Cyclic and Bicentric Quadrilaterals

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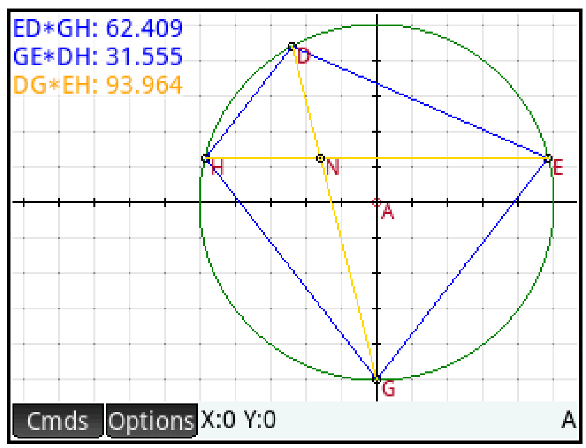
Abstract. *In this hands-on workshop, participants will use the HP Prime graphing calculator and its dynamic geometry app to explore some of the many properties of cyclic and bicentric quadrilaterals. The workshop will start with a brief introduction to the HP Prime and an overview of its features to get novice participants oriented. Participants will then use ready-to-hand constructions of cyclic and bicentric quadrilaterals to explore.*

Part 1: Cyclic Quadrilaterals

The instructor will send you an HP Prime app called *CyclicQuad* for this part of the activity. A cyclic quadrilateral is a convex quadrilateral that has a circumscribed circle.

1. Press ! to open the App Library and select the CyclicQuad app.

The construction consists DEGH, a cyclic quadrilateral circumscribed by circle A.



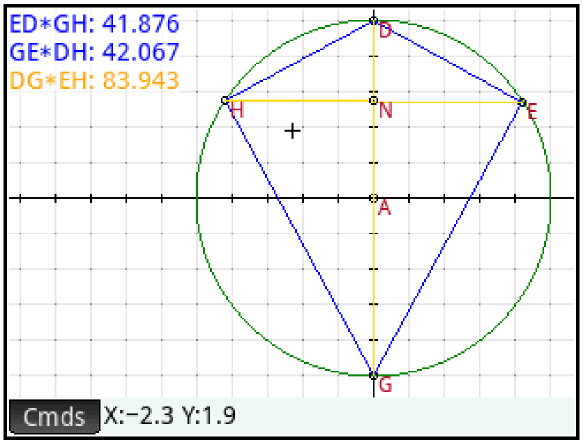
2. Tap and drag any of the points D, E, G, or H to change the quadrilateral. Which of the following can DEGH never be?
   - Square
   - Rhombus (non-square)
   - Rectangle (non-square)
   - Parallelogram (non-rhombus)
   - Isosceles trapezoid
   - Kite

Just move the points of the quadrilateral around enough to convince yourself for each one.

Notice ∢HDE and ∢HE are both inscribed angles that subtend the entirety of the circle; likewise with ∢DHG and ∢DEG. This leads us to a defining characteristic of cyclic quadrilaterals. Make a conjecture.

A quadrilateral is cyclic if and only if…

3. Make DEGH into a kite, similar to that shown to the right. Tap segment HE and press E to select it. Now use U and D to move the diagonal vertically. What do you notice about the angles of this kite? Press & to deselect segment DG. Make a conjecture to prove later:



If a cyclic quadrilateral is a kite,

then ________________________________

4. Notice that we have defined the following products as well:
   a. The product of the opposite side lengths ED and GH
   b. The product of the opposite side lengths GE and DH
   c. The product of the lengths of the diagonals DG and EH

What do you notice about these products? Make a conjecture:

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This conjecture is known as Ptolemy's Theorem. If DEGH is a rectangle, to what theorem does it become equivalent?

There is also a famous formula for the area of a cyclic quadrilateral. This formula looks a bit like Heron's formula for the area of a triangle. It is called Brahmagupta's Formula:

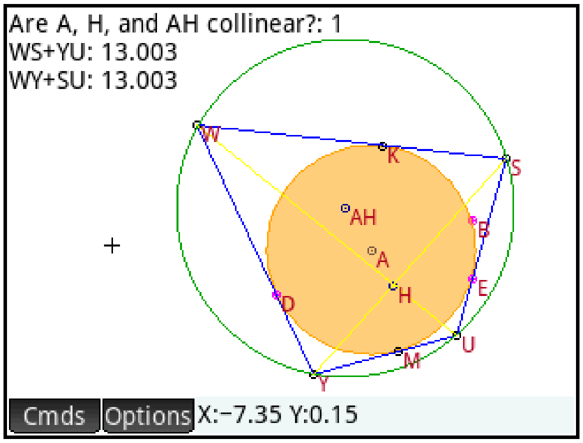
$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where s is the semi-perimeter of the quadrilateral and a, b, c, and d are the side lengths.

