# Case Studies in Experimental Mathematics

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**Abstract**

Experimental mathematics is now a well accepted genre of study. It is a field in which computer-assisted experimentation — the generation of numbers, sequences, graphs and so on, using powerful software — combined with theoretical study can yield a great deal. Though this sounds revolutionary, it isn't. Indeed, using numerical and graphical experimentation as a means to arrive at conjectures has been a standard mode of operation used by mathematicians for centuries. The only difference is that today vastly stronger technology is available to us.

In this talk we describe three problems in which substantial progress is accomplished when computer software is used: (i) a problem from number theory, featuring an iteration with an unusual conclusion; (ii) a problem dealing with the enumeration of integer-sided triangles; (iii) a problem where we study the variation in a function defined on the space of all triangles. We also give solutions to the problems.

## 1 A two-term number theoretic iteration

Iterations in number theory often yield surprising results of great beauty. The present one is no exception.

Let $a$, $b$ be two numbers. Define a sequence $u_n$ such that $u_n = (u_{n-1} + u_{n-2})/2$, the starting values being $u_1 = a$ and $u_2 = b$ (so the recurrence holds for $n > 2$). For example, with $a = 1$ and $b = 4$ this leads to the sequence

$$1, \quad 4, \quad \frac{5}{2}, \quad \frac{13}{4}, \quad \frac{23}{8}, \quad \frac{49}{16}, \quad \frac{95}{32}, \quad \frac{193}{64}, \quad \frac{383}{128}, \quad \dots,$$

and it is easy to show that the sequence converges to a limiting value of $(a + 2b)/3$. Note that this is a weighted mean of $a, b$.

Now let us alter the recurrence just a bit. Let $a, b$ be two *odd* positive integers, and let $v_n$ be a sequence such that $v_1 = a$, $v_2 = b$ and for $n > 2$,

$$v_n = \text{the largest odd divisor of } v_{n-1} + v_{n-2}. \tag{1}$$

Hence $v_n$ is of the form $(v_{n-1} + v_{n-2})/2^k$ where $2^k$ is a suitable power of 2 (specifically, the largest power of 2 that divides $v_{n-1} + v_{n-2}$). Stated this way, the similarity to the earlier iteration is easy to see (and, similarly, a distant resemblance to the Fibonacci recurrence); but the results are very different. Here are some data.

- If the starting numbers are 1 and 3, the sequence goes: $1, 3, 1, 1, 1, 1, \ldots$.
- If the starting numbers are 1 and 5, the sequence goes: $1, 5, 3, 1, 1, 1, 1, \ldots$.
- If the starting numbers are 9 and 13, the sequence goes: $9, 13, 11, 3, 7, 5, 3, 1, 1, 1, 1, \ldots$.

It is easy to see that the $v$-sequence consists only of odd positive integers, and if two successive values of $v$ are equal, then the sequence stays fixed at that value from that point on. Observe that in each instance (above) the sequence converged to a fixed value. Will this always be the case? The data suggest it is so. Assuming it is so, exactly how does this fixed value depend on the initial values? This is the question we study here. The pattern is not obvious, so it helps to generate data using a computer and then to look for patterns. We use the well known `Mathematica` package. Here are the commands:

```
ClearAll[f, g];
g[n_] := If[OddQ[n], n, g[n/2]];
(* The function g computes the largest odd divisor of the integer n. *)
f[a_, b_] := (ClearAll[v];
  SetAttributes[v, Listable];
  v[1] = a; v[2] = b;
  v[n_] := v[n] = g[v[n - 1] + v[n - 2]];
  (* The function v generates the sequence under study. *)
  c = 1;
  While[v[c] != v[c + 1], c++];
  c++;
  v[Range[1, c]])
```

## 1.1 Notes

The function $g$ computes the largest odd divisor of a given positive integer $n$, using recursion: if $n$ is odd then $g(n) = n$, and if $n$ is even then $g(n) = g(n/2)$. In the commands defining $f$, we first define the sequence $v$, then find the point at which two successive $v$-values are equal, and finally display all the $v$-values till that point. The fixed point is thus the last number in the string. Note that the code implicitly assumes that the $v$-sequence always reaches a fixed point! But as the initial experimentation has supported this belief we have gone ahead and set up the code as shown, without inserting any 'safety valve' — i.e., without any exit condition in case the program does not terminate.

