

Examples and Techniques of Morphing within CAS and DGS Environments (Cabri and TI-Nspire)

A Way of Enriching our Teaching at all Levels

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Abstract: *Proportionality has always been a very important part of all curricula in all countries. This concept is used both in geometry and algebra (similar triangles, linear functions for example). For middle school students, this concept is not simple to understand even if they solve a lot of problems based on it. This paper aims to give another approach of this concept in showing the power of dynamic geometry to create motivating activities for our students at all levels, from middle school to university. Morphing which is a “continuous transformation” of an object onto another is something very familiar to the students of the present generation: they are used to meeting images animated by morphing everywhere on their smartphones, their tablets and their games... As DGS software is mostly based on the continuity property, the other one is “determinism” ([5]), such software must be appropriate to help us modelling the morphing phenomenon in 2D and 3D. Cabri 2 Plus, Cabri 3D and the geometry application of TI-Nspire are the tools used to present my “morphing” examples with their associated techniques. At a higher level, the connection between the different software of the TI-Nspire and especially between CAS and geometry will allow us to model for example, problems of morphing in relation to curves in 2D and 3D ([1]). Knowing and practising these techniques open to students a window to a lot of realistic modelling. In doing so they improve their skills with the software used in this paper and improve deeply their math knowledge. At last, solving challenging problems coming from the real life increases their motivation for mathematics ([7]).*

1. Introduction: a survey of some challenging examples

1.1. The floating flag (Figure 1)

I have conducted a workshop with middle school students with the help of a French teacher of the French High school of Casablanca (research project between this highschool and my Institute). This workshop was about “creating animations with Cabri 3D”. Before letting students working on their own, we used to give them some techniques in order to make richer use of the software. The floating flag is one of the examples created in order to show some of these techniques. We can state that this way of conducting such workshop is really very powerful: the idea comes from a research work done by Kate Mackrell ([6], [8]). The flag represented below (Figure 1 on the left) is an example of morphing performed with *Cabri 3D* (3D DGS) where the starting point is the morphing of an arc onto another. The flags represented below in Figure 1 in the middle have been created in the geometry application of *TI-Nspire* (very similar to *Cabri 2 Plus*) and we used the military perspective ([12]) which is a good perspective for such representation with a 2D DGS ([2] and ([3]). The techniques we used are really the same as those used in the previous example.

This technique is illustrated in Figure 1 on the right: Two points A and B are created on a line (D) of the horizontal plane. $[MN]$ is a segment on line (D') perpendicular to (D) in the same plane where I is the common midpoint of segments $[AB]$ and $[MN]$. V is a point that can be animated along segment $[MN]$ and finally an arc is created with the three points A , V and B . The animation of V will generate the morphing of the arc defined by A , M and B onto the arc defined by A , N and B . This arc is our way of modelling the bottom of the flag. Central symmetries are used to create the

two other arcs of this bottom part. Translations are used to create the upper arcs. Points are created on the first arc and translated with the translation mapping A onto A' . The surface of the flag is modelled with polygons passing through the previous points (really rectangles).

This technique is presented in a YouTube video of my personal channel ([10]). A 12-year-old student showed to French math teachers this construction during the annual congress of our math association in October 2013 in Marseille (France). It was the confirmation that our way of conducting such initiation helps students to memorize 3D constructions when related to the real world, to enhance a conceptual approach of proportionality when using several times the same provided technique. This approach is successful even if students have to use math tools of an upper level hidden behind the tools of the software ([9] and [10]).

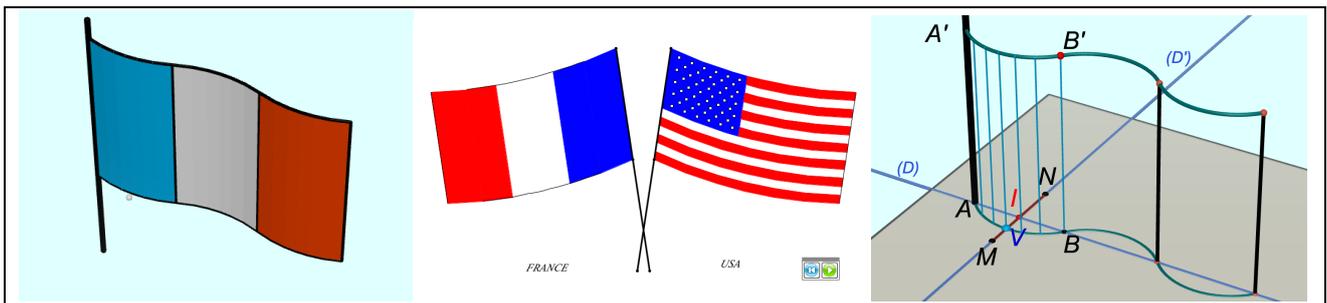


Figure 1: Floating flags (*Cabri 3D* and *TI-Nspire*)

Remark: middle school students noticed very quickly that, although the modelling of the flags was easy to understand and to perform, this modelling was incorrect according to math properties: they argued that the length of the flag was not constant because during the motion the size of the animated arc was changing. That was a very pertinent argument. We will solve in paragraph 4 this problem in using the CAS of *TI-Nspire* and the possible link between the **Note application** (with CAS) and the **Graphs & Geometry application** of this software. The solution is understandable at the University level.

