# Integration of Products using Differentials

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**Abstract**: *This paper is an attempt to increase students' performance in Integral Calculus by the inclusion in the curriculum of a new integration technique, Integration of Products using Differentials, introduced by Dr. Tilak de Alwis. The study was conducted in the only locally funded chartered university in the Philippines. The study employed mixed method research design where students performance and verbal feedbacks were analyzed. Results showed a ratio of 2:3 attempts versus no attempts. Findings showed that the performance of students who attempted to use the technique is not significantly different from the performance of students who did not attempt to use the technique. Among the attempts group, success rate is higher in items involving integration of products than their over-all success rate while for the no-attempts group success rate is not significantly different.*

## I. Introduction

According to the National Council of Teachers of Mathematics [1], a strong foundation in mathematics is increasingly important in almost all fields of study most especially in arts, physical sciences, engineering and, health sciences. Mathematics has become important tools for analysis and interpretation of the data they dealt with and, validation of the contrivances that they had produced.

Singapore Ministry of Education characterized mathematics as an excellent vehicle for the development and improvement of a person's intellectual competencies. In the job market for instance, as cited by US Department of Education Mathematics Equals Opportunity [2] workers who have strong mathematics and science backgrounds are more likely to be employed and generally earn more than workers with lower achievement. It is for this reason why performances of students in mathematics and science are measured regularly.

Trends in International Mathematics and Science Study (TIMSS) for instance, annually reports performances of students worldwide in the two mentioned subjects. For the past 20 years, TIMSS has measured trends in mathematics and science achievement at the fourth and eighth grades. It has been conducted on a regular 4-year cycle since 1995, making TIMSS 2011 the fifth assessment of mathematics and science achievement trends. TIMSS Advanced measures trends in advanced mathematics and physics for students in their final year of secondary school. Results of the TIMSS assessment have become a reference of all nations in developing their mathematics and science curriculum.

Carbello [3], reported that the Science and Education Institute study on Trends in Mathematics and Science Study in 2003 showed that Philippines' 8th grade (2nd year high school) students' skills and competencies in Math ranked a pitiful 42nd out of 46 participating countries while the Philippine 4th grade students placed 23rd out of 25 participating countries. This report is quite alarming because if this trend continues poor performance in mathematics in the tertiary level is expected.

In the study of Salleh and Zakaria [4], they raised concerns about the decline in student's performance in Integral Calculus. In this subject integration is an important concept in mathematics and together with differentiation, is one of the two main operations in calculus. Their study found out that in the departmental final examination, only 6.8% of the students attempted to answer the questions involving integral calculus. They proposed for the implementation an innovative change in the teaching and learning of mathematics, particularly in integral calculus.

Nowadays, ways to improve performance in Calculus are present. These include pedagogy, technology and instructional environment. In the study of Noinang, et al [5] entitled "Teaching-Learning Tool for Integral Calculus", they introduced a set of PowerPoint slides with Maple animation and interactive Mapletswith Maple worksheets. Results showed that PowerPoint slides with Maple animation helped instructors explain certain concepts and methods more effectively and clearly. The interactive Maplets and Maple worksheets reinforced students' conceptual understanding of integral calculus.

Dimicelli, Lang & Lock [6] presented another alternative method. They used Wolfram|Alpha as the platform for teaching calculus concepts in the lab setting. Wolfram|Alpha is a free, browser-based web service, developed by Wolfram Research, which dynamically calculates results to natural language queries by applying algorithms to its extensive internal database of facts.

Chappel & Killpatrick [7] investigated the effects of instructional environment(concept-based vs. procedure-based) on students' conceptual understanding and procedural knowledge of calculus. Results showed that the students enrolled in concept-based environment scored significantly higher than students enrolled in the procedure-based environment on conceptual understanding as well as procedural skills.

In the 2012 Asian Technology Conference in Mathematics which was held in Bangkok, Thailand, Dr. Tilak de Alwis presented a Novel Technique or Integrating Certain Products without using Integration by Parts. The Novel Integration Technique is based on the product rule for differentiation with the formula:

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

The procedure involves the following steps: (1) identifying an expression in product form, whose differential contains the given integrand; (2) getting the differential of the expression in step 1; (3) integrating the result of step 2 and; (3) applying the axioms of equality to obtain the desired integral. It is to be noted that an arbitrary constant must be added to give explanation to the anti-derivative of zero, which is equivalent to a constant. The procedure could be done repeatedly until the desired integral is found. Dr. Tilak [8] said that the method is quite useful when integrating products of polynomial and exponential functions, products of polynomial and trigonometric functions, odd powers of secant and cosecant functions, three fold products of elementary functions.

