Applications of Information Technology to the "Five Points" Conjecture

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Abstract: *Taking the "five points" conjecture for example, this paper shows some possibilities of using graphing calculators for exploring conjectures and offering promising directions for the proof of a conjecture. Through a process of making conjectures, evaluating experiments performed on a graphing calculator and theoretical proof in "five points", a pattern of information technology (IT) application can be experienced.*

1. Background

The development of computers, graphing calculators etc. has largely promoted science. Information technology makes it possible to solve problems that couldn't have been possible in the age of "paper and pencils". Let alone those experimental subjects such as physics, chemistry, and biology, even in mathematics, a subject which was featured by rigorous induction and deduction, numerical methods are more and more frequently used to find solutions for problems, no matter they are about differential equations, definite integrals or large data samples. Nowadays, mathematics is not only a combination of formulas, theorems and algorithms. It is intertwisted with technology. Since the development of the society depends on the development of technology and science, the future of human beings falls upon the shoulders of people who can master them both. And then IT is a significant indicator of nation strength.

2. Introduction

The "Five Points" Conjecture For any five points in a triangle, there must be three out of them constituting a triangle whose area is not bigger than a quarter of the original one's. The area is considered to be zero when three points are collinear.

For simplicity, we start with the nine points case.

Case 1.1 Scatter nine points into a triangle of area one, then there must be three points constituting a triangle whose area is not bigger than 1/4. Again, the area is considered to be zero when the three points are collinear.

This problem is relatively trivial. As shown in Figure 1, three median lines divide ΔABC into four parts of the same area. According to the drawer principle, there must be at least one part containing three points. And the area of the triangle formed by these points is of course less than or equal to 1/4.

However, the drawer principle does not suit cases of fewer points very well. We might as well consider the case of seven points, for if the conclusion holds in this case, then it naturally holds for the eight points case.

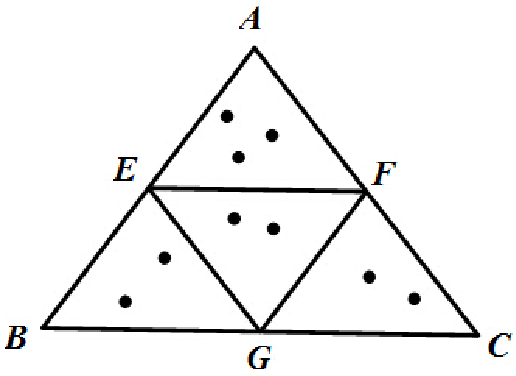


Figure 1

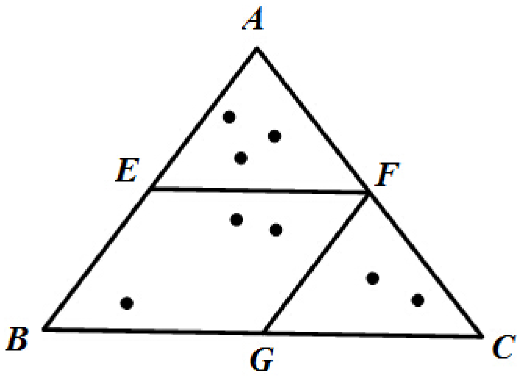


Figure 2

3. Problem Solving

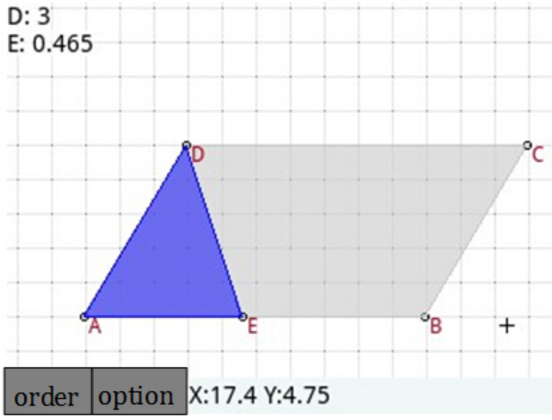
Case 1.2 Seven points are randomly chosen in a triangle of area one. Try to explore whether or not there exist three points which constitute a triangle whose area is not bigger than 1/4. And the area is considered to be zero when the three points are collinear.