1.2. Morphing a triangle onto a circle and *vice-versa*

Figure 2 on the left: The following example of morphing is a simple but very motivating one. In *Cabri 2 Plus*, we use the locus of point V on segment $[MN]$ when M moves along the sides of triangle PQR (not possible in *Cabri 3D* directly: see below). N is the intersection point between ray $[OM)$ and the red circle centred in O . As point V stays in a position on $[MN]$ such as ratio VM/MN is constant, the locus of V is an intermediate curve between the given triangle and the red circle. The animation of V gives the solution of the morphing of a triangle onto a circle. Changing the shape of the triangle, changing the position of O , changing the radius of the red circle provide to the experimenter a lot of opportunities for artistic or realistic performances ([11]).

Figure 2 in the middle: the problem to solve is the same as the previous one but here we work on the horizontal plane of *Cabri 3D*. Let us state first that loci on *Cabri 3D* can only be created when the commanding point belongs to a segment and secondly, if a point is created on a side of a triangle it cannot be dragged along the other sides. Because of these two technical problems, we have to modify the technique of morphing to solve the problem in *Cabri 3D*. We do so with the use of special sliders and the dilation tool. Here is the new technique:

Let us give the triangle PQR and the circle centred on O and passing through S . Point m on slider $[vw]$ commands a number between 0 and the perimeter of the circle. By the way, m commands the

animation of point M along the circle. A_1, A_2 and A_3 are respectively the intersection points between the ray $[OM)$ and the sides of the triangle PQR , $[PQ]$, $[QR]$ and $[RP]$. N_1, N_2 and N_3 are the images of point M with respect to the dilations centred respectively in A_1, A_2 and A_3 and having the same ratio (between 0 and 1); this ratio is commanded by point n on segment $[tu]$. The three loci of points N_1, N_2 and N_3 are altogether the intermediate curve between the triangle and the circle. We perform the expected morphing in dragging point n from t to u : the triangle is morphed continuously onto a circle.

Figure 2 on the right: the technique used in *TI-Nspire* is exactly the same as the one used in *Cabri 3D*. We have added the loci of two segments to represent in military perspective the morphing of a prism onto a cylinder and the morphing of a pyramid onto a cone.

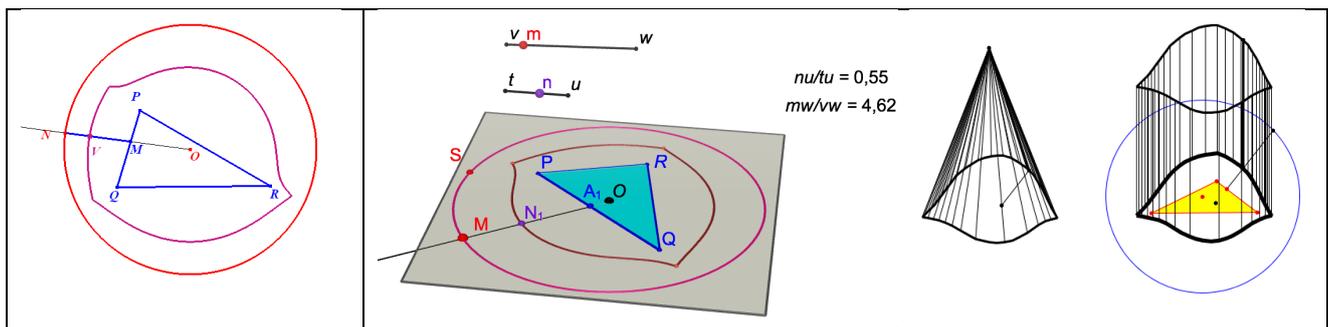


Figure 2: Morphing a triangle onto a circle with *Cabri 2 Plus*, *Cabri 3D* and *TI-Nspire*

1.3. Morphing a number onto another one

Figure 3 on the left: in this example presented during the defence of my PhD ([1]), I wanted to illustrate the concept of “figure”. I wanted to show that *Cabri* as a DGS software could help anyone to understand the difference between a figure and each of its representations (a “figure” is a program and one of its representations is an output of this program). The animation of the empty point of this figure along a segment morphs number 1 onto number 2. In this figure, apart from the initial and the end positions we can see an intermediate position of this morphing and my constructions. This construction was not a general construction for any morphing segment-segment or segment-arc but a tricky construction solving this particular task.

