unit quaternions a and b. French cruller is the best one to explain this calculation. Roughly speaking, if a and b are arranged as in Figure 3.1, then multiplication ab is given by sliding b along the pleat with distance between 1 and a. In this picture, great circle $\overline{1a}$ (axis of donut), and all pleats including great circle $\overline{b(ab)}$ are parallel to each other. In this sense, Clifford parallelism plays an important role in visualization of the multiplication of quaternions.

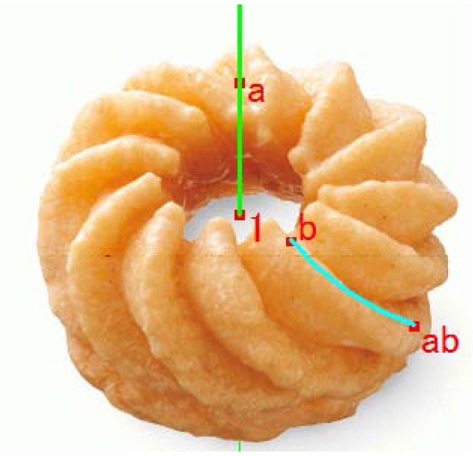

Figure 3.1 French cruller.

Definition 3.1 (Clifford parallel ([1] p.299)) Two great circles C, C' in S^3 are said to be *Clifford parallel* if $d(m, C')$ does not depend on $m \in C$.

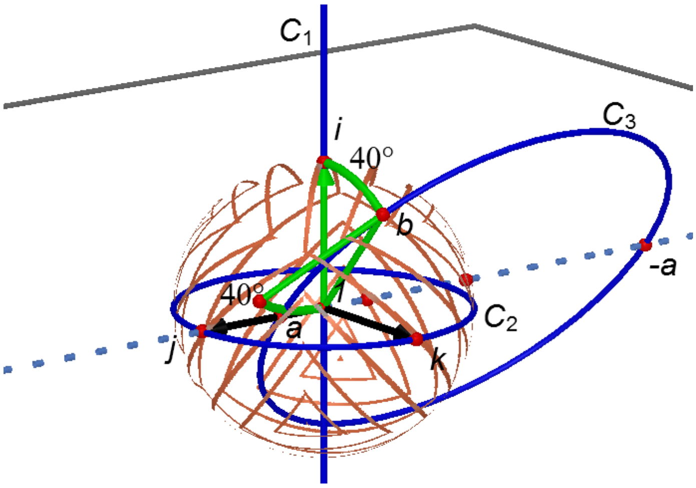

Figure 3.2 Clifford parallels.

In Figure 3.2, $\mathrm{d}(C_1, C_2) = 90°$, $\mathrm{d}(C_1, C_3) = 40°$, $\mathrm{d}(C_2, C_3) = 50°$. To construct Clifford parallel, it is convenient to use the relation between central angle and inscribed angle. In Figure 3.3, the angle between circle $aN(-a)S$ and circle NOS is the double of angle $\angle aNS$. This fact directly comes from inscribed angle theorem.

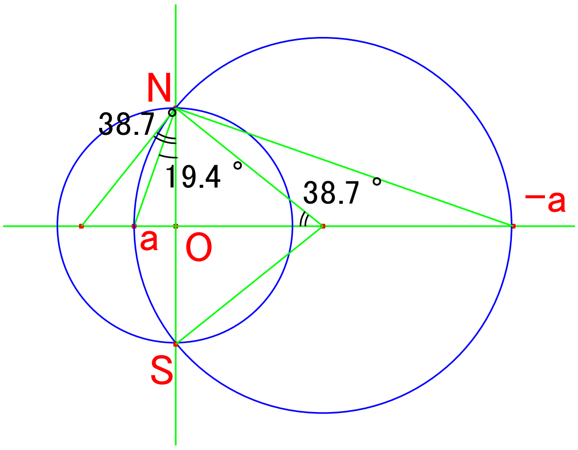

Figure 3.3 Central angle and inscribed angle.

