DIFFUSIVE MASS TRANSFER IN AN ECCENTRIC ANNULAR FLOW

[1]B.Umadevi , [2]Dinesh P.A. , [3]Indira. R.Rao and [1]C.V. Vinay

[1]Department of Mathematics, JSS Academy of Technical Education,Bangalore –60, India

[2]Department of Mathematics,M. S. Ramaiah Institute of Technology,Bangalore–54, India

[3]Department of Mathematics,NitteMeenakshi Institute of Technology,Bangalore–64,India

Corresponding Author : umadevi_gh@yahoo.com

ABSTRACT

The dispersion in an eccentric annulus region by taking blood as a Newtonian fluid with the investigation of oxygen transfer to the tissue cells in an eccentric catheterized artery is studied. The region bounded by eccentric circles in x-y plane is conformal mapping to concentric circles in ξ - η plane using a conformal mapping $z = c/1-\zeta$. The resulting governing equations are analytically solved by using transformation for the concentration. Numerical computations are carried out to understand the simultaneous effects of absorption parameter and eccentricity on the flow with respect to time. The observation through the numerical computation reveals that, as absorption parameter and eccentricity enhance, the solute concentration diminishes.

Key Words: Eccentric annulus, Dispersion coefficient, Absorption parameter, Eccentricity, Conformal mapping.

1. Introduction

Catheters are used extensively in contemporary medicine. Their purpose may be to accurately measure arterial pressure of pressure gradient, or to assist by injecting the inflatable balloon catheter – possibly even to clear short occlusions from the walls of a stenosed artery. Arterial constriction / stenosis is associated with significant changes in blood flow, pressure distribution and resistance to flow the damage caused to arterial wall due to getting old and stenos is increases permeability of solvents at the walls. One of the clinical procedures to treat atherosclerosis balloon angioplasty involves the insertion of a catheter with a tiny balloon at the end. The rate of flow of blood through an artery is however, governed by the need of the surrounding tissues for nutrients, such as oxygen when the supply of nutrients is not sufficient to meet the needs of the tissues the artery dilates and the pressure gradient increases until the rate of blood flow rises to a level which ensures an adequate supply of nutrients this process is known as auto regulation. Transport process other than permeation may control accumulation. The analysis of oxygen transport illustrates a correlation between zones of low wall shear stress and reduced oxygen flux into the wall. As the dynamics of flow between circular cylinders can be modified appreciable by displacement of axis it is necessary to consider eccentric annular region bounded by two circular cylinders.

Taylor [9] was the first person to study the dispersion of a bolus of a passive tracer in circular tube and showed that if a solute is injected into a solvent flowing steadily in a straight tube, the lateral molecular diffusion and the variation of velocity over the cross section would spread the solute diffusively with effective molecular diffusivity. Aris [1] included the effect of axial molecular diffusivity. Sankarasubramanian and Gill [6] have studied the Taylor diffusion in laminar flow in an eccentric annulus. Feldman and et al [3] have found a laminar developing flow in eccentric annular ducts. Nouri and Whitelaw [5] have studied the flow of Newtonian and non-Newtonian fluids in an eccentric annulus with rotation of the inner cylinder. Escudier and et al [2] have studied the fully developed laminar flow of purely viscous non-Newtonian liquids through annuli, including the effects of eccentricity and inner-cylinder rotation. Lorenzini and Casalena [4] have analyzed pulsatile blood flow in an atherosclerotic human artery with eccentric plaques.

Shivakumar and Ji [7], [8] have considered transport of Newtonian fluid flowing in an annular domain 'D' in the x-y plane bounded by two eccentric circles. The method is to map the eccentric circles in the x-y plane to concentric circles in a $\zeta - \eta$ plane conformally such that the boundary condition on the eccentric annulus is satisfied. In this investigation, the dispersion in an eccentric annulus region by taking blood as a Newtonian fluid with the investigation of oxygen transfer to the tissue cells in an eccentric catheterized artery.

2. Mathematical Formulation

The cross section of the artery is shown in Figure. 1. The annular region Ϙ is bounded by the circles C_1 and C_2 given by $C_1 : (x-h)^2 + y^2 = a^2$, $C_2 : (x-k)^2 + y^2 = b^2$, where a is the radius of inner tube (catheter), b is the radius of outer tube (artery) $a < b$, $h > k$.