Inspired by the method above, we can, as Figure 2 shows, remove one median line then three drawers are obtained. Hence there should be at least three points falling in one of these drawers. If that drawer happens to be ΔAEF or ΔAEF , then the conjecture is verified. So the problem has been reduced to that if the three points are located in parallelogram $EBGF$, they would constitute a triangle whose area is not bigger than 1/4, as stated in the lemma below.

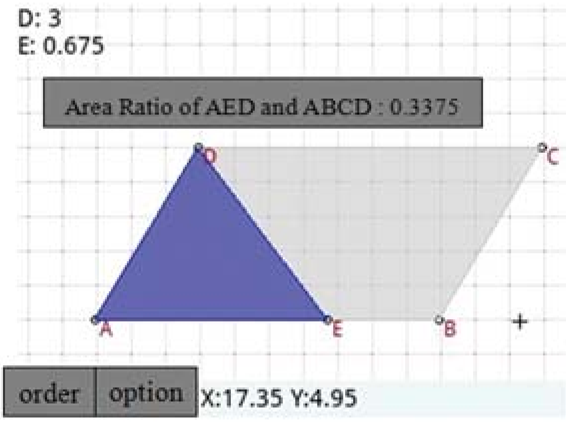
Lemma: Any three points in a parallelogram form a triangle whose area is not bigger than half of that of the parallelogram. (Collinear points give a triangle of area zero).

We now use the HP Prime graphing calculator Geometry app to offer a numerical solution. Note that if the triangle is in the interior of the parallelogram, then its area must be less than or equal to half of that of the parallelogram.

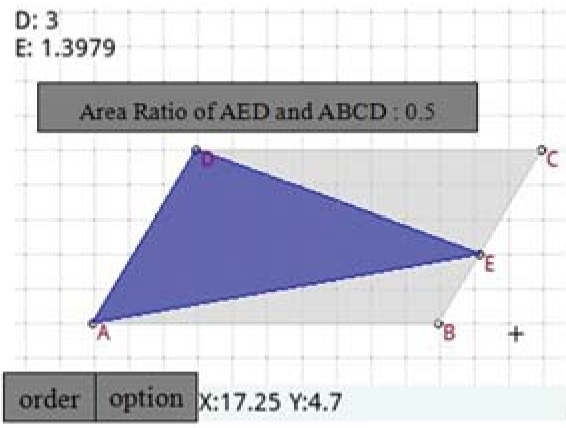
Test 1 First, fix A and D. Assume E is a moving point on other sides, as shown below:



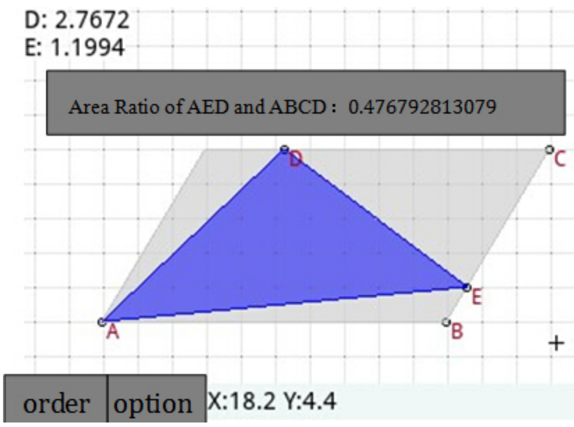
Apparently, the ratio (measured by HP Prime) of the areas of parallelogram $ABCD$ and ΔAED changes as E moves.



We can also make D a moving point. The maximum of the ratio can be directly observed as 0.5.



The conclusion shows that no matter where D and E are, the area ratio is always less than or equal to 1/2.



Is this the optimal version of the conjecture? Obviously we cannot reduce the problem to four-point cases. A counterexample is illustrated in Figure 3. Three points are chosen next to the three vertices and the fourth point is located at the barycenter of the triangle. Then three out of four triangles are of area close to 1/3 and the other one has area that is very close to one.

But can we prove the statement for five-point cases?

