Computing a Perturbation Bound for Preserving the Number of Common Zeros of a Polynomial System

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Abstract

We propose a method for computing a perturbation bound that preserves the number of common zeros in $(\mathbb{C}^{\times})^n$ of a polynomial system $(f_1, \ldots, f_n)$, where $\mathbb{C}^{\times} = \mathbb{C} \setminus \{0\}$ and $f_j \in \mathbb{C}[x_1, \ldots, x_n]$, by using Stetter's results on the nearest polynomial with a given zero, Bernshtein's theorem, and minimization techniques for rational functions such as sum of squares (SOS) relaxations.

1 Introduction

Solving systems of polynomial equations is important in both theory and practice. Traditionally, algebraic computation or symbolic computation treats only polynomials with exact coefficients. Thus, algorithms for solving polynomial systems also treat only such polynomials. However, in practical examples, the coefficients of polynomials can contain errors, because the coefficients are obtained through measurement or can be specified with only finite precision.

On the other hand, numeric computation treats polynomials with inexact coefficients or empirical polynomials. Therefore, symbolic and numeric computation, which is a research area of computational algebra that combines ideas from symbolic and numeric computation, has been of increasing interest over the past decade.

When solving a system of empirical polynomials, one might be concerned whether properties of zeros such as the number of zeros are preserved under perturbation of coefficients. This paper treats such a problem for a polynomial system with complex coefficients. More specifically, we consider the following type of problem.

> Given n polynomials $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ such that the number of common zeros in $(\mathbb{C}^\times)^n$ of the polynomial system $(f_1, \ldots, f_n)$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$, compute $\varepsilon > 0$ such that for every polynomial $\tilde{f}_j(x_1, \ldots, x_n)$ $(j = 1, \ldots, n)$ with an appropriate norm of the vector of coefficients of $\tilde{f}_j - f_j$ being less than ε, the polynomial system $(\tilde{f}_1, \ldots, \tilde{f}_n)$ has the same number of common zeros in $(\mathbb{C}^\times)^n$.

When a polynomial system $(f_1, \ldots, f_n)$ in the above problem is linear, we can write the system as $Ax + b$, where A is an $n \times n$ complex matrix and n-dimensional complex vectors x and b. In this case, the problem almost corresponds to the following.

> Compute the distance between A and the nearest singular matrix to A. Here, the distance between two matrices A_1 and A_2 is $\|A_2 - A_1\|$, where $\|\cdot\|$ is an appropriate matrix norm.

The distance can be described using $\|A\|$ and the condition number of A with respect to the norm $\|\cdot\|$ [9].

In general cases, that is, the cases where polynomials are not restricted to linear polynomials, we might be able to use comprehensive Gröbner bases/systems [11, 12, 13]. However, in most cases, the computational cost is rather high and this approach is impractical. Alternatively, we can use resultants. In [6], such a method is proposed for bivariate polynomial systems and slightly different problems, however, if the system has more than two variables, methods based on resultants might not be efficient.

To solve the problem, we use Bernshtein's theorem [1]. The basic idea is as follows. Let $Q_j \subset \mathbb{R}^n$ be the Newton polytope constructed from the support of f_j. For a hyperplane α in $\mathbb{R}^n$, $f_{j\alpha}$ denotes the polynomial made from f_j by taking terms whose exponents lie on α. Bernshtein's theorem (Theorem 9 in the present paper) states that if the system $(f_{1\alpha}, \ldots, f_{n\alpha})$, where none of $f_{1\alpha}, \ldots, f_{n\alpha}$ are zero polynomials, has no common zeros in $(\mathbb{C}^\times)^n$ for any hyperplane α in $\mathbb{R}^n$, then the original system $(f_1, \ldots, f_n)$ has exactly the same number of common zeros in $(\mathbb{C}^\times)^n$ as the mixed volume of $Q_1, \ldots, Q_n$. Furthermore, if the number of the common zeros is greater than 0, the converse also holds. From the theorem, our problem comes down to that of seeking the bound of perturbation of one (or finitely many) polynomial system(s) with no common zeros.

Our first contribution is that we show that the bound of perturbation can be computed by Stetter's approach for the nearest polynomial problem [8], which is briefly reviewed in 3.1.2. The detail will be described in Section 3.1. In terms of Gröbner basis, non-existence of common zeros means that the Gröbner basis contains 1. Along this line, our problem could be solved by something like a "perturbed" Gröbner basis or an "approximate" Gröbner basis such as [3], but we do not go into the detail in this paper.