## 1.2 Results

Here are the data generated by the above commands, for different pairs of starting numbers. (Do not forget that the numbers are supposed to be odd.) We shall use the symbol $L(a, b)$ to denote the limiting fixed value when the initial two numbers are $a$ and $b$ (in that order). We shall simultaneously derive an expression for $L(a, b)$ and show that it is well defined. Note that the sequence generated by $a, b$ is not the same as the sequence generated by $b, a$, so it is not clear at the outset whether or not $L(a, b)$ equals $L(b, a)$. Here are our findings:

- $L(a, a) = a$ for all odd $a$.
- $L(1, a) = 1$ for all odd $a$.
- $L(3, a) = 1$ when $a = 1, 5, 7, 11, 13, 17, \ldots$, and $L(3, a) = 3$ when $a = 3, 9, 15, 21, 27, \ldots$.
- $L(5, a) = 1$ when $a = 1, 3, 7, 9, 11, 13, 17, \ldots$, and $L(5, a) = 5$ when $a = 5, 15, 25, 35, \ldots$.

## 1.3 Conjecture

After some experimentation we are led to the following guess:

**Conjecture 1** *If $a, b$ are odd positive integers, then $L(a, b)$ equals the greatest common divisor of $a, b$. That is, $L(a, b) = \gcd(a, b)$.*

The conjecture is correct, and we now provide a proof. We make use of the following observation: *If $L(a, b) = c$ then $L(ka, kb) = kc$ for any odd positive integer $k$.*

## 1.4 Proof

In view of the observation made above, we may suppose with no loss that the initial odd numbers $a$ and $b$ are coprime; for, if $\gcd(a, b) = d$ then $d$ will be a divisor of every member of the $v$-sequence and hence may simply be divided out from the sequence. Hence what we need to show is: *If $a, b$ are two coprime odd positive integers, then $L(a, b) = 1$.* The proof is as argued out below.

**Step 1:** For odd positive integers $r, s$, the greatest odd divisor of $r + s$ is at most equal to $(r + s)/2$ and therefore at most equal to $\max(r, s)$. Hence if $v_1 = a$ and $v_2 = b$ then $v_n$ is bounded above by $\max(a, b)$. Therefore $v_n$ can take only finitely many values.

**Step 2:** Therefore the number of possibilities for pairs of successive values of the $v$-sequence is finite. It follows that after some point, a pair of successive values will recur. From this point onwards the $v$-sequence will necessarily be periodic.

**Step 3:** Therefore for any choice of the two starting odd numbers ($a$ and $b$) the $v$-sequence will ultimately settle into a periodic cycle.

**Step 4:** In the periodic cycle, let $r$ be the largest number, and let $s$ be the number immediately following it. If $s = r$ then the sequence from this point will be $r, r, r, \ldots$. Now consider the consequence of $s < r$. In this case the number following $s$ will be at most $(r+s)/2$ and hence strictly less than $r$. But with two consecutive numbers strictly less than $r$, every number after that will be strictly less than $r$, so there is no possibility of the sequence ever reaching $r$ again. But this contradicts the very notion of a cycle! Hence all the numbers in the cycle are the same, which actually means that the cycle consists of just one number.

**Step 5:** It follows that for any choice of the two starting odd numbers ($a$ and $b$) the $v$-sequence does ultimately reach some fixed value $L(a,b) = c$, say. That is, from some point on, the $v$-sequence reads $c, c, \ldots$ (here $c$ is an odd number). Let the last number before the fixed value is reached be $d$; by definition, $d \neq c$. (It cannot be that all numbers in the sequence are equal, so the number $d$ does exist.) Hence we have the following three consecutive terms of the $v$-sequence: $d, c, c$. This means that $c$ is the largest odd divisor of $d + c$. But this implies that $c$ is a divisor of $d$. It is easy now to deduce inductively that $c$ divides every member of the $v$-sequence; in particular, that $c$ divides $a$ and $b$. But we had assumed at the start that $a$ and $b$ are coprime. Hence $c = 1$. It follows that $L(a,b) = 1$. This conclusion is valid provided $a, b$ are coprime. We have shown what we set out to show.