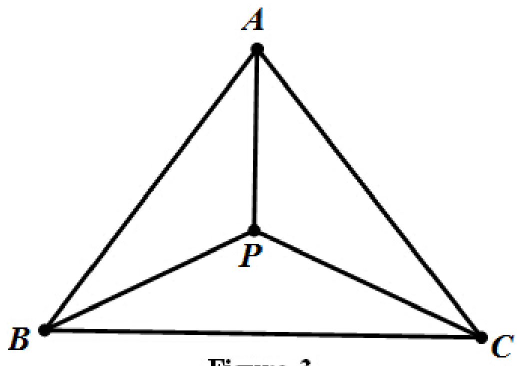


Figure 3

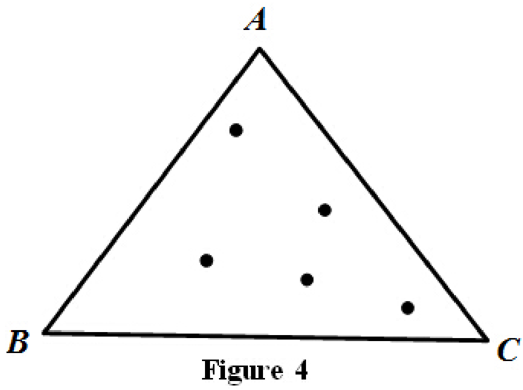


Figure 4

Case 1.3 As is shown in Figure 4, among any five points in the triangle of area one, are there three points that define a triangle whose area is no more than 1/4 ?(Collinear points give area zero.).

It is difficult to solve this problem theoretically, but again, we can conduct a numerical experiment here.

Case 2.1 Let the area of ΔABC be one. As shown in Figure 5, P_1 and P_2 are on the side BC, P_3 is on AB and P_4 on AC. P_1, P_2, P_3, P_4 give a partition of ΔABC. Then at least one of the four areas takes up no more than 1/4 of ΔABC.

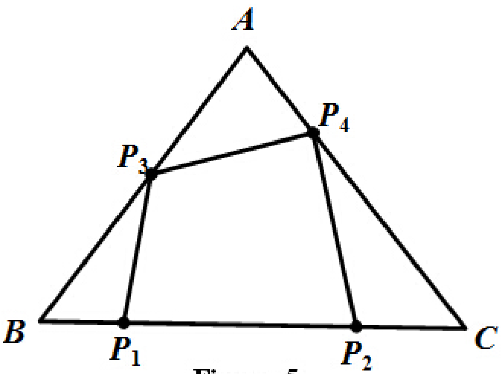


Figure 5

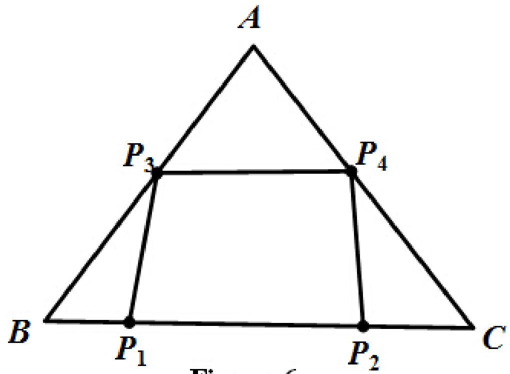


Figure 6

Generally speaking, segment P_3P_4 is not parallel to segment P_1P_2, but for simplicity, we study the parallel case first.

If $P_3P_4 \parallel P_1P_2$, then $\frac{AP_3}{AB} = \frac{AP_4}{AC} = \lambda$ for some , where $0 < \lambda < 1$.

Now $S_{\Delta P_1P_3P_4} = S_{\Delta BP_3P_4} = (1-\lambda)S_{\Delta ABC} \le \left[\frac{\lambda + (1-\lambda)}{2}\right]^2 = \frac{1}{4}$. For the general case shown in Figure 5, let's assume that the distance from P_3 to BC is less than that of from P_4 to BC.

Draw a line P_3Q with $P_3Q \parallel P_1P_2$ and connect P_1P_4, QP_4, and P_3C (shown in Figure 7). Now we are in the very position to use the HP Prime graphing calculator for Test 2.