Part 2: Bicentric Quadrilaterals

The instructor will send you an app named *BicentricQuad* for this part of the activity. A bicentric quadrilateral is one that has both an inscribed circle and a circumscribed circle. In other words, it is a cyclic quadrilateral that has an inscribed circle as well.

1. Press ! to open the App Library and select the BicentricQuad app.

In the construction, quadrilateral WSUY is a bicentric quadrilateral. Point A is the center of the inscribed circle and point AH is the center of the circumscribed circle. Point H is the intersection of the diagonals. You can move points D, H, and E, but not K and M.



2. In Part 1, you discovered that cyclic quadrilaterals can be right kites, squares, rectangles, and isosceles trapezoids. Which type of cyclic quadrilateral cannot be bicentric? Why?

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3. The sums of the lengths of opposite sides are also displayed. What do you notice about these two sums? Make a conjecture.

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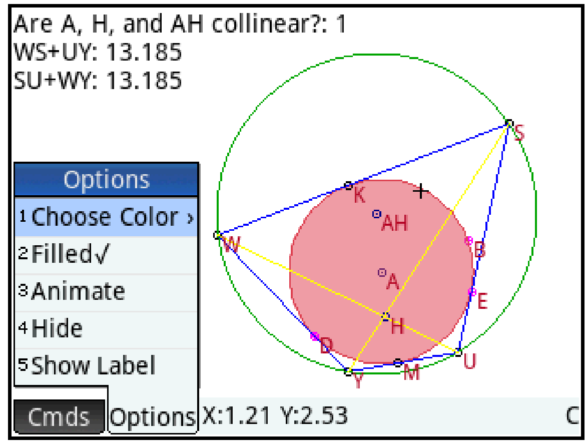
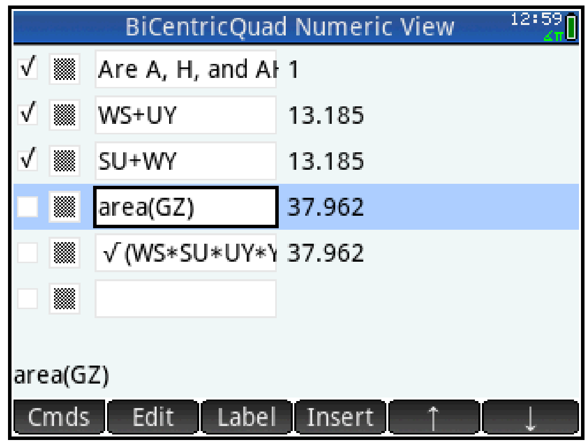
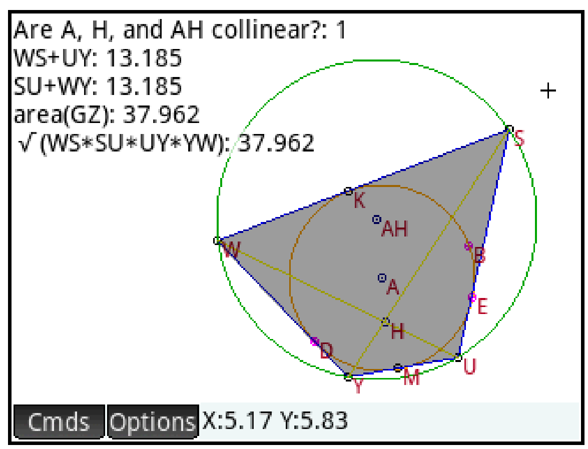
To support your conjecture, notice point W is a point exterior to circle A, with two tangent segments, $\overline{WK}$ and $\overline{WD}$, of equal length. The same is true of the other three vertices of quadrilateral WSUY. You can now write four equations and manipulate them to get the desired result.

4. Tap on Circle A and then tap Options. From the pop-up menu, tap Filled to deselect it.
5. Press N to open Numeric view. You will see that there are two items at the end of the list that are not selected; that is, they do not have check marks to their left. One is the area of the quadrilateral WSUY; the other is the square root of the product of the side lengths of WSUY. Check them both so that they appear in Plot view by selecting each one in turn and tapping √.
6. Press P to return to Plot view and move points H, E, or D. What do you notice about these two measurements? Make a conjecture.







In the first part of this activity, we ended with Brahmagupta's Formula:

$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$, where *s* is the semi-perimeter of the quadrilateral and *a*, *b*, *c*, and *d* are the side lengths.

Since the sums of the opposite sides are equal, the semi-perimeter, s, is equal to either of these sums: a+c or b+d.

7. Use this fact to simplify Brahmagupta's Formula to the conjecture you made in #6.

Finally, we note that the question at the top of Plot view always has a 1 after it. This is a test that returns 0 for false and 1 for true, so it seems the incenter, the circumcenter, and the intersection of the diagonals in a bicentric quadrilateral are always collinear.