Our second contribution is that we show that the perturbation bound is represented by infimum of one (or finitely many) rational function(s) with real coefficients. Thus, we can apply minimization techniques for rational functions such as sum of squares (SOS) relaxations [4] to compute the bound. In [7], a similar approach, that is, reducing the problem to computing the nearest polynomials, is described, however, no computation method for the reduced problem is proposed.

The rest of this paper is organized as follows. Section 2 explains the notations used throughout this paper and the problem we consider. Section 3 describes the proposed computation method. We divided the computation into two cases. First, we treat the case where the number of common zeros in $(\mathbb{C}^\times)^n$ of a given polynomial system is 0 and describe the relation to the nearest polynomial with a given zero. Then, we treat the case where the number of common zeros in $(\mathbb{C}^\times)^n$ of a given polynomial system is greater than 0. Using Bernshtein's theorem, we reduce this case to the first case. In both cases, the problem is reduced to minimization problems of rational functions with real coefficients. Finally, Section 4 concludes the paper by mentioning directions for future work.

2 Preliminaries

2.1 Notations

We introduce several notations used throughout this paper. We will introduce other notations where they are needed.

We write the ring of complex coefficient polynomials with n variables $x_1, \ldots, x_n$ as $\mathbb{C}[\mathbf{x}]$ and a monomial $x_1^{q_1} \ldots x_n^{q_n}$ as $\mathbf{x}^{\mathbf{q}}$. Hereafter, we consider nonzero polynomials in $\mathbb{C}[\mathbf{x}]$ unless otherwise stated. For $f = \sum_{\mathbf{q}} c_{\mathbf{q}} \mathbf{x}^{\mathbf{q}} \in \mathbb{C}[\mathbf{x}]$, we define $\mathrm{supp}(f)$, the support of f, as $\{\mathbf{q} \mid c_{\mathbf{q}} \neq 0\} \subset \mathbb{Z}^n_{\geq 0}$.

For a complex vector $\mathbf{c} = (c_1, \ldots, c_m) \in \mathbb{C}^m$, we write the 2-norm of $\mathbf{c}$, $\sqrt{\sum_{j=1}^m |c_j|^2}$, as $\|\mathbf{c}\|$. For a polynomial $f = \sum_{\mathbf{q}} c_{\mathbf{q}} x^{\mathbf{q}}$, we define the 2-norm of f as $\|(c_{\mathbf{q}})_{\mathbf{q} \in \mathrm{supp}(f)}\|$, the 2-norm of the vector of the coefficients of f.

Let $F = (f_1, \ldots, f_n)$ be a polynomial system. We denote the number of common zeros of F in $(\mathbb{C}^\times)^n$ with multiplicity counted by $\mathcal{N}(F)$. We consider common zeros not in $\mathbb{C}^n$ but those in $(\mathbb{C}^\times)^n$ because we apply Bernshtein's theorem (Theorem 9 in the present paper) to solving our problem.

2.2 Problem

First, we define a perturbation bound of a polynomial system to describe our problem.

Definition 1 (Perturbation bound) *Let $F = (f_1, \ldots, f_n)$ be a polynomial system with $\mathcal{N}(F) < \infty$. A positive real number ε is said to be a perturbation bound of F when it satisfies the following condition: $\mathcal{N}(F) = \mathcal{N}(G)$ holds for every polynomial system $G = (g_1, \ldots, g_n)$ such that* $\mathrm{supp}(g_j) = \mathrm{supp}(f_j)$ *($j = 1, \ldots, n$) and $\sum_{j=1}^n \|g_j - f_j\|^2 < \varepsilon$.*

Note that there is a case where a perturbation bound does not exist. That is, there is a polynomial system $(f_1, \ldots, f_n)$ such that for any $\varepsilon > 0$, there exists a polynomial system $(g_1, \ldots, g_n)$ with $\sum_{j=1}^n \|g_j - f_j\|^2 < \varepsilon$ and $\mathcal{N}(F) \neq \mathcal{N}(G)$. Consider the following example.

Example 2 *Let $f = x_1 + x_2 + 1$ and $g = x_1 + 2x_2 + 2$, and consider the polynomial system (f, g). The polynomial system (f, g) has no common zero in $(\mathbb{C}^\times)^2$ (the only common zero of the system is $(x_1, x_2) = (0, -1)$). For any $\varepsilon \neq 0,\ 1$, let $f_\varepsilon = x_1 + (1 + \varepsilon)x_2 + 1$. Then, the system (f_ε, g) has a common zero*

$$(x_1, x_2) = \left(\frac{2\varepsilon}{1 - \varepsilon}, \frac{-1}{1 - \varepsilon} \right) \in (\mathbb{C}^\times)^2.$$

We consider the following problem.