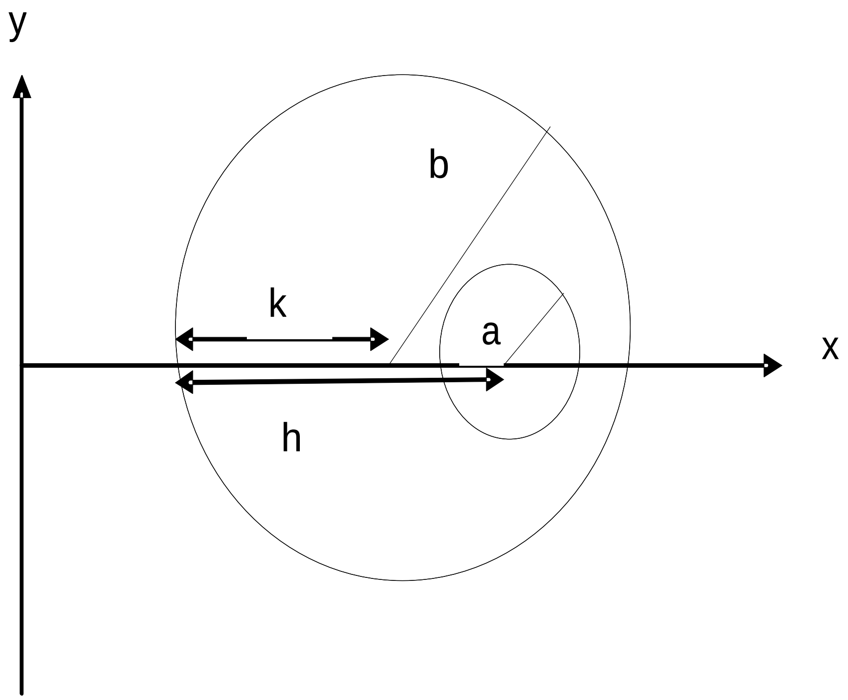

Fig. 1 Physical Configuration

The governing dispersion equation for the physical configuration in an eccentric annulus flow of a solute concentration $C(X,Y,t)$ is given with the initial and boundary condition

$$\frac{\partial C}{\partial t}=D\left(\frac{\partial^2 C}{\partial X^2}+\frac{\partial^2 C}{\partial Y^2}\right), \tag{1}$$

$$C(X,Y,0)=C_0, \tag{2}$$

$$\frac{\partial C}{\partial n}=0 \quad \text{No reaction at the inner wall } C_1. \tag{3}$$

$$\frac{\partial C}{\partial n}=-\kappa C \quad \text{First order reaction at the outer wall } C_2. \tag{4}$$

where C is solute concentration, C_0 is the initial solute concentration, n is the normal directional variable, D is the diffusion coefficient and κ is the reaction(absorption) parameter.

3. Method of the Solution

It is expedient to introduce the non-dimensional variables

$$c=\frac{C}{C_0},\ \tau=\frac{Dt}{R_0^2},\ x=\frac{X}{R_0},\ y=\frac{Y}{R_0} \tag{5}$$

In terms of these dimensionless quantities the governing diffusion equation with initial and boundary condition are given by

$$\frac{\partial c}{\partial \tau}=\left(\frac{\partial^2 c}{\partial x^2}+\frac{\partial^2 c}{\partial y^2}\right), \tag{6}$$

$$c(x, y, 0)=1, \tag{7}$$

$$\frac{\partial c}{\partial n}=0 \quad \text{No reaction at the inner wall } C_1. \tag{8}$$

$$\frac{\partial c}{\partial n}=-\kappa c \quad \text{First order reaction at the outer wall } C_2. \tag{9}$$

For the eccentric annulus, the boundary conditions can't be expressed in cylindrical coordinates at a constant value of one independent variable, therefore introducing the complex coordinates $z=x+i\,y,\ \bar{z}=x+i\,y$ in the Eq. (6), it will be converted as

$$\frac{\partial c}{\partial \tau}=\frac{\partial^2 c}{\partial z\,\partial \bar{z}}, \tag{10}$$

