Visualization of special orthogonal group SO(3) with dynamic geometry software

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Abstract: *In this paper, we try to visualize special orthogonal group $SO(3)$ with dynamic geometry software Cabri 3D. Every element in $SO(3)$ is an ordered triple of unit vectors orthogonal to each other. Every element in $SO(3)$ is also given as a rotation of the orthogonal bases around an axis with an angle. With this geometric aspect, multiplication of matrices is simply visualized. In this visualization, spin-like movement of the axis is observed. Through the investigation of this phenomenon, geometric meaning of multiplication becomes clear.*

1. Introduction

Special orthogonal group $SO(3)$ is one of the most familiar algebraic objects in mathematics ([1] p.16, [2] p.5, [3] p.67). It is visualized in three-dimensional Euclidean space as an ordered triple of unit vectors orthogonal to each other. Figure 1.1 shows that the orthogonal bases $i = (1,0,0), j = (0,1,0), k = (0,0,1)$ correspond to the identity matrix I. Denote A be an element in $SO(3)$ composed of three unit vectors ai, aj and ak. A naive construction of A is as the follows:

Construction 1.1 (element A in $SO(3)$)
1. ai, any unit vector such that the end point of ai is on the unit sphere.
2. aj, any unit vector which lies on the plane perpendicular to ai.
3. ak, cross product of ai and aj.

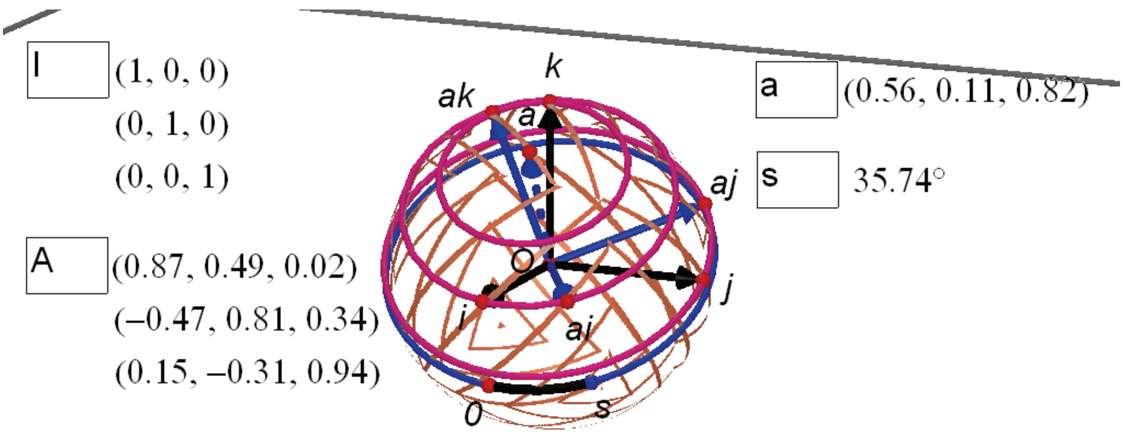

Figure 1.1 Element in $SO(3)$ as a rotation of the orthogonal basis.

For any element in $SO(3)$, one of the eigenvalues is always equal to one. The fact implies that any element A in $SO(3)$ is given as the rotation of I around an axis a with a certain angle s (a is the eigenvector). Actually, the axis a in Figure 1.1 is found out as the intersection of bisector plane of angle $\angle iOai$ and bisector plane of angle $\angle jOaj$ (and also bisector plane of $\angle kOak$). The angle of rotation s is measured on the equator of the pole a. In this sense, any element in $SO(3)$ is given by the following simple construction:

Construction 1.2 (element $A = R(a, s)$ in $SO(3)$)
1. unit vector a, axis of rotation.
2. two points 0 and s on the equator of the pole a.
3. $\{ai, aj, ak\}$, rotation of $\{i, j, k\}$ around a mapping 0 towards s.

With this construction, we can easily construct the multiplication of two matrices $A = R(a, s)$ and $B = R(b, t)$. In fact, matrix AB is given as the rotation of $A = \{ai, aj, ak\}$ around b with angle t as Figure 1.2.