Problem 3 *For a polynomial system $F = (f_1, \ldots, f_n)$ with $\mathcal{N}(F) < \infty$, compute a perturbation bound ε. Furthermore, compute $B(F) = \sup\{\varepsilon\}$, where ε runs over perturbation bounds of F.*

Note that $B(F)$ might not exist as described in Example 2 and that $B(F)$ might be infinity as follows. Consider the case where $n = 1$. Let $F = (f)$, where $f = \sum_{j=m}^{d} a_j x^j$ (a_m, $a_d \neq 0$). Then, f has $d - m$ zeros in $\mathbb{C}^\times$ and any polynomial g with $\mathrm{supp}(g) = \mathrm{supp}(f)$ has also $d - m$ zeros in $\mathbb{C}^\times$. Thus, $B(F) = \infty$ holds. This also shows that we can solve Problem 3 easily when $n = 1$.

3 Computation Method

We propose a computation method for Problem 3 in this section. Let F be a given polynomial system. We divide the problem into two cases: the case where $\mathcal{N}(F) = 0$ (Section 3.1) and the case where $1 \leq \mathcal{N}(F) < \infty$ (Section 3.2). In both cases, the computation is reduced to computing the infimums of rational functions with real coefficients.

3.1 The Case Where $\mathcal{N}(F) = 0$

In this subsection, we consider the case where $\mathcal{N}(F) = 0$. In this case, Problem 3 is directly related to Stetter's results on the nearest polynomials having a given zero [8].

3.1.1 Relaxed Problem

To compute a perturbation bound, we consider the following relaxed problem in which the conditions $\mathrm{supp}(g_j) = \mathrm{supp}(f_j)$ in Problem 3 are replaced with $\mathrm{supp}(g_j) \subset \mathrm{supp}(f_j)$.

Problem 4 *For a polynomial system $F = (f_1, \ldots, f_n)$ with $\mathcal{N}(F) = 0$, compute $\varepsilon > 0$ satisfying the following condition: $\mathcal{N}(G) = 0$ holds for every polynomial system $G = (g_1, \ldots, g_n)$ such that $\mathrm{supp}(g_j) \subset \mathrm{supp}(f_j)$ ($j = 1, \ldots, n$) and $\sum_{j=1}^{n} \|f_j - g_j\|^2 < \varepsilon$. Furthermore, compute $B_0(F) = \sup\{\varepsilon\}$, where ε runs over perturbation bounds of F.*

The following proposition guarantees that bounds in Problem 4 exist if bounds in Problem 3 exist.

Proposition 5 *Let F be a polynomial system with $\mathcal{N}(F) = 0$.*

1. *A positive ε for F in Problem 4 exists if and only if a perturbation bound of F exists.*
2. *$B_0(F)$ exists if and only if $B(F)$ exists.*
3. *If $B_0(F)$ and $B(F)$ exist, then the inequality $B_0(F) \leq B(F)$ holds.*

Proof. It is clear that if $\varepsilon > 0$ in Problem 4 exists then ε is a perturbation bound of F. Thus, if $B_0(F)$ exists then $B(F)$ exists and the inequality $B_0(F) \leq B(F)$ holds. These prove the third statement and the "only if" parts of the first and second statements.

Conversely, let $\varepsilon > 0$ be a perturbation bound of F. We write $f_j = \sum_{\mathbf{q}} c_{j\mathbf{q}} \mathbf{x}^{\mathbf{q}}$. Let δ be

$$\min\left\{\varepsilon,\ \min\left\{|c_{1\mathbf{q}}|^2 \mid \mathbf{q} \in \mathrm{supp}(f_1)\right\},\ \ldots,\ \min\left\{|c_{n\mathbf{q}}|^2 \mid \mathbf{q} \in \mathrm{supp}(f_n)\right\}\right\}.$$

Note that $0 < \delta \leq \varepsilon$. Let

$$P_1 = \left\{(g_1, \ldots, g_n) \,\middle|\, g_j \in \mathbb{C}[\mathbf{x}],\ \mathrm{supp}(g_j) = \mathrm{supp}(f_j),\ \sum_{j=1}^{n} \|g_j - f_j\|^2 < \delta\right\},$$

$$P_2 = \left\{(g_1, \ldots, g_n) \,\middle|\, g_j \in \mathbb{C}[\mathbf{x}],\ \mathrm{supp}(g_j) \subset \mathrm{supp}(f_j),\ \sum_{j=1}^{n} \|g_j - f_j\|^2 < \delta\right\}.$$