**Step 6:** In general, therefore, $L(a,b) = \gcd(a,b)$ for any pair of odd positive integers $a, b$.

# 2 Counting triangles

From number theory we switch to combinatorics, though this too is a problem with a number theoretic flavour. Now we count triangles! The problem is easily stated:

**Problem 2** *Find the number $f(n)$ of integer-sided triangles with perimeter $n$.*

We have: $f(1) = f(2) = 0$, $f(3) = 1$, $f(4) = 0$. The last equality comes as a surprise, and shows that $f$ is not monotone as might have been expected.

Let the lengths of the sides be $a, b, c$. To avoid duplication we impose the condition $a \geq b \geq c$. Invoking the fact that any two sides of a triangle exceed the third one we see that $f(n)$ is equal to the number of positive integer solutions to the following system:

$$\begin{cases} a + b + c = n, \\ a \geq b \geq c, \\ b + c > a. \end{cases} \tag{2}$$

This can be recast using only two variables by using the fact that $c = n - a - b$. We then see that $f(n)$ equals the number of positive integer pairs $(a, b)$ such that

$$\begin{cases} a \geq b, \\ a + 2b \geq n, \\ a + b \leq n - 1, \\ a < n/2. \end{cases} \tag{3}$$

It is easy to write the `Mathematica` code that will generate values of $f$. The code for formulation (2) is the following:

```
ClearAll[t, f];
t[n_] := t[n] = (s = {};
    Do[If[a + b + c == n, s = Append[s, {a, b, c}]],
     {a, 1, Floor[n/2]}, {b, 1, a}, {c, a - b + 1, b}];
    s);
f[n_] := f[n] = Length[t[n]];
SetAttributes[{t, f}, Listable]
```

Here is the code for formulation (3); it generates exactly the same results as the one above, but is significantly faster in its operation:

```
ClearAll[t, f];
t[n_] := (s = {};
   Do[If[a + 2 b >= n, s = Append[s, {a, b, n - a - b}]],
    {a, 1, Floor[(n - 1)/2]}, {b, 1, a}];
   s);
f[n_] := Length[t[n]];
SetAttributes[{t, f}, Listable]
```

## 2.1 Notes

$t(n)$ computes the strings $(a, b, c)$ corresponding to the side lengths of the triangles with perimeter $n$, while $f(n)$ counts their number. For example, for $n = 11$ we get the following strings: $(4, 4, 3)$, $(5, 3, 3)$, $(5, 4, 2)$, $(5, 5, 1)$. These give the four integer-sided triangles with perimeter 11.

## 2.2 Results

Here is a list (obtained after executing the above code) of the first hundred values of $f$, i.e., the values $f(1)$, $f(2)$, $f(3)$, …, $f(100)$:

> 0, 0, 1, 0, 1, 1, 2, 1, 3, 2, 4, 3, 5, 4, 7, 5, 8, 7, 10, 8, 12, 10, 14, 12, 16, 14, 19, 16, 21, 19, 24, 21, 27, 24, 30, 27, 33, 30, 37, 33, 40, 37, 44, 40, 48, 44, 52, 48, 56, 52, 61, 56, 65, 61, 70, 65, 75, 70, 80, 75, 85, 80, 91, 85, 96, 91, 102, 96, 108, 102, 114, 108, 120, 114, 127, 120, 133, 127, 140, 133, 147, 140, 154, 147, 161, 154, 169, 161, 176, 169, 184, 176, 192, 184, 200, 192, 208, 200, 217, 208.

Note the non-monotonicity! The irregularity would seem to make the task of finding a neat formula for $f$ quite a challenge.

Figure 1 shows a plot of the points $(n, f(n))$ for $n = 1, 2, 3, \ldots, 100$. It is indeed a very curious looking graph.