Applying the Laplace transformation technique on the diffusion equation given in Eq.(7) and on the boundary conditions given in Eqs. (8) & (9) are transformed as

$$4\frac{\partial^2 \bar{c}(s)}{\partial z\,\partial \bar{z}}-s\bar{c}(s)+1=0 \tag{11}$$

$$\frac{\partial \bar{c}}{\partial n}=0 \quad \text{No reaction at the inner wall } C_1. \tag{12}$$

$$\frac{\partial \bar{c}}{\partial n}=-\kappa \bar{c} \quad \text{First order reaction at the outer wall } C_2. \tag{13}$$

$\bar{c}(s)$ is a function of $z\bar{z}$ and $z+\bar{z}$ such that $\bar{c}$ is real function of $x \,\&\, y$.

$$\bar{c}(s)=f(z\bar{z})+g(z+\bar{z}) \tag{14}$$

By solving the Eq. (11) with the technique of series solution in terms of Bessel functions of first and second kinds, according to physical configuration as a cross section of cylindrical annulus for doubly connected region for the transformation of concentration profile is obtained as

$$\bar{c}(s) = A_1 J_0\left(\frac{\sqrt{sz\bar{z}}}{2}\right) + A_2 Y_0\left(\frac{\sqrt{sz\bar{z}}}{2}\right) + A_3 e^{\frac{\sqrt{s}}{2}(z+\bar{z})} + A_4 e^{\frac{-\sqrt{s}}{2}(z+\bar{z})} + \frac{4}{s}, \tag{15}$$

where

$$J_0\left(\frac{\sqrt{sz\bar{z}}}{2}\right) = \sum_{r=0}^{\infty} \frac{(-1)^r s^r \left(z\bar{z}\right)^r}{2^{2r}(r!)^2},$$

$$Y_0\left(\frac{\sqrt{sz\bar{z}}}{2}\right) = \frac{2}{\pi} J_0\left(\frac{\sqrt{sz\bar{z}}}{2}\right)\left(\log\frac{\sqrt{sz\bar{z}}}{2} - \psi(r+1)\right)$$

$$\psi(r+1) = -\gamma + 1 + \frac{1}{2} + \frac{1}{3} + - - - - - + \frac{1}{r}.$$

4. The Eccentric Annulus

The conformal mapping defined as $z = \dfrac{c}{1-\zeta}$, (16)

where $z = x + iy$ and $\zeta = \xi + i\eta$, to transform conformally the annular space enclosed by two eccentric circles to a concentric circles annular region bounded by two circles of radius ($|\zeta| = \rho$) ρ_1 and ρ_2 with

$C_1 : (x-h)^2 + y^2 = a^2$, $C_2 : (x-k)^2 + y^2 = b^2$, $a < b, h < k$ then

$\rho_1 = \dfrac{a}{h}, \rho_2 = \dfrac{b}{k}$ and $c = h - \dfrac{a^2}{h} = k - \dfrac{b^2}{k}$, h and k must satisfy $k - h = \dfrac{a^2}{h} - \dfrac{b^2}{k}$,

where a is the radius of the catheter, b is the radius of the artery, ε is the eccentricity parameter.

Using the conformal transformation given in Eq.(16) the boundary conditions on concentration become for eccentric annulus region transformed as concentric annulus region

$$\frac{\partial \bar{c}}{\partial \rho} = 0 \quad \text{No reaction at the inner wall } C_1. \tag{17}$$

$$\frac{\partial \bar{c}}{\partial \rho} = -\kappa \bar{c} \text{ First order reaction at the outer wall } C_2. \tag{18}$$

Transforming the Eq. (15) by using the conformal mapping on Eq. (16) and applying the boundary conditions given in Eqs. (17) & (18) then the transformed concentration profile is obtained as

$$\bar{c}(s) = A_1 \Lambda_1 \left(\zeta \bar{\zeta}\right) + A_2 \Lambda_2 \left(\zeta \bar{\zeta}\right) + A_3 \Lambda_3 \left(\zeta \bar{\zeta}\right) + A_4 \Lambda_4 \left(\zeta \bar{\zeta}\right) + \frac{4}{s}. \tag{19}$$

where

$$\Lambda_1\left(\varsigma\bar{\varsigma}\right) = \sum_{r=0}^{\infty} \alpha_r \left(1 + \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \beta_1 (r-1, p, m) \rho^{2p} \left[\zeta^m + \frac{\rho^{2m}}{\zeta^m} \right] \right);$$