We will show $P_1 = P_2$. It is clear that $P_1 \subset P_2$. Conversely, $P_1 \supset P_2$ can be proved as follows. Take arbitrary $(g_1, \ldots, g_n) \in P_2$ and write $g_j = \sum_{\mathbf{q}} d_{j\mathbf{q}} \mathbf{x}^{\mathbf{q}}$. From the definitions of δ and P_2, for any k $(1 \leq k \leq n)$ and $\mathbf{q} \in \mathrm{supp}(f_k)$, the inequalities

$$|c_{k\mathbf{q}}|^2 \geq \delta > \sum_{j=1}^{n} \|g_j - f_j\|^2 \geq \|g_k - f_k\|^2 \geq |d_{k\mathbf{q}} - c_{k\mathbf{q}}|^2$$

hold. Thus, $|c_{k\mathbf{q}}| > |d_{k\mathbf{q}} - c_{k\mathbf{q}}| \geq |c_{k\mathbf{q}}| - |d_{k\mathbf{q}}|$ and therefore, $|d_{k\mathbf{q}}| > 0$, that is, $d_{k\mathbf{q}} \neq 0$ holds. This means $\mathrm{supp}(g_k) = \mathrm{supp}(f_k)$ and thus $(g_1, \ldots, g_n) \in P_1$. Therefore, $P_1 = P_2$ holds.

Thus, $\mathcal{N}(G) = 0$ for any $G \in P_2$. Therefore, $\delta > 0$ satisfies the condition in Problem 4 and thus the "if" part of the first statement holds. This also proves that the existence of $B(F)$ implies that of $B_0(F)$. ■

3.1.2 Nearest Polynomials

Below, we describe a representation of $B_0(F)$ that is convenient for computation.

For a finite set $S \subset \mathbb{Z}^n_{\geq 0}$ and $\mathbf{z} \in \mathbb{C}^n$, we define $Z(S; \mathbf{z}) \subset \mathbb{C}[\mathbf{x}]$ as $\{f \in \mathbb{C}[\mathbf{x}] \mid \mathrm{supp}(f) \subset S,\ f(\mathbf{z}) = 0\}$. For $f \in \mathbb{C}[\mathbf{x}]$ and $\mathbf{z} \in \mathbb{C}^n$, we define $d(f; \mathbf{z})$ as $\min\{\|g - f\| \mid g \in Z(\mathrm{supp}(f); \mathbf{z})\}$, the distance between f and $\tilde{f}$ measured by the 2-norm, where $\tilde{f}$ is the nearest polynomial to f such that $\mathrm{supp}(\tilde{f}) \subset \mathrm{supp}(f)$ and $\tilde{f}(\mathbf{z}) = 0$. Then, the following proposition is immediate from the definitions.

Proposition 6 *Let F be a polynomial system with $\mathcal{N}(F) = 0$. Then, the following equality holds.*

$$B_0(F) = \inf\left\{\sum_{j=1}^{n} d(f_j; \mathbf{z})^2 \,\middle|\, \mathbf{z} \in (\mathbb{C}^\times)^n\right\}. \tag{1}$$

Thus, it is sufficient to compute the right-hand side of Equation (1). To do this, we use the following theorem, which is a part of Stetter's results [8].

Theorem 7 (Stetter) *For $f \in \mathbb{C}[\mathbf{x}]$ and $\mathbf{z} \in \mathbb{C}^n$,*

$$d(f; \mathbf{z}) = \frac{|f(\mathbf{z})|}{\|(\mathbf{z}^{\mathbf{q}})_{\mathbf{q} \in \mathrm{supp}(f)}\|}$$

holds if $\|(\mathbf{z}^{\mathbf{q}})_{\mathbf{q} \in \mathrm{supp}(f)}\| \neq 0$.

Remark 8 *If we use other norms than the 2-norm, there is a minor omission in Stetter's results [8]. See [5] for details.*

From Theorem 7, it is sufficient to compute

$$\inf\left\{\sum_{j=1}^{n}\frac{|f_j(\mathbf{z})|^2}{\|(\mathbf{z}^{\mathbf{q}})_{\mathbf{q}\in\mathrm{supp}(f_j)}\|^2}\;\middle|\;\mathbf{z}\in(\mathbb{C}^\times)^n\right\}. \tag{2}$$

Note that $\|(\mathbf{z}^{\mathbf{q}})_{\mathbf{q}\in\mathrm{supp}(f_j)}\| \neq 0$ if $\mathbf{z}\in(\mathbb{C}^\times)^n$. We will describe a computation method for (2) in Section 3.3.

3.2 The Case Where $1 \leq \mathcal{N}(F) < \infty$

In this subsection, we consider the case where $1 \leq \mathcal{N}(F) < \infty$. In this case, we can reduce the problem to the case where $\mathcal{N}(F) = 0$ by using Bernshtein's theorem [1].