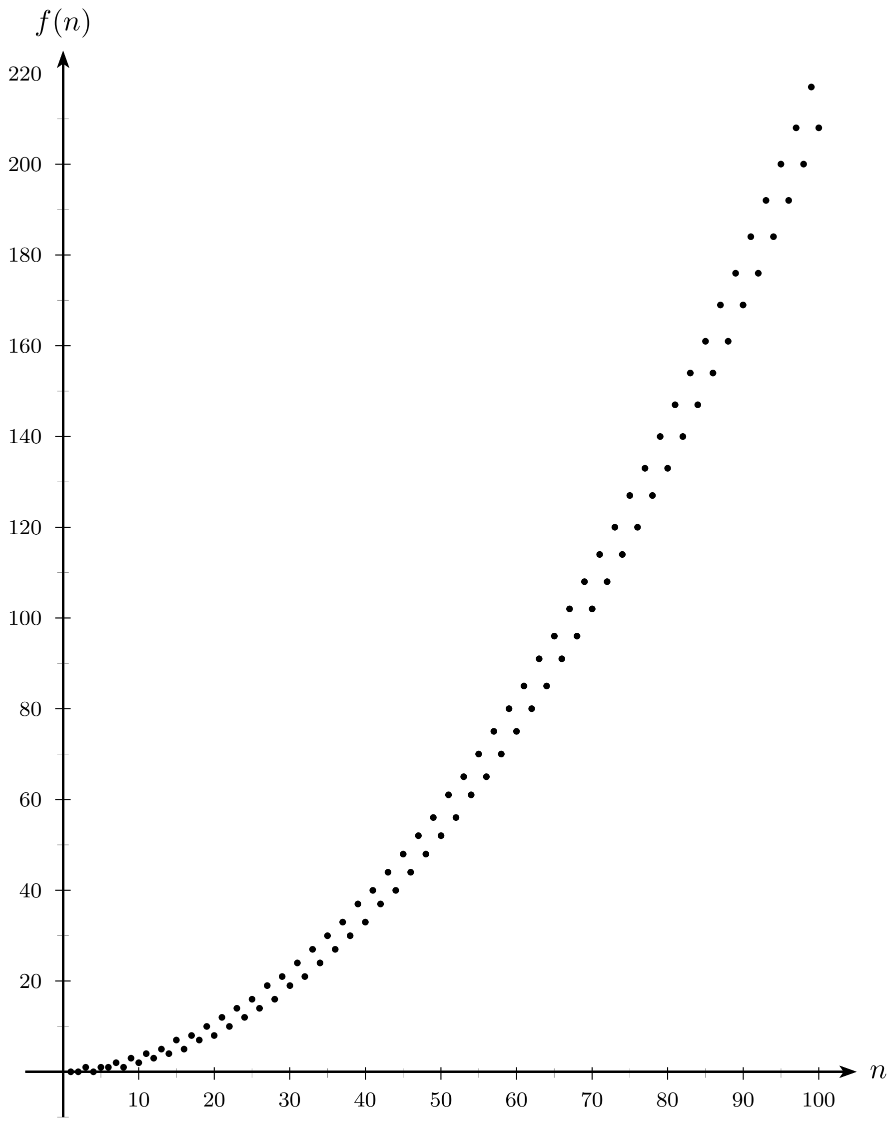


Figure 1: *A strange looking graph*

The appearance of two overlaid curves suggests that the formula for $f(n)$ will depend on the class of $n$ relative to some modulus. The shape of the curve(s) suggests a quadratic relationship. A simple way of getting a handle on the relationship is to examine the values of $r(n) := n^2/f(n)$ for a few values of $n$. Here's what we find: $r(90) \approx 47.9$, $r(100) \approx 48.1$. The evidence strongly suggests that $f(n)$ is close to $n^2/48$. Once this is noticed, it is easy to see why it must be so. We had found earlier that $f(n)$ is the number of positive integer pairs $(a, b)$ such that $a \geq b$, $a+2b \geq n$, $a + b \leq n - 1$ and $a < n/2$. The last condition may be replaced by $a \leq \lfloor n/2 \rfloor$. If we plot these inequalities on a graph, we get the shape shown in Figure 2.