$$\Lambda_2\left(\varsigma\bar{\varsigma}\right) = \sum_{r=0}^{\infty} \alpha_r \left(1 + \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \beta_1 (r-1, p, m) \rho^{2p} \left[\zeta^m + \frac{\rho^{2m}}{\zeta^m} \right] \right)$$

$$\left(S_r + \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \left[\zeta^n + \frac{\rho^{2n}}{\zeta^n} \right] \right);$$

$$\Lambda_3\left(\varsigma\bar{\varsigma}\right) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \gamma_r \left(\begin{array}{l} 2\sum_{p=0}^{\infty} \beta_2 (n, r-1, p) \rho^{2p} \\ + \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \beta_3 (n, r-1, m, p) \rho^{2p} \left[\zeta^m + \frac{\rho^{2m}}{\zeta^m} \right] \end{array} \right);$$

$$\Lambda_4\left(\varsigma\bar{\varsigma}\right) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} (-1)^n \gamma_r \left(\begin{array}{l} 2\sum_{p=0}^{\infty} \beta_2 (n, r-1, p) \rho^{2p} \\ + \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \beta_3 (n, r-1, m, p) \rho^{2p} \left[\zeta^m + \frac{\rho^{2m}}{\zeta^m} \right] \end{array} \right);$$

$$\rho_1 = \frac{a}{h},\ \rho_2 = \frac{b}{k},\ \lambda = h - \frac{a^2}{h},\ \gamma = -.577215666,\ S_r = \psi(r+1) - \log\left(\frac{\sqrt{s}\,\lambda}{2} \right),$$

$$\alpha_r = \frac{(-1)^r \left(\frac{s}{2}\right)^{2r} \lambda^{2r}}{(r!)^2} \qquad \gamma_r = \frac{\lambda^{n+2r}}{r!\, n+r!}$$

$$\beta\left(r-1,p\right)=\frac{\left(r-1+p\right)!}{(r-1)!\,p!},\ \beta\left(n,r-1,p\right)=\frac{\left(n+r-1+p\right)!}{(n+r-1)!\,p!},$$

$$\beta_1\left(r-1,p,m\right)=\beta\left(r-1,p\right)\beta\left(r-1,m\right),$$

$$\beta_2\left(n,r-1,p\right)=\beta\left(n,r-1,p\right)\beta\left(r-1,p\right),$$

$$\beta_3\left(n,r-1,p,m\right)=\beta\left(n,r-1,p\right)\beta\left(r-1,m\right)+\beta\left(n,r-1,m\right)\beta\left(r-1,p\right),$$

5. Results and Discussions

Numerical solutions are investigated with graphical presentations to understand the simultaneous effects of absorption parameter and eccentricity on the transformed concentration profile given in Eq. (7.4.4) is evaluated for absorption parameter $\kappa = 0.001, 0.1, 1, 10$, eccentricity parameter $\varepsilon = 0, 0.1, 0.5$. These values of κ & ε are taken as typical although actual values can be used according to the physical situations. The changing nature of the concentration profiles are incorporated in figures 2 to 5.

It is observed from the figures 2 & 3 for various values of eccentricity $\varepsilon = 0, 0.1, 0.5$ the concentration profile along X-axis shrinks from parabolic view as absorption parameter increases. On the other hand it can be prudent from the figures 2 & 3 as the eccentricity enhances the concentration diminishes. It is further interesting to note that the influence of eccentricity is more on concentration.

It is noticed from the figures 4 & 5 for various values of eccentricity $\varepsilon = 0, 0.1, 0.5$ the concentration profile along Y-axis is diminishing as absorption parameter increases. It is further interesting to note that the the influence of eccentricity is more on concentration along both the axes.