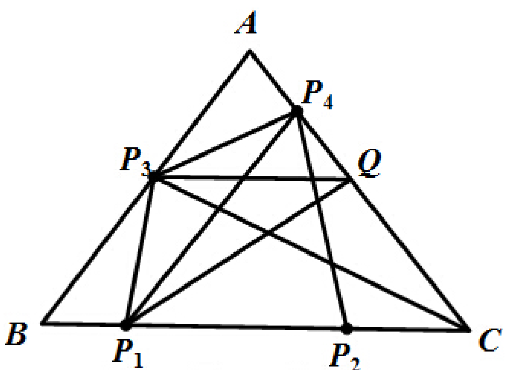


Figure 7

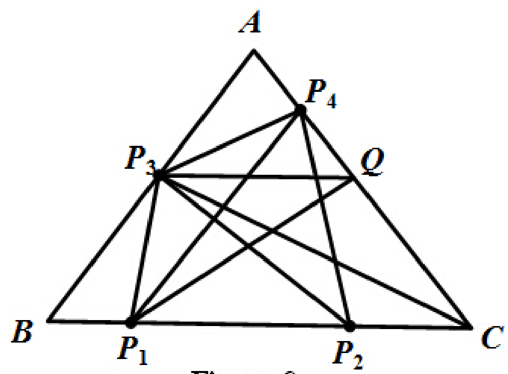
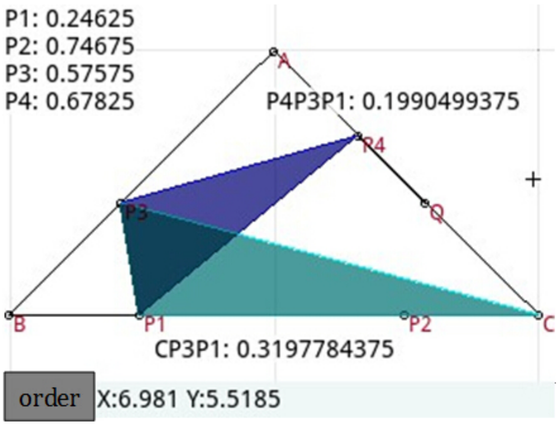


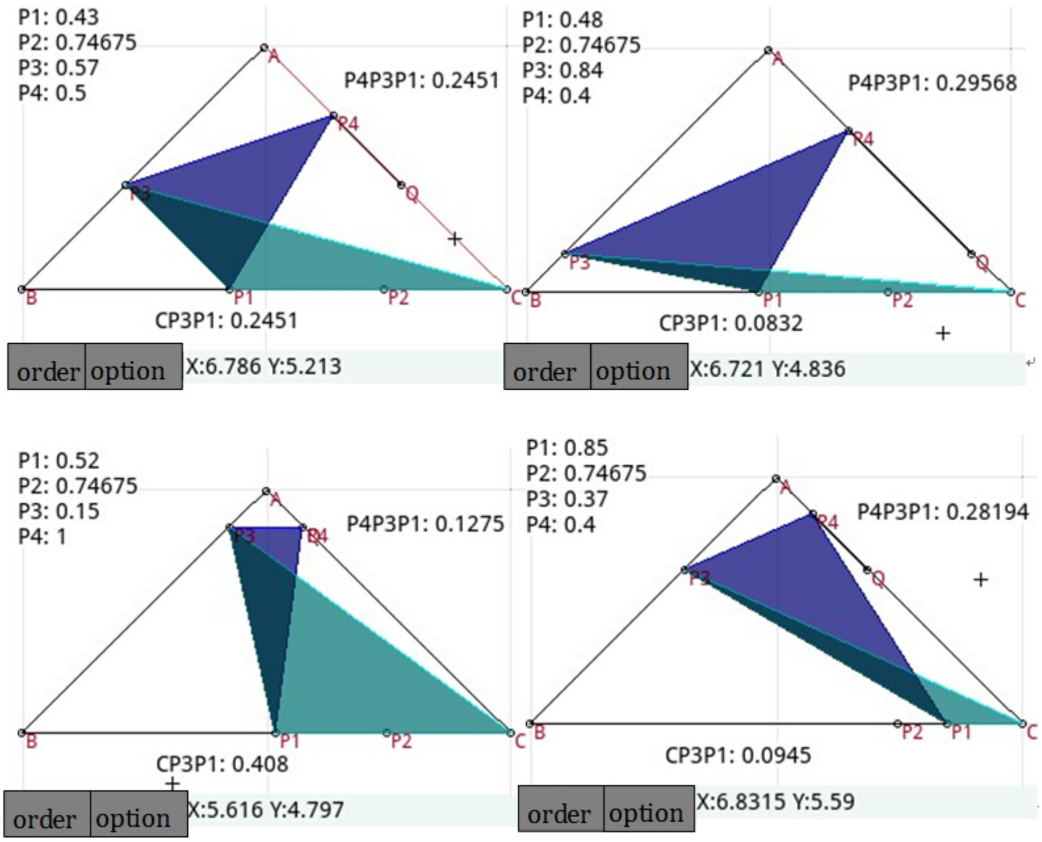
Figure 8

Test 2 (Figure 7) Show that one of $\Delta P_1P_3P_4, \Delta CP_3P_1$ has area that is less than or equal to 1/4.