Figure 3 in the middle (with *TI-Nspire*): here we can see that the heights of number 1 and number 3 are the same. The technique created for this morphing is a mix between the previous techniques: it uses alignments of points (points on 1 and points on 3) with a special point which is the intersection point between the line [first point-first point] and the line [last point-last point] of the two objects to morph. But below,, these two lines are parallel so we had to create a slider to control the motions of the three intermediate points defining our intermediate curve. This curve is defined with three loci depending from three points moving along the three segments of initial number 1. The intermediate point in each case is defined as a dilated point with a dilation having as a ratio a number between 0 and 1.

Figure 3 on the right (with *TI-Nspire*): this example uses the general technique we can use for each problem of morphing where the two objects have the same number of parts and where these parts are either segments or arcs. The idea of the technique can be understood in the right part of the illustration of the morphing of number 4 onto number 2. Let us explain how to morph segment $[cd]$ onto arc CD : we measure first their lengths, we create a point p on $[cd]$, measure cp , evaluate cp/cd and then multiply this number by the length of arc CD , transfer this number on the arc from C to obtain point P . We create a slider with a segment generating a number between 0 and 1. We use this

number as the ratio of the dilation centred on c to obtain point $I3$ as the image of point P . The locus of $I3$ when the point of the slider moves on its segment is the third part of the intermediate curve morphing 4 onto 2. The technique used for the other parts is strictly the same. In these examples the case of the morphing of an arc onto an arc has not been treated (there is a difficulty about the measurement of an arc where one of its vertices is defined on another arc).

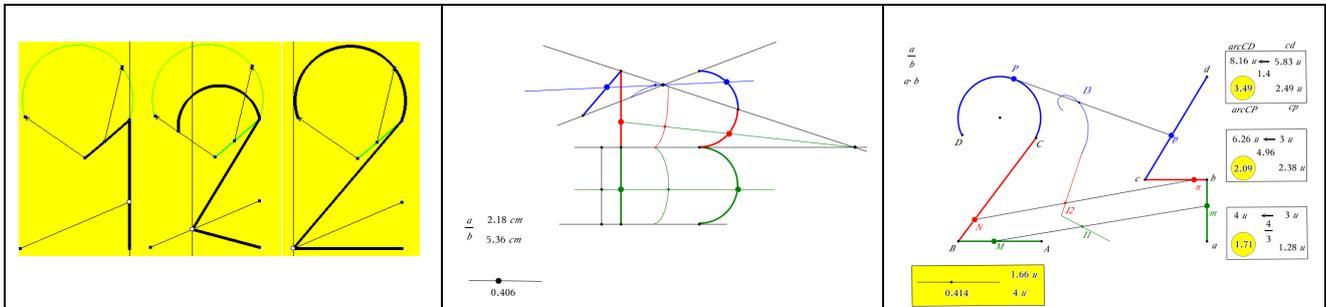


Figure 3: Morphing a number onto another number (*Cabri 2 Plus* and *TI-Nspire*)

2. Some interesting morphings fore some original conjectures

2.1. Morphing a segment onto a segment (*Cabri 2 Plus*)

We perform the morphing of segment $[ab]$ onto segment $[AB]$ where the intermediate curve is the locus of V when point k moves on its slider (controlling a number between 0 and 1). The visualisation of a set of intermediate curves (Figure 4 on the left) leads to the conjecture stating that the intermediate curves are segments. But the next experiment with other locations and sizes for the two segments immediately invalidates this conjecture (Figure 4 on the right)

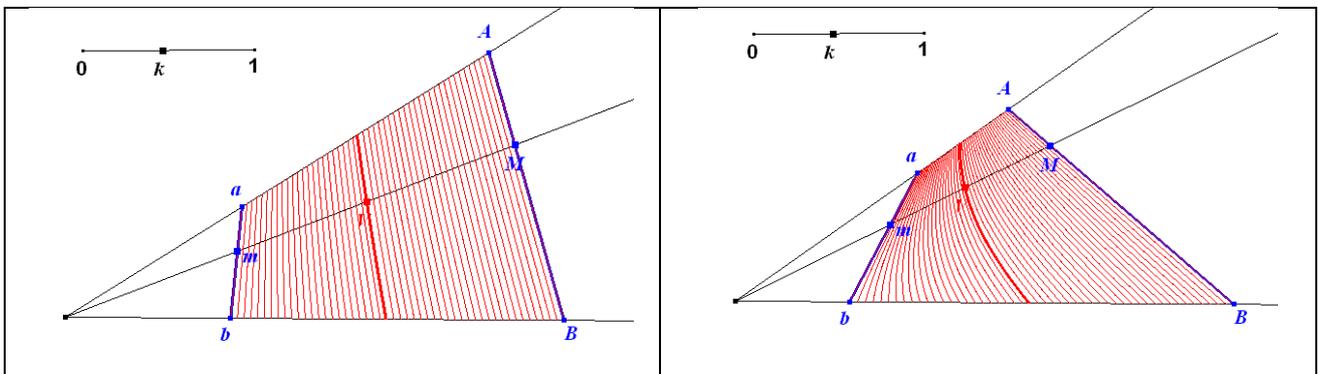


Figure 4: Morphing a segment onto another one (*TI-Nspire*)

2.2. Morphing a segment onto an arc (*TI-Nspire*)

The same investigation can be undertaken when morphing a segment onto an arc when using the same technique. In Figure 5 on the left we conjecture that the intermediate curve is an arc. This conjecture is invalidated in Figure 5 on the right because the intermediate curve in another case is similar to the letter S upside down. It could be interesting to experiment in order to discover when the intermediate curve is always an arc and to prove it (not done here).