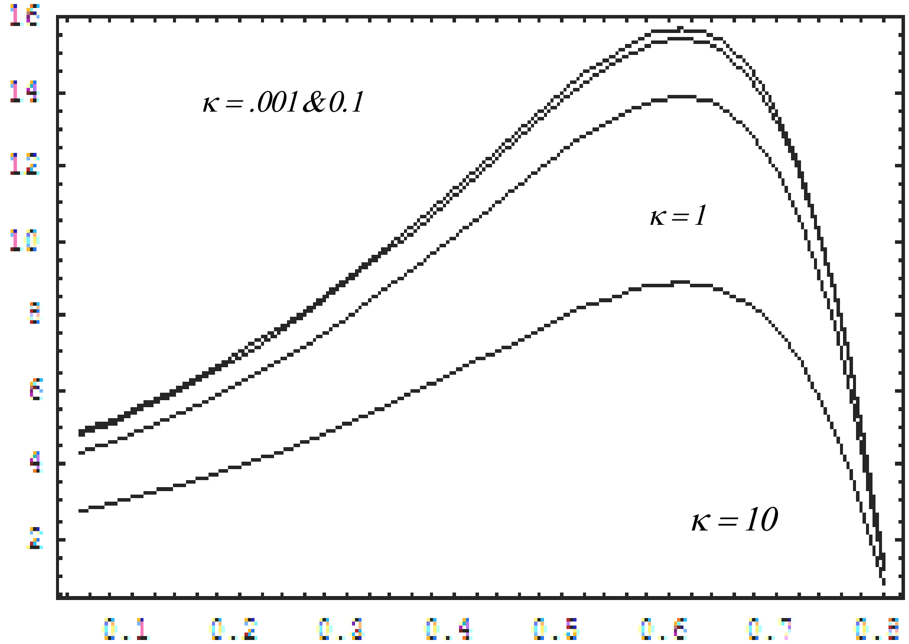

Fig.2 Concentration profile $(\bar{c})$ along X-axis for various values of absorption parameter (κ) at $(0.001, .1, 1, 10)$ for eccentricity $\varepsilon = 0$

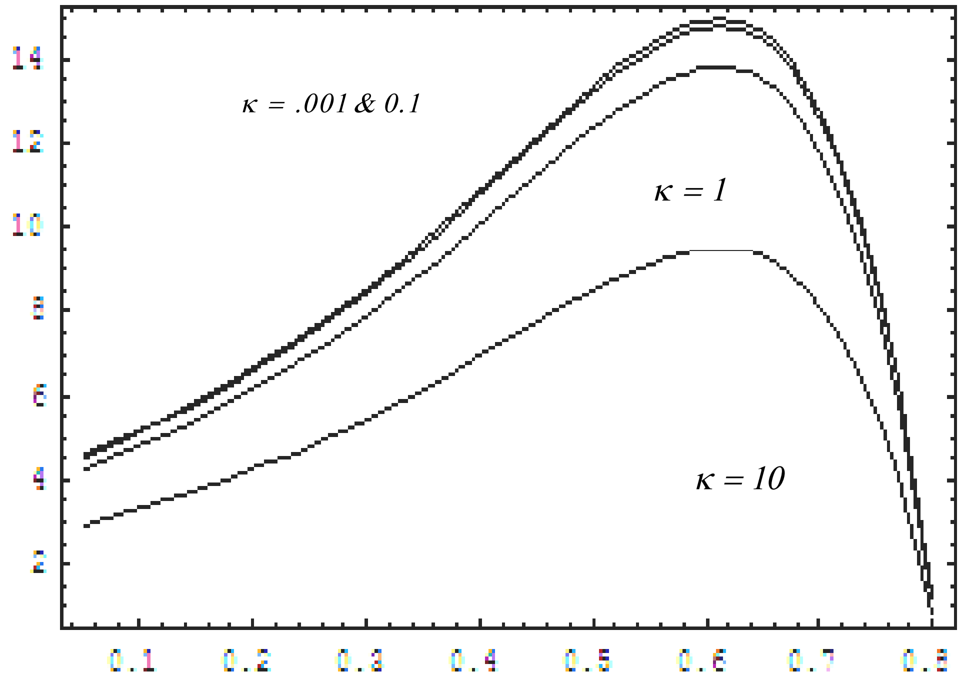

Fig.3 Concentration profile $(\bar{c})$ along X-axis for various values of absorption parameter (κ) at $(0.001, .1, 1, 10)$ for eccentricity $\varepsilon = 0.1$