Now we are ready to construct Clifford parallel as follows (for convenience, a great circle is a Euclidean line $\overline{1i}$, and point a is on the jk-plane) :

Construction 3.1 (Clifford parallel passing through a parallel to $\overline{1i}$) (Figure 3.4)

0. Input: great circle $\overline{1i}$ and point a is on the jk-plane.
1. m: midpoint of i and a.
2. m': rotation of m around $\overline{1i}$ mapping j towards k.
3. i' : half turn of i around segment $\overline{1m'}$
4. $-a$: antipodal point of a (Construction 2.1)
5. Output: circle $ai'(-a)$ is the great circle parallel to $\overline{1i}$ passing through a.

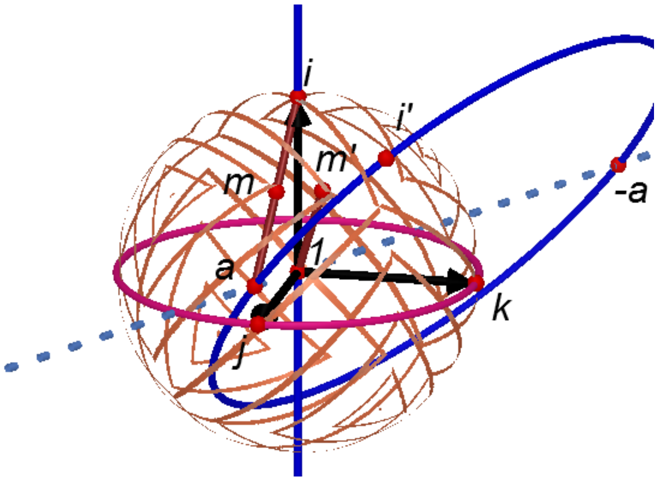

Figure 3.4 Construction of Clifford parallel.

4. Multiplication of quaternions

In fact, geometric construction of multiplication of unit quaternions in the stereographic projection is done by three Euclidean planar reflections and one Euclidean inversion with respect to a certain sphere. To understand this construction, we should recall several properties of quaternions, conjugation, inner product, norm, and so on ([1] p.187, [3]). The conjugate of $a = s + a_1 i + a_2 j + a_3 k$ is the quaternion $\bar{a} = s - a_1 i - a_2 j - a_3 k$. Note that $\overline{ab} = \bar{b}\bar{a}$. The canonical inner product is $(a|b) = \frac{1}{2}(\bar{a}b + \bar{b}a)$. Hence in particular, $\|a\| = \sqrt{a\bar{a}}$, and $\|ab\| = \|a\| \cdot \|b\|$. Regarding (s, a_1, a_2, a_3) as a point in R^4, $\{1, i, j, k\}$ is an orthogonal basis of R^4. It is easy to check that $\{u, ui, uj, uk\}$ is also an orthogonal basis.

Proposition 4.1 (reflection in R^4 with respect to unit quaternion)
Let u be a unit quaternion. Then $q \to -u\bar{q}u$ is a reflection in the hyper plane passing through the origin perpendicular to u.

Proof. Let $f(q) = -u\bar{q}u$. Then,

$$f(uq) = -u\overline{uq}u = -u\bar{q}\bar{u}u = -u\bar{q},$$

hence, $f(u) = -u$, $f(ui) = ui$, $f(uj) = uj$, and $f(uk) = uk$.Therefore, f is a reflection in the hyper plane subtended by ui, uj, and uk. ■

For simplicity, let $a = e^{\theta i}$ in the following argument. Multiplication ab is realized by the following four reflections f_i $(i = 1,2,3,4)$ in R^4. Let f_i $(i = 1,2,3,4)$ be four reflections perpendicular to four unit quaternions such as

$$u_1 = j, \quad u_2 = j\,cos\frac{\theta}{2} + k\,sin\frac{\theta}{2} = e^{\frac{\theta}{2}i}j = je^{-\frac{\theta}{2}i}, \quad u_3 = i, \quad u_4 = ie^{\frac{\theta}{2}i}.$$

Then,

$$f_1(b) = -j\bar{b}j, \quad f_2(b) = -e^{\frac{\theta}{2}i}j\bar{b}je^{-\frac{\theta}{2}i},$$

hence,

$$f_2 \circ f_1(b) = -e^{\frac{\theta}{2}i}j\overline{(-j\bar{b}j)}je^{-\frac{\theta}{2}i} = e^{\frac{\theta}{2}i}be^{-\frac{\theta}{2}i}.$$