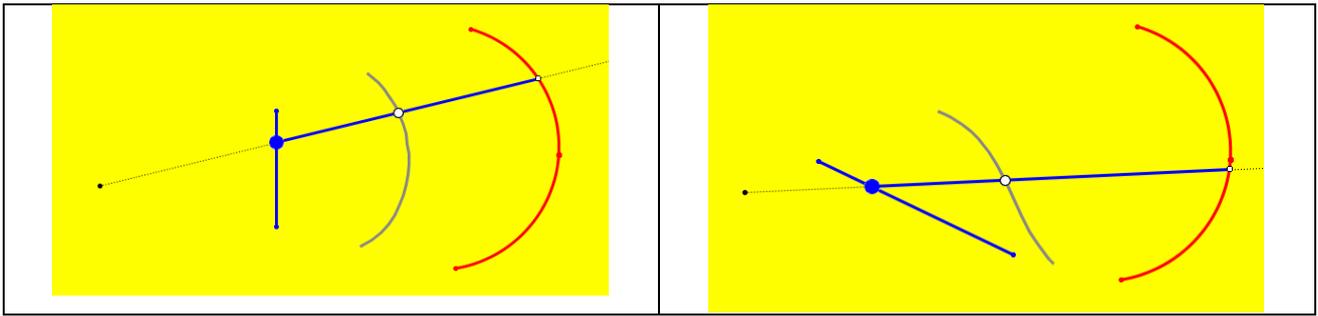


Figure 5: Morphing a segment onto an arc (*TI-Nspire*)

3. Several examples illustrating the “barycentre” technique

3.1. Dilations and barycentres (centroids)

If d is the dilation centred in A having k as a ratio such that $M = d(B)$, it means $\overline{AM} = k \cdot \overline{AB}$, which is equivalent to $(1 - k) \cdot \overline{MA} + k \cdot \overline{MB} = \vec{0}$ and so, M is the barycentre of the system $\{(A, 1-k), (B, k)\}$.

3.2. Examples (with *TI-Nspire*)

Figure 6 on the left: we display the two curves of functions f_1 and f_2 , a vertical segment $[AB]$ between these two curves (controlled by point V moving along the x -axis). The locus of a random point of segment $[AB]$ when V moves along the x -axis is the intermediate curve solution of our problem (dotted curve). The morphing is performed when dragging point M from A to B . According to paragraph 3.1., we can obtain the same result with function $f_3 = u \cdot f_1 + (1-u) \cdot f_2$ where u is commanded by a slider ($0 \leq u \leq 1$): the curve is the bold one.

Figure 6 in the middle: same technique in the 3D graphing tool where

$$f_1(x,y) = 2\sin(x) \text{ and } f_2(x,y) = 4.5.$$

Figure 6 on the right (with polar functions): we can first investigate the shape of $r_1(\theta) = \cos(k \cdot \theta)$ and $r_3(\theta) = \sin(k \cdot \theta) - k \cos(\theta)$ in changing the values of k with its slider (k is an integer).

Then, as $r_2 = u \cdot r_1 + (1-u) \cdot r_3$, changing u with its slider between 0 and 1 morphs the curve of r_3 onto the curve of r_1 .

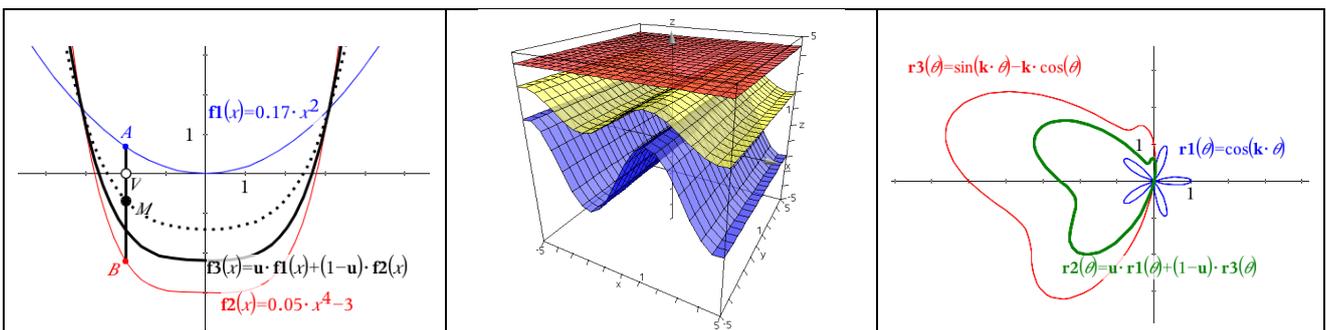


Figure 6: examples with *TI-Nspire*

4. The flag problem: a solution with *TI-Nspire*

4.1. Folding a segment onto an arc having the same length (and finally onto a circle)

The problem is: how to fold a segment onto an arc to model the bottom of the floating flag. Point V is a given point of segment $[OA]$. We want to find point $H(0,h)$ on the positive y -axis so that the length of the arc OSV is equal to $l = OA$.