First, we construct and measure triangles by using the HP Prime Geometry app (for visual simplicity, we hide P_3Q). The construction is shown below:



Next, let P_1, P_3, P_4 move on AB, BC, AQ respectively and make sure they have different velocities so that different location situations are obtained and the ordinate of P_4 is greater than that of P_3). The changing of triangles' area is quite observable, as shown below:



After we analyze the results of Test 2, it's not difficult to find the essence of the result: the distance from point Q to line P_1P_3 is between the distances from point P_4 to line P_1P_3 and point C to P_1P_3. As shown in Figure 8, we may thus demonstrate,

$$\min(S_{\Delta P_1P_3P_4}, S_{\Delta P_1P_3C}) \leq S_{\Delta P_1P_3Q} = \frac{1}{4}$$

Connect P_3P_2, we get Figure 8 and it is easy to prove $S_{\Delta P_3P_2P_1} \leq S_{\Delta CP_3P_1}$, so at least one of the area of $\Delta P_1P_3P_4$ and $\Delta P_2P_3P_4$ is not bigger than 1/4, and so the proof ends.

Based on Case 2.1, it is not difficult to move on to the following Case 2.2.

Case 2.2 Four vertices of a convex quadrangle are on the three sides of a triangle ABC with area one. Prove that at least one of the areas of the four triangles $\Delta P_1P_3P_4$, $\Delta P_1P_2P_4$, $\Delta P_1P_3P_4$, $\Delta P_2P_3P_4$ is not bigger than 1/4.

Since the four vertices of the convex quadrangle are on three sides of ΔABC, which has an area of one, there exist two possibilities. One of them is shown in Figure 5 and the other one is in Figure 9. Move line BC until BC and P_1P_2 coincide, just as shown in Figure 5. Thus, Case 2.2 is absolutely true.

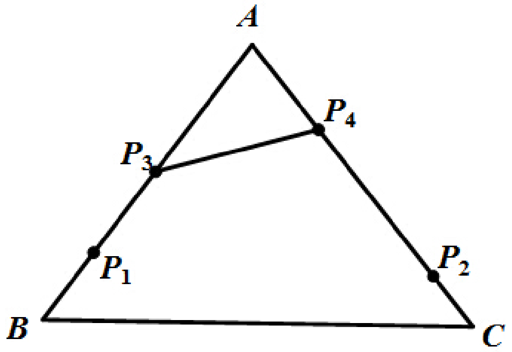


Figure 9

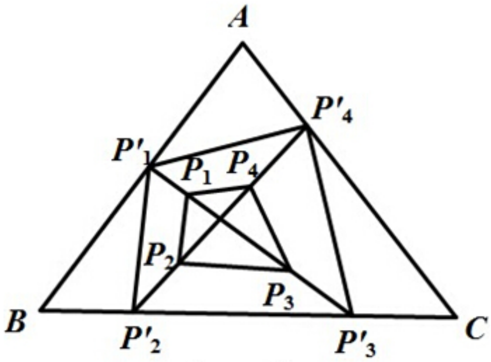


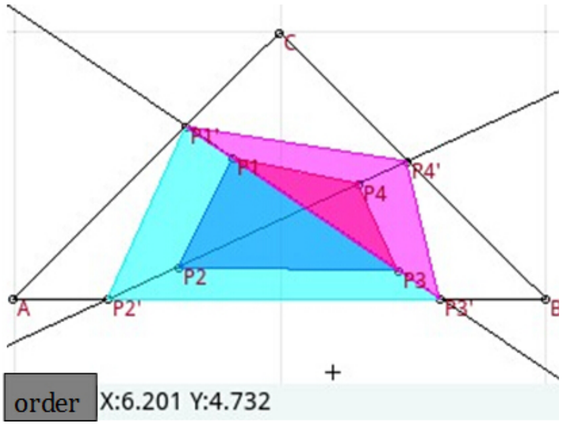
Figure 10

Case 2.3 Four vertices of a convex quadrangle are interior points of ΔABC whose area is one. Prove that at least one of the areas of the four triangles $\Delta P_1P_2P_3$, $\Delta P_1P_2P_4$, $\Delta P_1P_3P_4$, $\Delta P_2P_3P_4$ is not bigger than 1/4.