On the other hand,

$$f_3(b) = -i\bar{b}i, \quad f_4(b) = -ie^{\frac{\theta}{2}i}\bar{b}ie^{\frac{\theta}{2}i},$$

hence,

$$f_4{\circ}f_3(b) = -ie^{\frac{\theta}{2}i}\overline{(-i\bar{b}i)}ie^{\frac{\theta}{2}i} = e^{\frac{\theta}{2}i}be^{\frac{\theta}{2}i}.$$

Therefore,

$$f_4{\circ}f_3{\circ}f_2{\circ}f_1(b) = e^{\theta i}b = ab.$$

In this way, we can realize the multiplication ab with four reflections in R^4. In the stereographic projection, the composition of reflections $f_2{\circ}f_1$ is a rotation around Euclidean line $\overline{1i}$. On the other hand, the composition of reflections $f_4{\circ}f_3$ is a "*rotation*" around Euclidean circle $jk(-j)(-k)$. Note that in the stereographic projection, reflection is realized by inversion with respect to great sphere. In particular, if a unit quaternion u is a pure quaternion (Re $u = 0$), then reflection is a simple Euclidean planar reflection in the stereographic projection. Three unit quaternions u_1, u_2 and u_3 are pure quaternions, however, only u_4 is not pure. So, we can construct the multiplication of unit quaternions with three Euclidean reflections and one Euclidean inversion as follows:

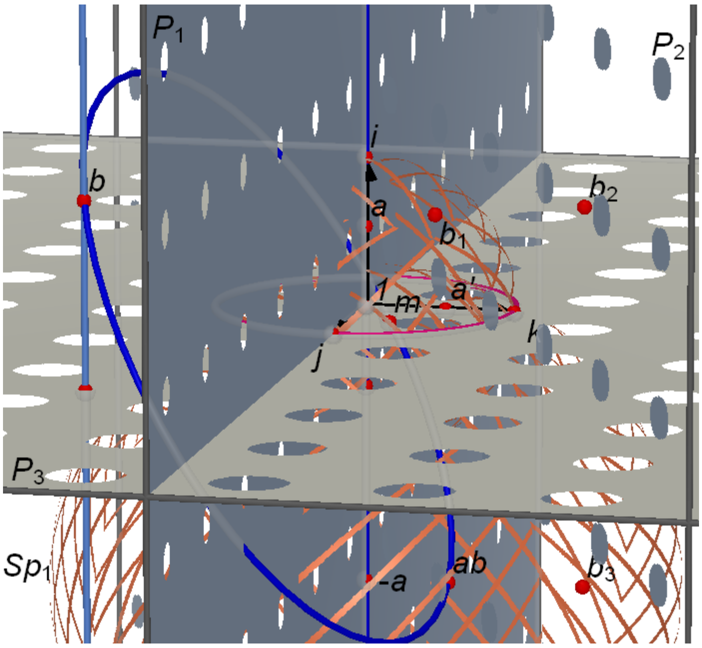

Figure 4.1 Construction of multiplication of unit quaternions.

Construction 4.1 (multiplication of unit quaternions) (Figure 4.1)

0. Input: unit quaternion a on great circle $\overline{1i}$, and unit quaternions b at any place.
1. a': rotation of a around $\overline{1j}$ mapping i towards k.
2. m: midpoint of a' and j.
3. $P1$: plane including $\overline{1i}$ and j.
4. $P2$: plane including $\overline{1i}$ and m.
5. $b1$: reflection of b in plane $P1$.
6. $b2$: reflection of $b1$ in plane $P2$.
7. $P3$: plane including $1, j$, and k.
8. $-a$: antipodal point of a (Construction 2.1)

9. $Sp1$: sphere centered at $-a$ through j.
10. $b3$: reflection of $b2$ in plane $P3$.
11. Output ab : inversion of $b3$ with respect to $Sp1$.

In the case that both a and b are at any place, the similar procedure leads the construction of ab. We can easily check that great circle $\overline{b(ab)(-b)}$ is Clifford parallel to great circle $\overline{1i}$.

References

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