3.2.1 Bernshtein's Theorem

For a polynomial $f \in \mathbb{C}[\mathbf{x}]$, we define the Newton polytope $\mathcal{P}(f)$ of f as the convex hull of $\mathrm{supp}(f)$ in $\mathbb{R}^n$. Let $F = (f_1, \ldots, f_n)$ be a polynomial system and $Q_j = \mathcal{P}(f_j)$. The mixed volume $\mathcal{M}(F)$ is the coefficient of $u_1 \cdots u_n$ in the homogeneous polynomial $V(u_1Q_1+\cdots+u_nQ_n)$, where V is the Euclidean volume, and

$$Q_1 + \cdots + Q_n = \{x_1 + \cdots + x_n \mid x_j \in Q_j\ (j = 1, \ldots, n)\}$$

denotes the Minkowski sum of polytopes.

To reduce the case where $1 \leq \mathcal{N}(F) < \infty$ to the case where $\mathcal{N}(F) = 0$, we use Bernshtein's theorem [1]. Let $\alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$. For a compact subset S of $\mathbb{R}^n$, let $m(\alpha, S) = \min\{\langle\alpha, \mathbf{q}\rangle \mid \mathbf{q} \in S\}$, where $\langle\cdot,\cdot\rangle$ is the standard inner product on the vector space $\mathbb{R}^n$. We write the set $\{\mathbf{q} \in S \mid \langle\alpha, \mathbf{q}\rangle = m(\alpha, S)\}$ as S_α. We define $f_\alpha = \sum_{\mathbf{q}\in S_\alpha} c_{\mathbf{q}} x^{\mathbf{q}}$ for $f = \sum_{\mathbf{q}\in S} c_{\mathbf{q}} x^{\mathbf{q}}$, where $S = \mathrm{supp}(f)$, and define $F_\alpha = (f_{1\alpha}, \ldots, f_{n\alpha})$ for $F = (f_1, \ldots, f_n)$. The following theorem is due to Bernshtein [1].

Theorem 9 (Bernshtein's theorem (polynomial version)) *Let $F = (f_1, \ldots, f_n)$ be a polynomial system with $\mathcal{N}(F) < \infty$.*

1. *$\mathcal{N}(F) \leq \mathcal{M}(F)$.*
2. *If $\mathcal{N}(F_\alpha) = 0$ for all $\alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$, then $\mathcal{N}(F) = \mathcal{M}(F)$.*
3. *If $\mathcal{N}(F) = \mathcal{M}(F) \geq 1$, then $\mathcal{N}(F_\alpha) = 0$ for all $\alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$.*

Remark 10

1. *Bernshtein's theorem is also valid for Laurent polynomials, that is, polynomials in $\mathbb{C}[x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n]$ [1].*

2. *In the second statement, it is sufficient to examine only finitely many α because $\{F_\alpha \mid \alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}\}$ is a finite set. For example, for each face H of the Newton polytopes $\mathcal{P}(f_j)$ $(j = 1, \ldots, n)$, take an $\alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$ such that $\mathrm{supp}(f_j) \cap H = \mathrm{supp}(f_{j\alpha})$. Let A be the set of all such α. Then, $\{F_\alpha \mid \alpha \in A\} = \{F_\alpha \mid \alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}\}$ holds. The results in [2] tell us that it is not necessary to examine $\mathcal{N}(F_\alpha) = 0$ for all $\alpha \in A$ to decide whether $\mathcal{N}(F) = \mathcal{M}(F)$ holds.*

3. *The equality $\mathcal{N}(F) = \mathcal{M}(F)$ holds for generic choices of the coefficients of polynomials in F [1].*

3.2.2 Reduction to the Case Where $\mathcal{N}(F) = 0$

We assume that $1 \le \mathcal{N}(F) < \infty$ and $\mathcal{N}(F) = \mathcal{M}(F)$. Note that the equality $\mathcal{N}(F) = \mathcal{M}(F)$ holds in general as we write in Remark 10.

The following theorem holds. Note that the assumption that $1 \le \mathcal{N}(F) = \mathcal{M}(F) < \infty$ implies $\mathcal{N}(F_\alpha) = 0$ for all $\alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}$.

Theorem 11 *Let $F = (f_1, \ldots, f_n)$ be a polynomial system with $1 \le \mathcal{N}(F) = \mathcal{M}(F) < \infty$. Let A be a finite subset of $\mathbb{Q}^n \setminus \{\mathbf{0}\}$ such that $\{F_\alpha \mid \alpha \in A\} = \{F_\alpha \mid \alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}\}$. Then, $B(F) = \min\{B(F_\alpha) \mid \alpha \in A\}$ holds.*

Proof. The assumption that $1 \le \mathcal{N}(F) = \mathcal{M}(F) < \infty$ implies that $\mathcal{N}(F_\alpha) = 0$ for all $\alpha \in A$.