- On a **Graphs & Geometry** page of *TI-Nspire*: the length of segment $[OA]$ is stored in variable called l . A point V is created on segment $[OA]$ and its abscissa is stored in variable xv . The ordinate h of point H defining with O and V the folded arc is the unknown number. Thanks to 6 on the left, we can find the equation which solution is h (same equation if C has a positive ordinate).

In the right angle triangle SOS'' , $\tan \alpha = (2h/xv)$ and $\alpha = \tan^{-1}(2h/xv)$.

In the right angle triangle SOS' , $OW^2 = WS.WS'$ which is $\frac{xv^2}{4} = k.(2k + h)$.

The length of arc OS is: $l/2 = (k + h).2\alpha$.

Eliminating k in the the system of the two previous equations leads to the relation linking h and xv :

$$\frac{xv^2}{4} - h \left(\frac{l}{\tan^{-1}(2h/xv)} - h \right).$$

Before using the CAS, we create in the geometry page a number stored immediately in a variable called **h** (it is a trick to create a variable in the actual geometry page that can be commanded from another page; in this case the software does **not italicized the variable** which is **displayed in bold**)

- We use now the CAS of *TI-Nspire* to solve this equation with respect to h in another page of the same problem (a **Note** page). Each command is put in a **Math box** of this Note page (in doing so, the value of xv and the result **h** is refreshed at any change of xv in the same problem where we will insert the graphs and geometry page of our modelling). A copy of the Note page is shown in Figure 7 on the right.

- We create point S having $xv/2$ and h as coordinates. The arc OSV is the solution of our problem. When V is on A , our solution must be a flat arc (a segment) but the software does not create an arc defined by three collinear points. When V in on O , our solution must be the circle centred at point $(0,1/2\pi)$ but our solution disappears because this arc is defined in this case only with two points O and S at the ordinate $1/\pi$. So, if we want to achieved a complete modelling of the folding-unfolding of a segment onto a circle via intermediate arc positions (all these objects having the same length), we need to use conditional constructions allowing segment $[OA]$ to appear when V reaches A and allowing a circle to appear when V reaches O . We will do it in 4.3..

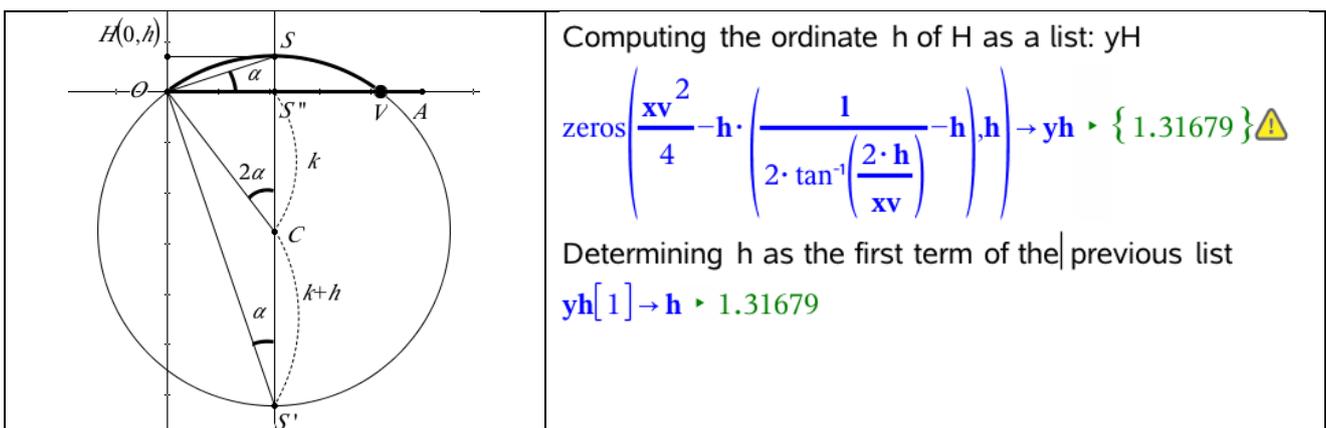


Figure 7: Using the CAS of *TI-Nspire*

4.2. How to obtain the symmetric motion (Figure 8)

$[O'A']$ is a segment parallel to segment $[OA]$ and its length is $2l$. W belongs to $[O'A']$ and commands point V on $[OA]$. When W moves from O' to B , V moves from O to A such as $xw = xv$ (xv is the abscissa of V and xw the abscissa of W). When W moves from B to A' , V moves from B to A such as $\frac{xv+xw}{2} = l$. The general formula giving xv with respect to xw for xw belonging to $[0,2l]$ is: $xv = l - |l - xw|$. We evaluate on the geometry page the expression $\text{sign}(l-xw)$ and store it in variable s . Then the CAS page is modified as shown in Figure 8 on the right. In order to obtain an arc located under the x-axis when W moves along $[BA']$ from B to A' , we need change the sign of h . After that the constructions are the same as the previous one.