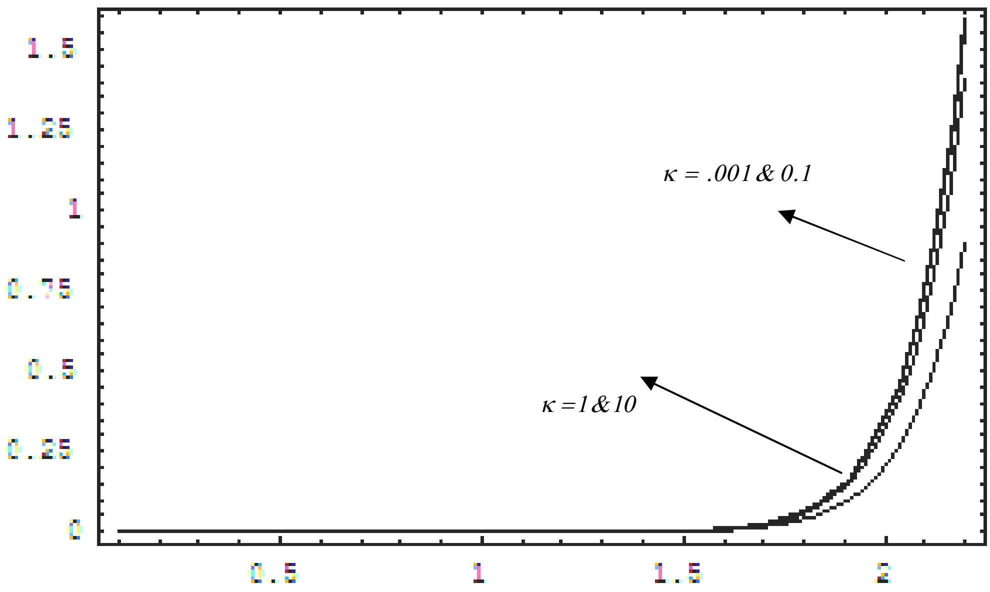

Fig.4 Concentration profile $(\bar{c})$ along Y-axis for various values of absorption parameter

(κ) at $(0.001, .1, 1, 10)$ for eccentricity $\varepsilon = 0$

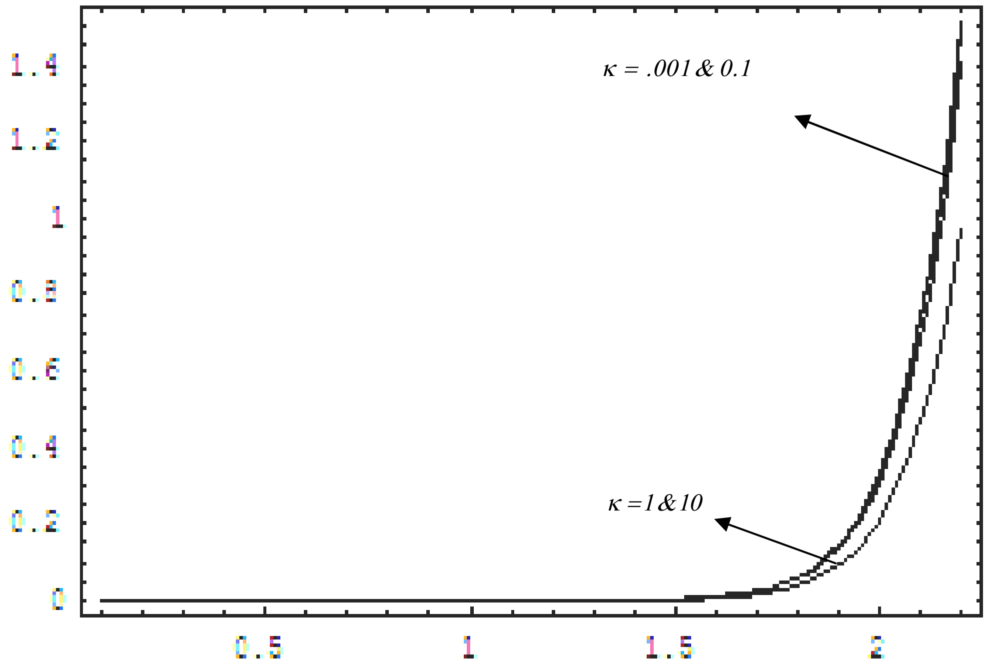

Fig.5 Concentration profile $(\bar{c})$ along Y-axis for various values of absorption parameter

(κ) at $(0.001, .1, 1, 10)$ for eccentricity $\varepsilon = 0.1$

6. Conclusions

The present problem is concerned with the investigations of the concentration in an eccentric annulus region. The investigations reveal that as absorption parameter and eccentricity enhances, the diffusion and solute concentration diminishes. As $\varepsilon \to 0$ gives results of concentric cylinders.

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