Let $0 < \varepsilon \le B(F)$ and take an arbitrary $\alpha \in A$. For this α, take an arbitrary polynomial system $\tilde{F} = (\tilde{f}_1, \ldots, \tilde{f}_n)$ such that $\mathrm{supp}(\tilde{f}_j) = \mathrm{supp}(f_{j\alpha})$ $(j = 1, \ldots, n)$ and $\sum_{j=1}^n \|\tilde{f}_j - f_{j\alpha}\|^2 < \varepsilon$. Furthermore, take a polynomial system $G = (g_1, \ldots, g_n)$ such that

$$g_j = \sum_{\mathbf{q} \in S_{j\alpha}} \tilde{c}_{\mathbf{q}} \mathbf{x}^{\mathbf{q}} + \sum_{\mathbf{q} \in S_j \setminus S_{j\alpha}} c_{\mathbf{q}} \mathbf{x}^{\mathbf{q}},$$

where $S_j = \mathrm{supp}(f_j)$, $f_j = \sum_{\mathbf{q}} c_{j\mathbf{q}} \mathbf{x}^{\mathbf{q}}$, and $\tilde{f}_j = \sum_{\mathbf{q}} \tilde{c}_{\mathbf{q}} \mathbf{x}^{\mathbf{q}}$.

Noting that $G_\alpha = \tilde{F} = \tilde{F}_\alpha$ and $\mathrm{supp}(f_j) = \mathrm{supp}(g_j)$ $(j = 1, \ldots, n)$, we have

$$\sum_{j=1}^n \|g_j - f_j\|^2 = \sum_{j=1}^n \|g_{j\alpha} - f_{j\alpha}\|^2 = \sum_{j=1}^n \|\tilde{f}_{j\alpha} - f_{j\alpha}\|^2 = \sum_{j=1}^n \|\tilde{f}_j - f_{j\alpha}\|^2 < \varepsilon.$$

Thus, $1 \le \mathcal{N}(G) = \mathcal{N}(F) < \infty$ holds. The inequality $1 \le \mathcal{N}(G) < \infty$ and Bernshtein's theorem imply $\mathcal{N}(G_\alpha) = 0$. Since $G_\alpha = \tilde{F}$, we have $\mathcal{N}(\tilde{F}) = \mathcal{N}(G_\alpha) = 0 = \mathcal{N}(F_\alpha)$. Therefore, $\varepsilon \le B(F_\alpha)$. Thus, $\varepsilon \le \min\{B(F_\alpha) \mid \alpha \in A\}$.

Conversely, let $0 < \varepsilon \le \min\{B(F_\alpha) \mid \alpha \in A\}$. Take an arbitrary polynomial system $G = (g_1, \ldots, g_n)$ such that $\mathrm{supp}(f_j) = \mathrm{supp}(g_j)$ $(j = 1, \ldots, n)$ and $\sum_{j=1}^n \|g_j - f_j\|^2 < \varepsilon$. Since $\mathrm{supp}(f_j) = \mathrm{supp}(g_j)$ $(j = 1, \ldots, n)$ holds, we have $\{G_\alpha \mid \alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}\} = \{G_\alpha \mid \alpha \in A\}$. The inequalities

$$\sum_{j=1}^n \|g_{j\alpha} - f_{j\alpha}\|^2 \le \sum_{j=1}^n \|g_j - f_j\|^2 < \varepsilon$$

imply $\mathcal{N}(G_\alpha) = 0$ for all $\alpha \in A$. Thus, the equality $\mathcal{N}(G) = \mathcal{M}(G)$ follows from Bernshtein's theorem. The equality $\mathcal{M}(G) = \mathcal{M}(F)$ holds because $\mathrm{supp}(f_j) = \mathrm{supp}(g_j)$ $(j = 1, \ldots, n)$ holds.

The assumption that $\mathcal{M}(F) = \mathcal{N}(F)$ thus implies $\mathcal{N}(G) = \mathcal{N}(F)$. Therefore, the inequality $\varepsilon \le B(F)$ holds. ■

From Theorem 11, computation of $B(F)$ is reduced to those for $B(F_\alpha)$ $(\alpha \in A)$. As written in the proof, the assumption that $1 \le \mathcal{N}(F) = \mathcal{M}(F) < \infty$ implies $\mathcal{N}(F_\alpha) = 0$. Thus, the case where $\mathcal{N}(F) \ge 1$ is reduced to the case where a polynomial system has no common zeros in $(\mathbb{C}^\times)^n$.