The following examples illustrate how the procedure works and were given as items in the exams in Calculus.

**Case 1. Integration of products of the form $x^n e^u$, where u is a function of x.**

The first step in integrating products using differentials instead of integration by parts is to look for an expression whose differential contains the given integrand. For integrand of the form $x^n e^u$, this expression is the integrand itself, i.e. $x^n e^u$. The second step is to use the product rule for differentiation. The differential of $x^n e^u$ is as follows:

$$d(x^n e^u) = \boldsymbol{x^n e^u} du + n x^{n-1} e^u dx \qquad (1.1)$$

Notice that $x^n e^u$ is in the first term of the differential as shown in (1.1). The next step is to integrate both sides of (1.1)

$$\int d(x^n e^u) = \int \boldsymbol{x^n e^u} du + \int n x^{n-1} e^u dx$$

$$x^n e^u = \int \boldsymbol{x^n e^u} du + n \int x^{n-1} e^u dx \qquad (1.2)$$

Notice that the left side of (1.2) contains the integrand $x^n e^u$. The procedure should be repeated until the second component of (1.2) can be integrated using the fundamental integration formulae. To illustrate, let us consider the following examples.

1. Calculate $\int xe^{3x} dx$ without using integration by parts

   First we differentiate $xe^{3x}$ with respect to $x$ using the product rule for differentiation:

$$d\left(xe^{3x}\right) = \left(3xe^{3x} + e^{3x}\right)dx \qquad (1.3)$$

Then we integrate both sides of (1.3),

$$\int d(xe^{3x}) = 3\int \boldsymbol{xe^{3x}} dx + \int e^{3x} dx \qquad (1.4)$$

Notice that the second term in (1.4) is already integrable using the fundamental integration formulae . Integrating it yields

$$xe^{3x} = 3\int xe^{3x} dx + \frac{1}{3} e^{3x}$$

By applying the axioms of equality, we get the following:

$$3\int xe^{3x} dx = xe^{3x} - \frac{1}{3} e^{3x} \qquad (1.5)$$

Then we divide both sides of (1.5) by 3 and add the arbitrary constant C .

$$\int xe^{3x} dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$$

2. Calculate $\int x^2e^{5x}dx$

Similar to example 1, we differentiate the expression $x^2e^{5x}$ with respect to *x*. This will yield

$$d\left(x^2e^{5x}\right)=\left(5x^2e^{5x}+2xe^{5x}\right)dx \qquad (2.1)$$

Then integrating both sides of the equation we obtain:

$$x^2e^{5x}=5\int x^2e^{5x}dx+2\int xe^{5x}dx \qquad (2.2)$$

Since the second term in (2.2) is not integrable using the fundamental integration formulae, we repeat the process on $\int xe^{5x}dx$. We start with the differential of $xe^{5x}$,

$$d\left(xe^{5x}\right)=\left(5xe^{5x}+e^{5x}\right)dx \qquad (2.3)$$

Then integrating both sides of (2.3) we obtain

$$xe^{5x}=5\int xe^{5x}dx+e^{5x} \qquad (2.4)$$

Next we consider (2.2) and (2.4). Our target is to eliminate the term involving $\int xe^{5x}dx$ by adding or subtracting the equations. Applying $\frac{1}{2}$ (2.2) $-$ $\frac{1}{5}$(2.4) yields

$$\frac{1}{2}x^2e^{3x}-\frac{1}{5}xe^{5x}=\frac{5}{2}\int x^2e^{5x}dx-e^{5x}$$

By applying the axioms of equality, we get the following:

$$\frac{5}{2}\int x^2e^{5x}dx=\frac{1}{2}x^2e^{3x}-\frac{1}{5}xe^{5x}-e^{5x}$$

$$\int x^2e^{5x}dx=\frac{1}{5}x^2e^{3x}-\frac{2}{25}xe^{5x}-\frac{2}{5}e^{5x}+C$$

**Case 2. Integration of Product of Polynomial and Trigonometric Functions**

In integrating products of a polynomial and trigonometric function, we consider the derivatives of trigonometric functions. For instance if we are to integrate products of the form $x^n\cos ax$, where a is constant, we have to think of the trigonometric function whose derivative is $\cos ax$. From the formula of the derivative of trigonometric function,

$$\frac{d(\sin u)}{dx}=\cos u\frac{du}{dx}.$$

So, to integrate products of the form $x^n \cos ax$, start with $x^n \sin ax$. To illustrate, let us work on the following examples,