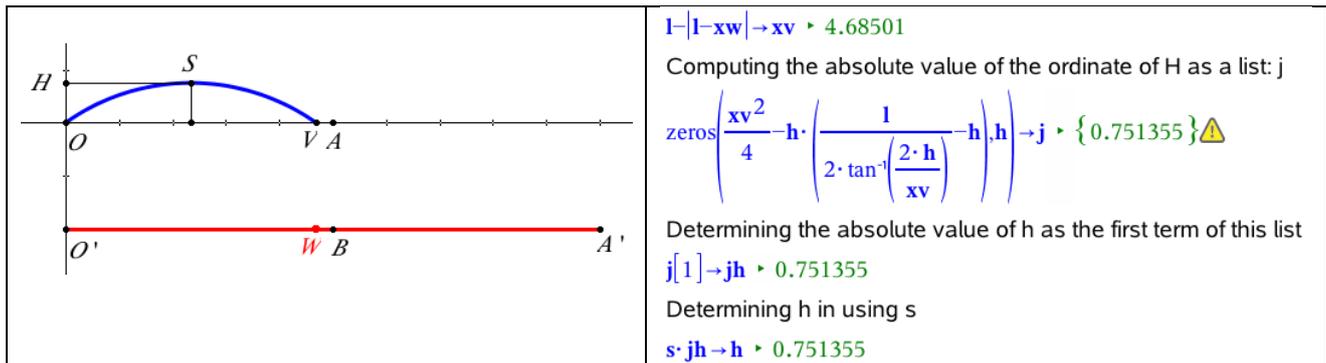


Figure 8: Linking CAS and Geometry (TI-Nspire)

4.3. Obtaining the last remaining cases

When W is superimposed to O , we construct the circle centred at point $(0,r)$ ($r = \pi/2$); when W is superimposed to A' , we construct the circle centred at point $(0,-r)$ ($r = \pi/2$) and when W is superimposed to B , we construct the segment superimposed to segment $[OA]$. The two circles are programmed by their equations:

First circle: $(x - \sqrt{-xw})^2 + (x - r)^2 = r^2$ (exist only when $xw = 0$)

Second circle: $(x - \sqrt{xw - 2l})^2 + (x + r)^2 = r^2$ (exist only when $xw = 2l$)

The segment is programmed as a parametric curve: $x(t) = l \cdot t$ and $y(t) = \sqrt{l - xw} \cdot \sqrt{xw - l}$ for $0 \leq t \leq 1$ and $step = 0.01$ (exist only when $xw = l$).

The animation of W along $[O'A']$ generates the complete folding and unfolding of an arc upside and downside (Figure 9). This technique will be used to model the floating flag in the next paragraph.

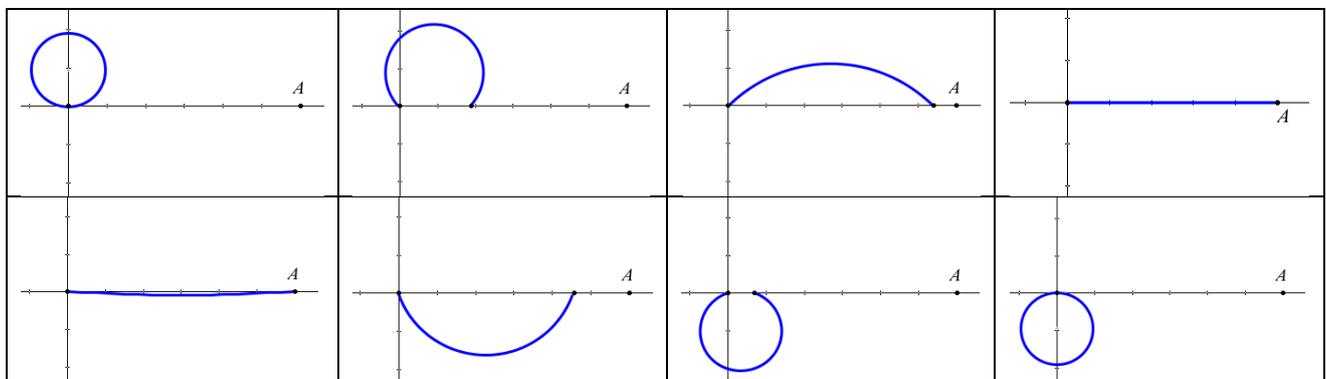


Figure 9: All the stages of the flag folding-unfolding

4.4. The French floating flag

To model the flag, the animation of point W (Figure 8) is limited to a small segment around B . The five other arcs are constructed with symmetries and translations. Filling the blue and red parts is obtained with loci of segments as shown in Figure 10.