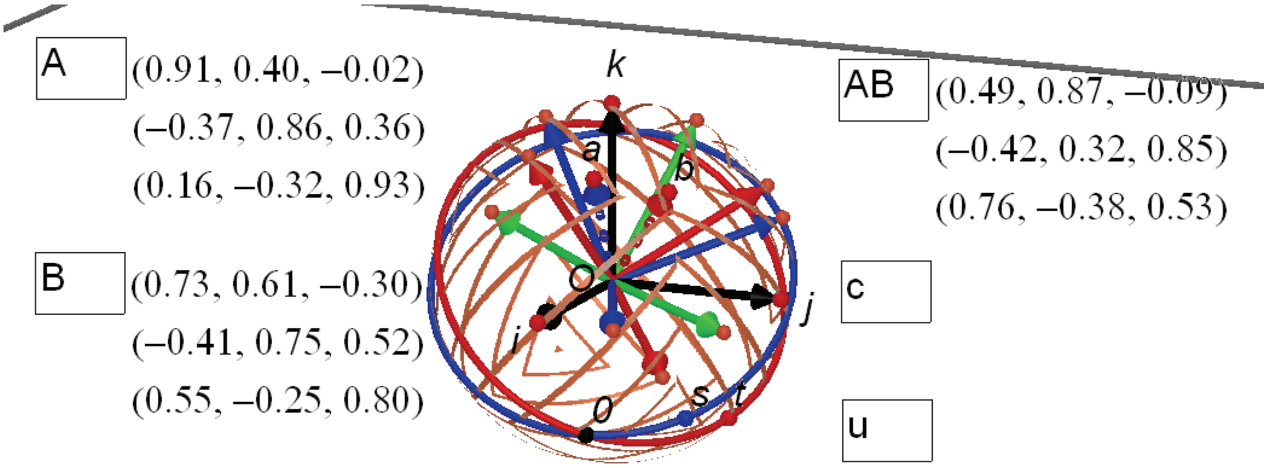

Figure 1.2 Multiplication of matrices.

AB is also an element in $SO(3)$, hence AB is also represented as $AB = R(c, u)$. Here are simple questions: What is the axis c of rotation of AB, and how much is the angle u of rotation of AB ?

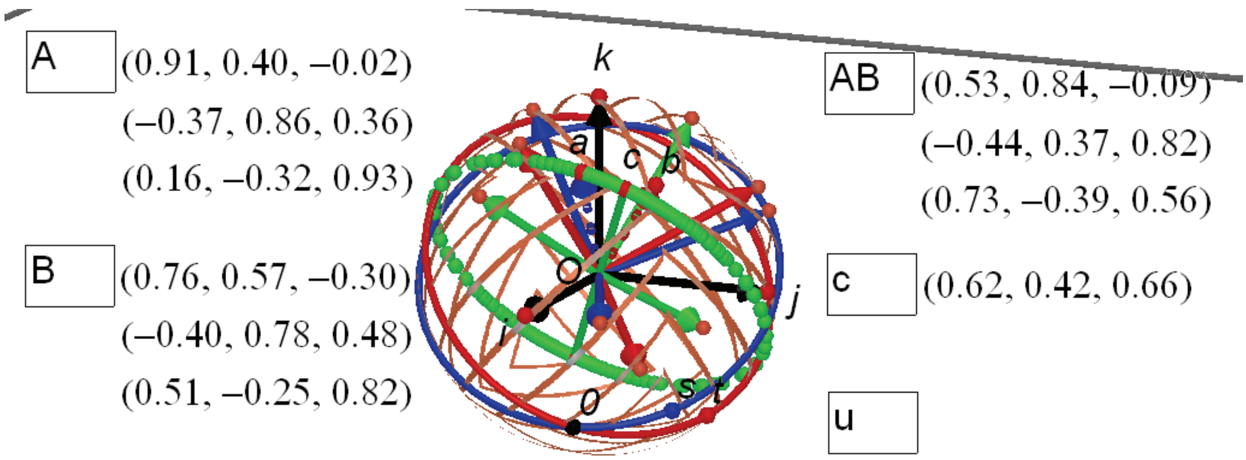

Figure 1.3 Trajectory of the axis of c changing the value of t.

By the way, Figure 1.3 shows the trajectory of axis c when we change the value of t. The trajectory sweeps a great circle passing through a. Moreover, this experiment with dynamic geometry software shows that one rotation of t corresponds to one half rotation of c, in other words, when we change the value of t from 0 degree to 720 degrees, the axis c rotates 360 degrees and returns to the original position. The similar phenomenon is observed when we change the value of s. These spin-like phenomena are also our motivation of this research.