Similar to the case where $\mathcal{N}(F) = 0$, the following proposition holds.

Proposition 12 *Let $F = (f_1, \ldots, f_n)$ be a polynomials system with $1 \le \mathcal{N}(F) < \infty$.*

1. $\min\{B_0(F_\alpha) \mid \alpha \in A\}$ *exists if and only if* $\min\{B(F_\alpha) \mid \alpha \in A\}$ *exists.*
2. *The inequality* $\min\{B_0(F_\alpha) \mid \alpha \in A\} \le \min\{B(F_\alpha) \mid \alpha \in A\}$ *holds if the both sides of the inequality exist.*

Proof. The statements follow from Proposition 5. ■

Thus, Problem 3 is reduced to Problem 4. More precisely, instead of $\min\{B(F_\alpha) \mid \alpha \in A\}$, we compute

$$\min\{B_0(F_\alpha) \mid \alpha \in A\} = \min\left\{ \inf_{\mathbf{z}\in(\mathbb{C}^\times)^n} \sum_{j=1}^{n} \frac{|f_{j\alpha}(\mathbf{z})|^2}{\|(\mathbf{z}^{\mathbf{q}})_{\mathbf{q}\in\mathrm{supp}(f_{j\alpha})}\|^2} \;\middle|\; \alpha \in A \right\}. \tag{3}$$

3.3 Minimization of Rational Functions

As described in Sections 3.1.2 and 3.2.2, computation of a perturbation bound of a polynomial system $F = (f_1, \ldots, f_n)$ is reduced to

1. computing $B_0(F)$, which is equal to the infimum of the square of the distance between F and perturbed polynomial systems $\tilde{F} = (\tilde{f}_1, \ldots, \tilde{f}_n)$ of F with $\mathrm{supp}(\tilde{f}_j) \subset \mathrm{supp}(f_j)$ $(j = 1, \ldots, n)$ and $\mathcal{N}(\tilde{F}) \ge 1$, when $\mathcal{N}(F) = 0$, or
2. computing finitely many $B_0(F_\alpha)$ $(\alpha \in A)$, where each $B_0(F_\alpha)$ is equal to the infimum of the square of the distances between F_α and perturbed polynomial systems $\tilde{F}_\alpha = (\tilde{f}_1, \ldots, \tilde{f}_n)$ of F_α with $\mathrm{supp}(\tilde{f}_j) \subset \mathrm{supp}(f_{j\alpha})$ $(j = 1, \ldots, n)$ and $\mathcal{N}(\tilde{F}_\alpha) \ge 1$, when $1 \le \mathcal{N}(F) < \infty$.

In each case, it is sufficient to compute one or finitely many quantities of the following type:

$$\inf\left\{ \sum_{j=1}^{n} \frac{|f_j(\mathbf{z})|^2}{\|(\mathbf{z}^{\mathbf{q}})_{\mathbf{q}\in\mathrm{supp}(f_j)}\|^2} \;\middle|\; \mathbf{z} \in (\mathbb{C}^\times)^n \right\}. \tag{4}$$

Put $\mathbf{z} = (z_1, \ldots, z_n) = (s_1 + t_1 \cdot i, \ldots, s_n + t_n \cdot i)$, where $s_j, t_j \in \mathbb{R}$ $(j = 1, \ldots, n)$. Then,

$$\begin{aligned} |f_j(\mathbf{z})|^2 &= f_j(s_1 + t_1 \cdot i, \ldots, s_n + t_n \cdot i) \cdot \overline{f_j}(s_1 - t_1 \cdot i, \ldots, s_n - t_n \cdot i), \\ \|(\mathbf{z}^{\mathbf{q}})_{\mathbf{q}\in\mathrm{supp}(f_j)}\|^2 &= \sum_{(q_1,\ldots,q_n)\in\mathrm{supp}(f_j)} (s_1^2 + t_1^2)^{q_1} \cdots (s_n^2 + t_n^2)^{q_n}, \end{aligned}$$

where $\overline{f_j}(\mathbf{x}) = \sum_{\mathbf{q}} \overline{c_{\mathbf{q}}}\mathbf{x}^{\mathbf{q}}$ for $f_j(\mathbf{x}) = \sum_{\mathbf{q}} c_{\mathbf{q}}\mathbf{x}^{\mathbf{q}}$. Thus, $\sum_{j=1}^{n} d(\tilde{f}_j; \mathbf{z})^2 \in \mathbb{R}(s_1, \ldots, s_n, t_1, \ldots, t_n)$ when we regard s_j and t_j $(j = 1, \ldots, n)$ as variables. Thus, we can apply minimization techniques for rational functions such as sum of squares (SOS) relaxations [4] to computing a guaranteed lower bound of (4).