Example 3. Calculate $\int x^3 \cos 5x dx$

To get the integral, we will start with the differential of $x^3 \sin 5x$.

$$d\left(x^3 \sin 5x\right) = \left(5x^3 \cos 5x + 3x^2 \sin 5x\right)dx \tag{3.1}$$

Integrating (3.1) results to

$$x^3 \sin 5x = 5\int x^3 \cos 5x dx + 3\int x^2 \sin 5x dx \tag{3.2}$$

However, the second term of (3.2) cannot be integrated by using the fundamental integration formulae but is also a product of a polynomial and trigonometric function. Hence we can apply the same procedure as follows:

$$d\left(x^2 \cos 5x\right) = \left(-5x^2 \sin 5x + 2x \cos 5x\right)dx \tag{3.3}$$

Integrating (3.3) yields

$$x^2 \cos 5x = -5\int x^2 \sin 5x dx + 2\int x \cos 5x dx \tag{3.4}$$

Again $\int x \cos 5x dx$ is integration of product so we will differentiate $x \sin 5x$ as follows:

$$d\left(x \sin 5x\right) = \left(5x \cos 5x + \sin 5x\right)dx \tag{3.5}$$

Integrating (3.5), we get

$$x \sin 5x = 5\int x \cos 5x dx + \sin 5x \tag{3.6}$$

Now, we consider (3.2), (3.4) and (3.6). First we will eliminate $\int x^2 \sin 5x dx$ from (3.2) and (3.4) by applying the following operations

$$\frac{1}{3}\ (3.2) + \frac{1}{5}(3.4)$$

$$\frac{1}{3}x^3 \sin 5x + \frac{1}{5}x^2 \cos 5x = \frac{5}{3}\int x^3 \cos 5x dx + \frac{2}{5}\int x \cos 5x dx \tag{3.7}$$

Next, we will eliminate $\int x \cos 5x dx$ from (3.6) and (3.7),

$$\frac{5}{2}(3.7) - \frac{1}{5}(3.6)$$

$$\frac{5}{6}x^3 \sin 5x + \frac{1}{2}x^2 \cos 5x - \frac{1}{5}x \sin x = \frac{25}{6}\int x^3 \cos 5x dx - \frac{1}{5}\sin x$$

Applying the rules for equality yields,

$$\frac{25}{6}\int x^3 \cos 5x dx = \frac{5}{6}x^3 \sin 5x + \frac{1}{2}x^2 \cos 5x - \frac{1}{5}x \sin x + \frac{1}{5}\sin x$$

$$\int x^3 \cos 5x dx = \frac{x^3 \sin 5x}{5} + \frac{3x^2 \cos 5x}{25} - \frac{6x \sin 5x}{125} + \frac{6 \sin 5x}{125} + C$$

**Case 3. Integration of Odd Powers of Secant and Cosecant**

The new method is easier to implement than integration by parts on the integral of odd powers of secant and cosecant[8].

Example 4. Calculate the $\int csc^3 x dx$

To calculate the integral of $csc^3 x$, we have to think of a trigonometric function that when we differentiate will give a result, which contains the expression $csc^3 x$. Since the derivative of cot x is - $csc^2 x$, we consider $csc\, x\, cot\, x$. Differentiating yields

$$d(csc\, x\, cot\, x) = (-csc^3 x - cot^2 x\, csc\, x)dx \qquad (4.1)$$

By integrating equation (4.1), we get

$$csc\, x\, cot\, x = -\int csc^3 x dx - \int cot^2 x\, csc\, x dx \qquad (4.2)$$

In (4.2) we notice that the expression $cot^2 x\, csc\, x$ cannot be integrated easily. So we have to get the equivalent expression of $cot^2 x$ from the Pythagorean Identities, which is $csc^2 x - 1$, and substitute this to $cot^2 x$. The resulting equations are as follows:

$$csc\, x\, cot\, x = -\int csc^3 x dx - \int (csc^3 x - csc\, x)dx \qquad (4.3)$$

$$csc\, x\, cot\, x = -\int csc^3 x dx - \int csc^3 x dx + \int csc\, x dx \qquad (4.4)$$

By combining like terms and integrating $csc\, x$ will yield

$$csc\,x\,cot\,x = -2\int csc^3\,xdx - ln\left|csc\,x + cot\,x\right| \quad (4.5)$$

$$\int csc^3\,xdx = -\frac{csc\,x\,cot\,x}{2} - \frac{ln\left|csc\,x + cot\,x\right|}{2} + C \quad (4.6)$$

This new method can be applied in other cases of products of elementary functions. The simplicity and elegance of this Novel Integration Technique as presented in the conference prompted the researchers to experiment on its inclusion in the curriculum of Integral Calculus.