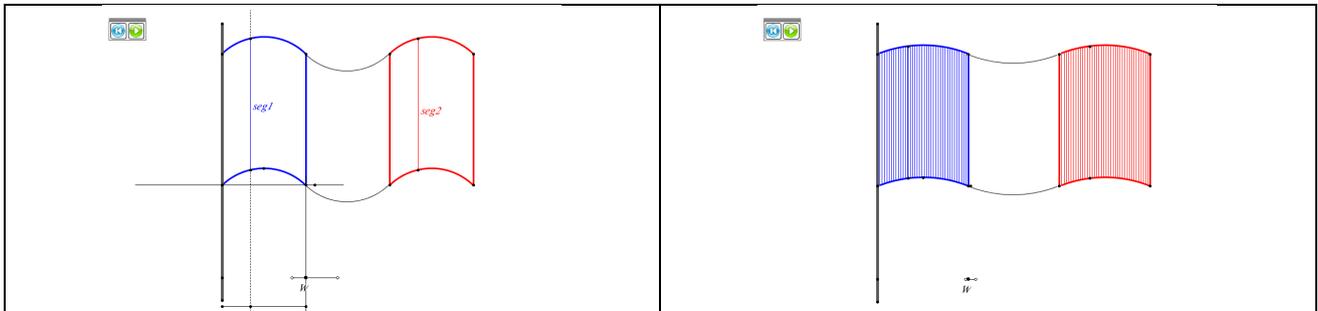


Figure 10: French floating flags

5. The flag problem: an approach of the solution with *Cabri (2 Plus and 3D)*

We know that it is not possible to construct with compass and ruler what we did with the help of the CAS of *TI-Nspire*. As *Cabri 2 Plus* and *Cabri 3D* do not include CAS capabilities, this problem cannot be solved exactly with these software tools. Nevertheless, we will show a very good approximation of this modelling in using the dynamic net of a regular polygon, which can be folded and unfolded easily geometrically. All the following constructions are shown in *Cabri 2 Plus* (Figure11).

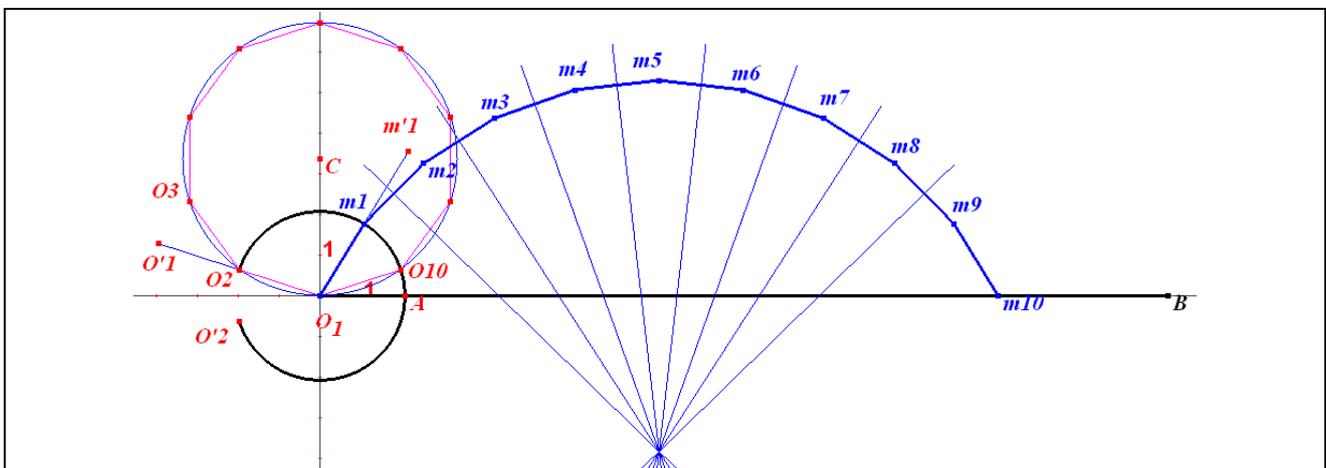


Figure 11: Folding and unfolding a regular decagon (*Cabri 2 Plus*)

Segment $[O_1B]$ is considered as an unfolded decagon (could consider a regular n -agon) and the first side is $[O_1A]$ ($O_1A = O_1B/10$ or O_1B/n). To fold this side, we consider point m_1 moving on an arc of the circle centred at O_1 and passing through A . This arc is defined by $O_2AO'_2$ where O_2 and O'_2 are symmetric with respect to the x -axis and angle $\angle AO_1O_2$ is evaluated by 162° (or by $180^\circ - 180^\circ/n$). Now the trick is how to construct the second side $[m_1m_2]$ of the unfolded decagon (n -agon) knowing the first one $[O_1m_1]$. Let us use point O'_1 symmetric of O_1 with respect to O_2 and m'_1 symmetric of