If the four vertices of a convex quadrangle are inside of ΔABC, which has an area of one (as shown in Figure 10), we connect the diagonal P_1P_3, P_2P_4 and extend until they intersect at P_1', P_2', P_3', P_4' of triangle ΔABC. We unify and study Case 2.3 and the following Test 3.

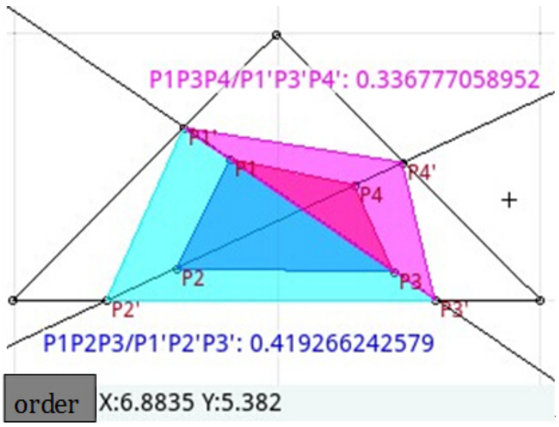
Test 3 Prove that the areas of $\Delta P_1P_2P_3$, $\Delta P_1P_2P_4$, $\Delta P_1P_3P_4$, $\Delta P_2P_3P_4$ are not bigger than $\Delta P_1'P_2'P_3', \Delta P_1'P_2'P_4', \Delta P_1'P_3'P_4', \Delta P_2'P_3'P_4'$ successively.

Draw a picture according to the assumptions in the Geometry app.

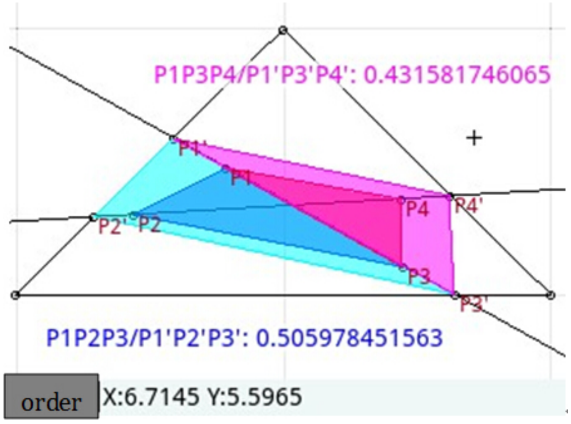


For concise visual experiences, here we just prove for $\Delta P_1P_2P_3$, $\Delta P_1P_2P_4$, $\Delta P_1P_3P_4$, $\Delta P_2P_3P_4$ on one of the diagonals.

Find the area ratio of each pair of triangles directly.



It can be found that the ratios are constantly less than one when P_2 or P_4 undergo arbitrary drags (keeping them inside the triangle).



The result shows that Case 2.3 is true. By this conclusion, we can redo the process of proving for Case 1.3, for you can always find four points out of any of the 5 points in the triangle and thus can use them as vertices to construct the convex quadrilateral. This proof is familiar to us and graphing calculator can again be used; thus it is omitted here.

4. Conclusion

Here the "Five Points" conjecture was solved by using deductive and plausible reasoning that we adopted from information technology, and specifically by using the dynamic geometry and measure functions of the graphing calculator, which provides us with an alternative way of solving problems in the future. With the development of information technology, computers and graphing calculators are going to be more and more important in testing conjectures and raising new methodology. It will bring lots of advantages and convenience to students' work and life and meanwhile will strengthen their ability to apply math. This is definitely a revolution in math education. Students do not need to remember many theorems, proofs and conclusions and they will know how to use computers to figure out mortgage calculation or to count the expense of house-decorating; and will learn to choose the best route with a navigator or to make right strategy about company resources. All of these demonstrate the power of mathematics.

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