## 2. Methodology

The study employed a mixed method design. Mixed method research design is a procedure for collecting, analyzing, and “mixing” both quantitative and qualitative research and methods in a single study [9]. Quantitative analysis was done on the scores of the respondents and their qualitative feedbacks on the technique were processed for deeper analysis of the results.

The locale of the study in a locally funded chartered university in the City of Manila, Philippines. The research was conducted in one section of Integral Calculus class.

The study made use of two research instruments: the final examination and the student feedback questionnaire. The first instrument is a departmental examination that covered all integration techniques and had undergone content validation. The student feedback questionnaire on the other hand is an open-ended questionnaire, which asks the students to enumerate their comments about the new integration technique.

## 3. Results and Discussion

**Performance of the Respondents**

The respondents were categorized into two: those who attempted to utilize the new technique and those who did not attempt at all. The ratio of attempts to no attempts is 12:18. Scores in the integration of products were extracted from the final examination scores. Then, scores of the two groups were compared and analyzed.

Table 1.Result of the Analysis of Performance of the Experimental Group

| Ratio of Attempts to No Attempt: 12:18 | | | | | | | |
|---|---|---|---|---|---|---|---|
| | Group | N | Mean | t | df | p-value | Conclusion |
| Products | No attempt | 18 | 42.59% | -0.695 | 28 | 0.493 | no significant difference |
| | Attempted | 12 | 50.00% | | | | |
| Over-all Score | No attempt | 18 | 32.08% | 1.607 | 28 | 0.119 | no significant difference |
| | Attempted | 12 | 26.46% | | | | |

Table 1 shows that out of 30 students in the experimental group, 12 attempted to use the Novel Integration Technique and 8 students did not attempt to use the Novel Technique of Integration. Those who had attempted to use the Novel Technique had an average percentage score of 50.00% on integration of products while those who did not attempt to use the Novel Integration

Technique had an average percentage score of 42.59%. This shows that the average percentage scores of those who used the technique on integration of products was more than 7% higher than the average percentage score of those who did not apply the new technique. However this difference in mean score was found to be not significant . In terms of the over-all final examination scores, the attempts group got a mean score of 26%, which is lower than the 32% mean score of the no attempts group. However, the difference was again found to be not significant.

**Comparison of the Performance in the Integration of Products and Over-all Performance**

Table 2.Result of Comparison of the Performance in the Integration of Products and Over-all Performance

| | GRP | Mean | t | df | p value | conclusion |
|---|---|---|---|---|---|---|
| no attempt | product | 42.59 | 1.632 | 17 | 0.121 | no significant difference |
| | total | 32.08 | | | | |
| attempted | product | 50.00 | 3.062 | 11 | 0.011 | significant difference |
| | total | 26.46 | | | | |

Table 2 exhibits the comparison of performance in the integration of products and the over-all performance in integration techniques of both the no attempts and the attempts groups The t-tests for the group who did not attempt to use the Novel Technique, resulted to a p-value of 0.121 indicated that there is no significant difference between the performance in the integration of products and the over-all performance. This means that the conventional technique has a weak impact in improving the over-all performance of the students. On the other hand, the result of the t-test for the group who attempted to use the Novel Technique yielded a p-value of 0.011 indicating that there is a significant difference in the performance in integration of products and the over-all performance. This shows that success rate of the attempts group was higher on items were they used the Novel Integration Technique. This is a demonstration of the new technique's effectiveness in improving student's performance.

**Feedback on the Novel Integration Technique**

Student feedback on the Novel Integration Technique is classified into two: the positive feedback and the negative feedback. The positive feedback includes the following:

1. "Better because it is easier to differentiate than to integrate."
2. "Easier because it's difficult to identify the proper u and dv functions in integration by parts."
3. "The procedure is better because it helps the student recall and properly use the differentiation formulae."
4. "With the Novel Integration Technique you don't have to recall all trigonometric identities when dealing with trigonometric functions."
5. "Easy and the procedure does not change unlike in other differentiation technique."

Negative feedback on the other hand includes the following:

1. "Confusing because I don't know any more when to use the Novel Integration Technique and when to use Integration By Parts."
2. "Sometimes it's hard to think of a function to differentiate."

## 4. Conclusion

Results showed that the method received good verbal feedbacks from students signifying that the method was easy to understand. This was also confirmed by the higher success rate of students who used the method on integration of products versus their over-all success rate. This higher success rate suggests that the Novel Integration Technique when included as an alternative technique in integration of certain producs may be further developed to help students facilitate integration of functions.

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