O_1 with respect to m_1 . We know that when point m_1 moves from A to O_2 , angle $\angle A O_1 m_1$ goes from 0 to 162° and angle $\angle m_2 m_1 m'_1$ goes from 0 to $\angle O'_1 O_2 O_3 = 36^\circ$ (or $180^\circ/n$). If $x = \angle A O_1 m_1$ is an intermediate position and y the corresponding value of $\angle m_2 m_1 m'_1$, respecting the proportionality rule which states: $\frac{y}{36} = \frac{x}{162}$, so y can be evaluated with the Cabri calculator with $y = \frac{36x}{162}$. This number is the absolute value of the angle of the rotation centred at m_1 transforming m'_1 onto m_2 . If s is the sign of the ordinate of point m_1 , the angle of the rotation is given by $-s.y$ evaluated by the software. That allows us to construct $[m_1 m_2]$ even if m_1 has a negative ordinate. To go further in this construction we get $[m_2 m_3]$ as the symmetric of $[O_1 m_1]$ with respect to the perpendicular bisector of segment $[m_1 m_2]$. We iterate the process until we get m_{10} .

Accuracy of the method: the length of the unfolded polygon equals here $l = O_1 B$; If we measure the length of arc $O_1 m_5 m_{10}$ ($= l'$) which is the arc we want to model, the ratio $\frac{l'-l}{l}$ is less than 5.10^{-4} (which means less than 0.05%) when we stay in the domain of modelling the flag (m_{10} close to B).

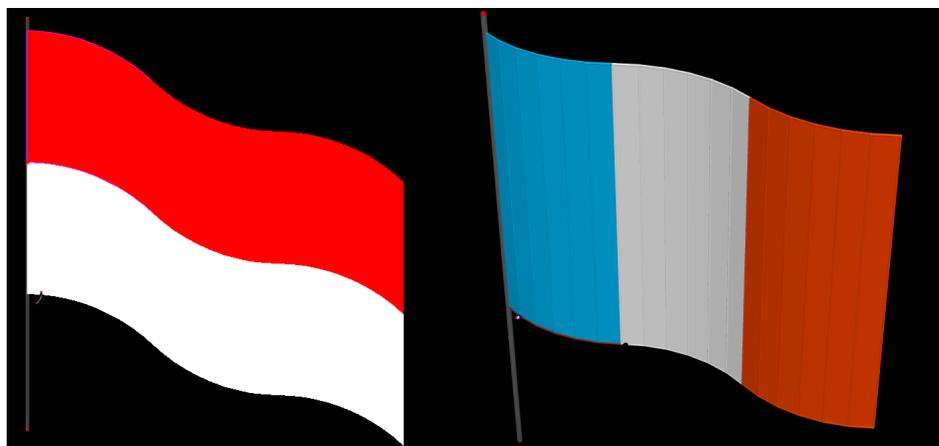


Figure 12: Floating Indonesian and French flags

If we use this technique in *Cabri 2 Plus* and complete the file to create the Indonesian flag we get the result shown in Figure 12 on the left. The same technique has been used in *Cabri 3D* to recreate the French floating flag (Figure 12 on the right: here we have used an unfolded regular hexagon with which the accuracy is still very good in the domain of the flag modelling)

6. Conclusion

In this paper, we have illustrated how proportionality can be used in dynamic geometry software to model morphing situations involving curves and surfaces. A geometrical approach is possible within the *Cabri 2 Plus* and *Cabri 3D* environments using the appropriate mathematical knowledge (barycentre, dilation, locus) associated with the appropriate tools of the software (invariance of the ratio defined by a point on a segment, sliders, loci, calculator, measurement transfer...). More difficult problems like “modelling a flag” (by the animation of an arc having a constant length) justify the use of a CAS connected to the geometry or the graphs page in a tool like *TI-Nspire*. The richness of the examples that can be treated with these techniques at all levels must enhance teachers to promote in their teaching such a practice in order to motivate their students with really challenging problems ([4], [7]). At last, knowing these techniques enhances students to create on

their own modelling situations using them. In a future paper, we will show how to model a real morphing of the curves of the intermediate Fourier sums S_n from S_1 to its Fourier sum S , part of the challenge is to morph with a value of n which is not an integer to obtain a real continuity.

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- [10] YouTube video: “Initiation Cabri 3D Drapeau flottant au vent » at <http://youtu.be/7cHEyIdjvDw>
- [11] YouTube video: **Morphing_triangle_cercle** at <http://youtu.be/F0ZSW81VhF0>
- [12] YouTube videos about military perspective
“T3_Chicago_MP” at <http://youtu.be/vYStmrDNT8E>
“T3_Chicago_1” at <http://youtu.be/yXDhiOD1fvw>
“T3_Chicago_2” at <http://youtu.be/3rJZZMzbtng>

Software :

Cabri 2 Plus and *Cabri 3D* by Cabrilog at <http://www.cabri.com>

TI-Nspire by Texas Instruments at <http://education.ti.com/en/us/home>