2. Axis of rotation

2.1. Geometric setting

From the investigation in the previous section, we found out that any element in $SO(3)$ is composed of a unit vector (axis) and a scalar (angle). This fact implies that we can forget the triple of vectors, even orthogonal bases $\{i, j, k\}$. Figure 2.1 is a geometric setting of multiplication of matrices.

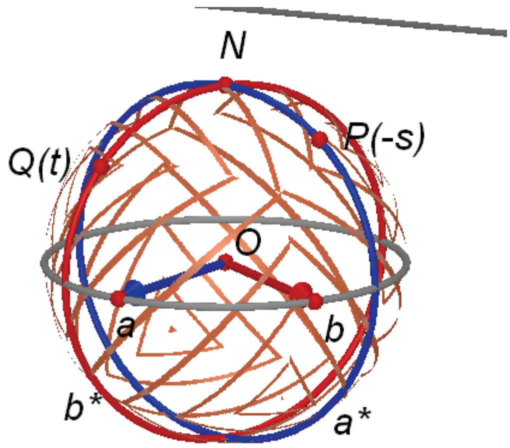

Figure 2.1 Geometric setting of $A = R(a, s)$ and $B = R(b, t)$.

Set two axes a and b are on the equator of the unit sphere, and let a^* (resp. b^*) be the polar equator of the pole a (resp. b). Then, a^* and b^* intersect at the north pole N. Let $P(-s)$ be the point such that $P(-s)$ is mapped to N by $A = R(a, s)$. In the similar way, let $Q(t)$ be the point such that N is mapped to $Q(t)$ by $B = R(b, t)$. Then, $P(-s)$ is mapped to $Q(t)$ by $AB = R(c, u)$. Our first aim is to find out the axis c geometrically. Notice that c is fixed by the rotation $R(c, u)$. Therefore, our problem is reduced to find out the fixed point under the rotation $R(c, u)$.

2.2. Geometric construction of the axis

Figure 2.2 shows the idea of finding the fixed point under the rotation $R(c, u)$. According to symmetry with respect to great circle ab, the fixed point c is given as the vertex of spherical triangle $\varDelta abc$ in the southern hemi-sphere such that $\angle bac = s/2$ and $\angle abc = t/2$.

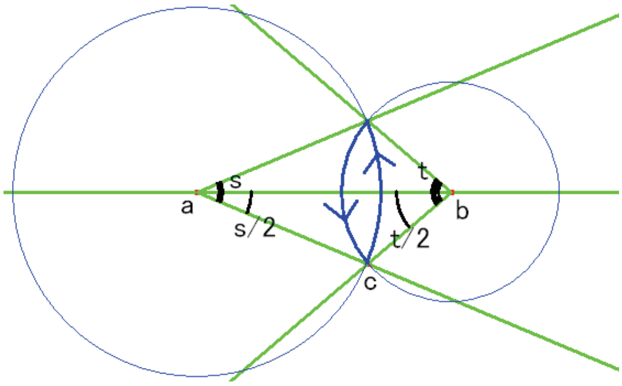

Figure 2.2 Construction of the fixed point.

To draw great circle ac, it is convenient to use $P(-s/2)$ which is the midpoint between N and $P(-s)$ on the unit sphere. Now the construction of the axis c of $AB = R(c,u)$ is given as follows (Figure 2.3):

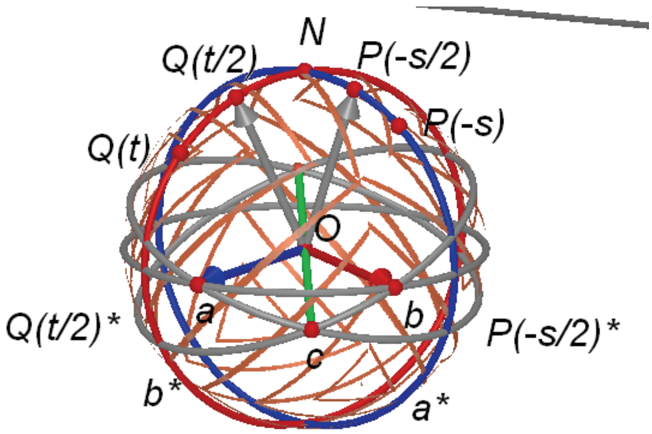

Figure 2.3 Construction of the axis.