3.4 Algorithm

We summarize the computation method in an algorithmic style.

Algorithm 13
Input: *A polynomial system* $F = (f_1, \ldots, f_n)$ *and* $\mathcal{N}(F) < \infty$. *When* $\mathcal{N}(F) \geq 1$, *we require the condition* $\mathcal{N}(F) = \mathcal{M}(F)$.
Output: *A perturbation bound* ε *for* F.

(1) If $\mathcal{N}(F) = 0$ *go to Step (2).*
Otherwise go to Step (3).

(2) Compute ε *with* $0 < \varepsilon \leq B_0(F)$ *and return* ε.

(3) Construct $A \subset \mathbb{Q}^n \setminus \{\mathbf{0}\}$ *such that* $\{F_\alpha \mid \alpha \in \mathbb{Q}^n \setminus \{\mathbf{0}\}\} = \{F_\alpha \mid \alpha \in A\}$.
Compute ε *with* $0 < \varepsilon \leq \min\{B_0(F_\alpha) \mid \alpha \in A\}$ *and return* ε.

In Steps (2) and (3), to compute guaranteed lower bounds of $B_0(F)$ and $B_0(F_\alpha)$, we use minimization techniques for rational functions such as sum of squares (SOS) relaxations [4]. In Step (3), to construct A, see Remark 10 (2).

Example 14 *Consider the polynomial system* $F = (f_1, f_2) = (2x_1 + x_2 - 1, x_1 + 2x_2 - 1)$. *The Newton polytopes* $\mathcal{P}(f_1)$ *and* $\mathcal{P}(f_2)$ *are the triangle whose vertices are* $(0,0)$, $(1,0)$, *and* $(0,1)$, *which correspond to the monomials* 1, x_1, *and* x_2, *respectively.* F *satisfies the condition* $\mathcal{N}(F) = \mathcal{M}(F) = 1$. *We can take* $A = \{(1,0), (-1,0), (0,1), (0,-1), (1,1), (-1,-1)\}$. *Then,* F_α *and* $B_0(F_\alpha)$ $(\alpha \in A)$ *are as follows.*

$$\begin{array}{lll} F_{(1,0)} = (x_2 - 1, 2x_2 - 1), & F_{(-1,0)} = (2x_1, x_1), & F_{(0,1)} = (2x_1 - 1, x_1 - 1), \\ F_{(0,-1)} = (x_2, 2x_2), & F_{(1,1)} = (-1, -1), & F_{(-1,-1)} = (2x_1 + x_2, x_1 + 2x_2), \end{array}$$

$$B_0(F_{(1,0)}) = B_0(F_{(0,1)}) = \frac{7 - 3\sqrt{5}}{2} = 0.145898\ldots,$$

$$B_0(F_{(-1,0)}) = B_0(F_{(0,-1)}) = 5, \quad B_0(F_{(1,1)}) = 2, \quad B_0(F_{(-1,-1)}) = 1.$$

Thus, $\min\{B_0(F_\alpha) \mid \alpha \in A\} = (7 - 3\sqrt{5})/2$. *Indeed, the polynomial system*

$$(g_1, g_2) = \left(2x_1 + \frac{5 + 3\sqrt{5}}{10}x_2 - \frac{5 + \sqrt{5}}{10},\ x_1 + \frac{5 + 2\sqrt{5}}{5}x_2 - \frac{5 + 3\sqrt{5}}{10}\right)$$

has the unique zero $(x_1, x_2) = (0,\ (\sqrt{5} - 1)/2) \notin (\mathbb{C}^\times)^2$ *and* $\|g_1 - f_1\|^2 + \|g_2 - f_2\|^2 = (7 - 3\sqrt{5})/2$.

4 Conclusion

We propose a computation method for a perturbation bound preserving the number of common zeros in $(\mathbb{C}^\times)^n$ of a polynomial system $F = (f_1, \ldots, f_n)$. The method reduces the problem to computing the distance to the nearest polynomial having a zero in $(\mathbb{C}^\times)^n$. This reduction is done directly when $\mathcal{N}(F) = 0$ and through Bernshtein's theorem when $1 \leq \mathcal{N}(F) < \infty$. The distance can be computed by using Stetter's results and minimization techniques of rational functions.

Extending the computation method to a perturbation bound preserving the number of common zeros in $\mathbb{C}^n$ is a direction of future research. The results in [10] will be utilized.

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