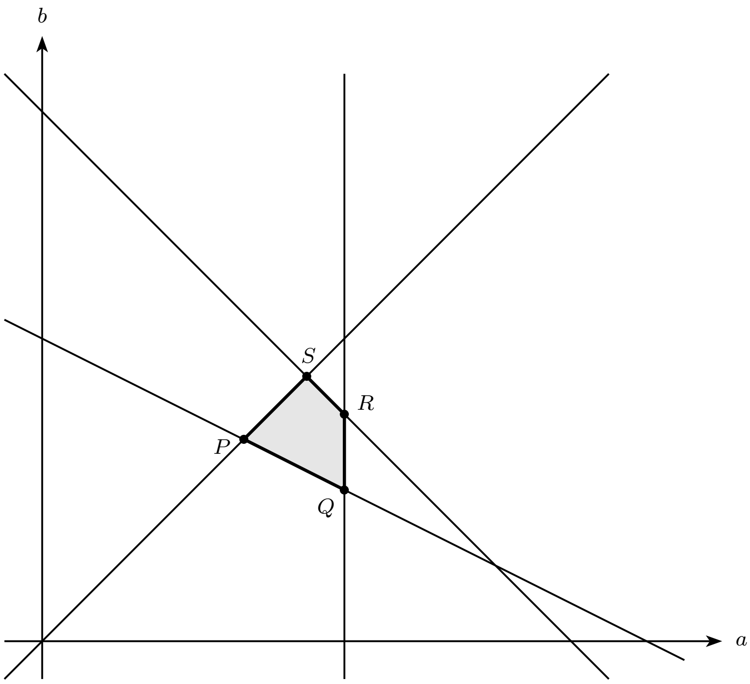


Figure 2: *The region produced by the four inequalities*

The coordinates of the vertices $P, Q, R, S$ of quadrilateral $PQRS$ are:

$$\begin{cases} P = \left(\frac{n}{3},\ \frac{n}{3}\right), & Q = \left(\left\lfloor \frac{n}{2} \right\rfloor,\ \frac{n}{2} - \frac{1}{2}\left\lfloor \frac{n}{2} \right\rfloor\right), \\ R = \left(\left\lfloor \frac{n}{2} \right\rfloor,\ n - 1 - \left\lfloor \frac{n}{2} \right\rfloor\right), & S = \left(\frac{n-1}{2},\ \frac{n-1}{2}\right). \end{cases} \tag{4}$$

If we approximate $\lfloor x \rfloor$ by $x$ (true for large $x$) then we may replace these by:

$$\begin{cases} P = \left(\frac{n}{3},\ \frac{n}{3}\right), & Q = \left(\frac{n}{2},\ \frac{n}{4}\right), \\ R = \left(\frac{n}{2},\ \frac{n}{2} - 1\right), & S = \left(\frac{n-1}{2},\ \frac{n-1}{2}\right). \end{cases} \tag{5}$$

The number of lattice points in this region (which is the quantity $f(n)$ for which we want a formula) is approximately equal to the area of the quadrilateral. Using determinants to compute area, we find that the area of triangle $PQR$ is $n(n-4)/48$, and that of triangle $QRS$ is $(n-4)/16$. Hence the area of $PQRS$ is

$$\frac{n(n-4)}{48} + \frac{n-4}{16} = \frac{(n+3)(n-4)}{48}, \tag{6}$$

and to a first approximation this is equal to $n^2/48$.

## 2.3 Curious relationships

Noting the irregularity of the sequence, let us study some subsequences of $\{f(n)\}$, corresponding to some modular classes. Here are the values of $f(1)$, $f(3)$, $f(5)$, ...:

$$0,\ 1,\ 1,\ 2,\ 3,\ 4,\ 5,\ 7,\ 8,\ 10,\ 12,\ 14,\ 16,\ 19,\ 21,\ 24,\ 27,\ 30,\ 33,\ 37,\ 40,\ 44,\ 48,\ 52,\ 56,\ \ldots.$$

And here are the values of $f(2)$, $f(4)$, $f(6)$, ...:

$$0,\ 0,\ 1,\ 1,\ 2,\ 3,\ 4,\ 5,\ 7,\ 8,\ 10,\ 12,\ 14,\ 16,\ 19,\ 21,\ 24,\ 27,\ 30,\ 33,\ 37,\ 40,\ 44,\ 48,\ 52,\ 56,\ \ldots.$$

The two sequences are completely identical, except for the initial 0 in the second sequence which produces a one-term offset. How curious. Is it possible, then, that $f(n) = f(n+3)$, identically? Well, not quite. Let's run the following `Mathematica` code:

```
s1 = {};
Do[If[f[n] == f[n + 3], s1 = Append[s1, n]], {n, 1, 100}];
s1
```

Here is the output:

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, 51, 53, 55, 57, 59, 61, 63, 65, 67, 69, 71, 73, 75, 77, 79, 81, 83, 85, 87, 89, 91, 93, 95, 97, 99.