Construction 2.1 (axis c of $AB = R(c,u)$)
1. $P(-s/2)$, midpoint between N and $P(-s)$.
2. $P(-s/2)^*$, great circle perpendicular to $P(-s/2)$.
3. $Q(t/2)$, midpoint between N and $Q(t)$.
4. $Q(t/2)^*$, great circle perpendicular to $Q(t/2)$.
5. the axis c, intersection of $P(-s/2)^*$ and $Q(t/2)^*$.

At this stage, we can understand the spin-like phenomena introduced in the previous section. When we change the value of t, point $Q(t/2)$ changes its position, therefore, axis c moves on great circle $P(-s/2)^*$. Moreover, the speed of $Q(t/2)$ is half of the speed of $Q(t)$, this is the reason why axis c behaves spin-like.

3. Angle of Rotation

In Section 2, we have found out the axis of rotation geometrically. The final question is to find out the value u of rotation $AB = R(c,u)$. Actually, the value u is given as

$$u = 2\,\angle P(-s/2)OQ(t/2).$$

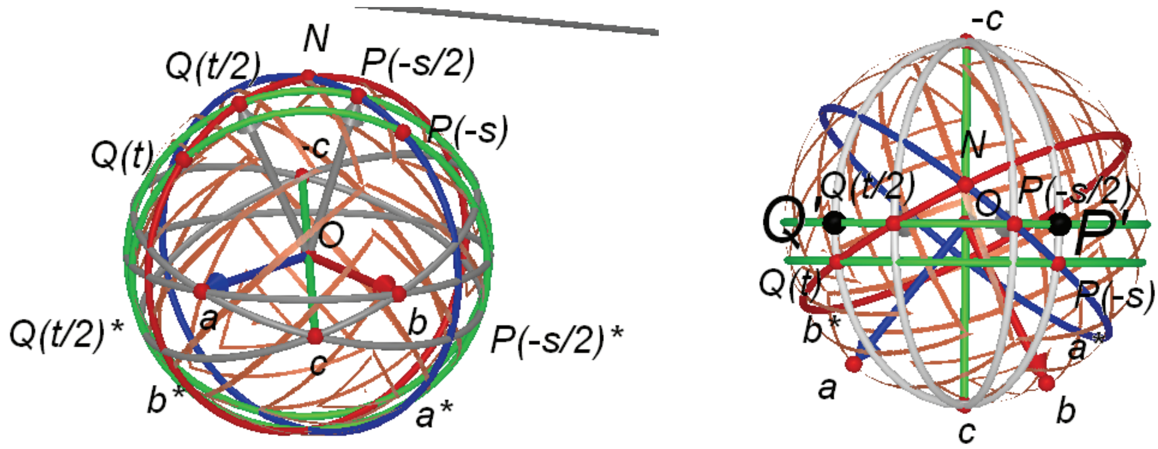

Figure 3.1 Angle of rotation.

To see this, recall that rotation $AB = R(c, u)$ maps $P(-s)$ to $Q(t)$. Figure 3.1(left) shows the orbit of rotation from $P(-s)$ to $Q(t)$. Notice that this orbit is a small circle in general, so the angle u is not equal to $\angle P(-s)OQ(t)$. The angle of rotation has to be measured on the equator of axis c, that is, the great circle passing through $P(-s/2)$ and $Q(t/2)$. As in Figure 3.1(right), let P'(resp. Q') be the point on the equator with the same longitude as $P(-s)$(resp. $Q(t)$) with respect to axis c. Then, the angle u is equal to $\angle P'OQ'$. Since $P(-s/2)N = P(-s/2)P(-s)$ and $Q(t/2)N = Q(t/2)Q(t)$, it is easy to see $\angle P'OQ'$ is double of $\angle P(-s/2)OQ(t/2)$, that is,

$$u = 2\,\angle P(-s/2)OQ(t/2).$$

In this way, we can summarize this study as follows:

1. Special orthogonal group $SO(3)$ is the set of rotation in the form of $A = R(a, s)$.
2. Multiplication of matrix is completely understood geometrically.
3. Spin-like phenomena of the axis come from half of angles.

References

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[3] Knapp, A., *Lie groups beyond an introduction, Second edition*, 2002, Progress in Mathematics 140, Birkhauser Boston Inc., Boston, MA.