Precisely the set of odd numbers! So we arrive at a most surprising finding:

**Conjecture 3** *The equality $f(n) = f(n+3)$ is true for every odd positive integer $n$. Otherwise expressed, we have: $f(2n-3) = f(2n)$ for every positive integer $n$.*

We shall prove the conjecture presently. If we study more such classes (every 3-rd term, every 4-th term, ...) we find that choosing a modulus of 12 yields dividends: by picking every 12-th term we find very pleasing patterns. Thus, the values of $f(1)$, $f(13)$, $f(25)$, $f(37)$, ... are:

$$0,\ 5,\ 16,\ 33,\ 56,\ 85,\ 120,\ 161,\ 208,\ \ldots;$$

the values of $f(2)$, $f(14)$, $f(26)$, $f(38)$, ... are:

$$0,\ 4,\ 14,\ 30,\ 52,\ 80,\ 114,\ 154,\ 200,\ \ldots;$$

the values of $f(3)$, $f(15)$, $f(27)$, $f(39)$, ... are:

$$1,\ 7,\ 19,\ 37,\ 61,\ 91,\ 127,\ 169,\ 217,\ \ldots;$$

and the values of $f(4)$, $f(16)$, $f(28)$, $f(40)$, ... are:

$$0,\ 5,\ 16,\ 33,\ 56,\ 85,\ 120,\ 161,\ 208,\ \ldots.$$

For each subsequence, all the second differences are equal to 6. From this observation emerges another one:

**Conjecture 4** *For all positive integers $n$ we have:*

$$f(n+24) - 2f(n+12) + f(n) = 6.$$

The fact that the subsequences have constant second difference allows us to find the formula we seek, though it will actually be a collection of different quadratic formulas (one for each class mod 12; the formulas will all have the same second degree term). Consider the values $f(1)$, $f(13)$, $f(25)$, $f(37)$, ...:

$$0,\ 5,\ 16,\ 33,\ 56,\ 85,\ 120,\ 161,\ 208,\ \ldots.$$

Let $a_k = f(12k+1)$; the leading term in the formula for $a_k$ will be $3k^2$ as the constant second difference ($= 6$) is 3 times the constant second difference for the sequence of squares. Subtracting $3k^2$ from $a_k$ we get the sequence $0, 2, 4, 6, \ldots$ whose $k$-th term is $2k$. (The first term here corresponds to $k = 0$.) Hence $a_k = 3k^2 + 2k$, and we can check that this fits. Going through each class mod 12 and repeating this analysis we readily obtain the following:

$$\begin{cases} f(12k) &= 3k^2, \\ f(12k+1) &= 3k^2+2k, \\ f(12k+2) &= 3k^2+k, \\ f(12k+3) &= 3k^2+3k+1, \\ f(12k+4) &= 3k^2+2k, \\ f(12k+5) &= 3k^2+4k+1, \\ f(12k+6) &= 3k^2+3k+1, \\ f(12k+7) &= 3k^2+5k+2, \\ f(12k+8) &= 3k^2+4k+1, \\ f(12k+9) &= 3k^2+6k+3, \\ f(12k+10) &= 3k^2+5k+2, \\ f(12k+11) &= 3k^2+7k+4. \end{cases} \tag{7}$$

These are the formulas found empirically, using the values of $f(n)$ generated by `Mathematica`. As such they cannot be considered as mathematically proved — not till we have also supplied a proper theoretical analysis. This can be done in various ways, but we do not include the analysis in this paper.

## 2.4 Another formula for f

Before closing we mention the following curious formula which has been found:

$$f(n) = \begin{cases} \text{Integer closest to } \dfrac{n^2}{48}, & \text{if } n \text{ is even,} \\[2ex] \text{Integer closest to } \dfrac{(n+3)^2}{48}, & \text{if } n \text{ is odd.} \end{cases} \tag{8}$$

Example: $f(5) = 1$, as the integer closest to $64/48$ is 1.

## 2.5 Generating function for f

By making use of the recurrence relations for $\{f(n)\}$ it is possible to prove the following remarkable formula which gives the generating function for the sequence:

$$1 + f(1)x + f(2)x^2 + f(3)x^3 + f(4)x^4 + \cdots = \frac{x^3}{(1-x^2)(1-x^3)(1-x^4)}. \tag{9}$$

## 2.6 Proof of conjecture 3

We shall now prove that $f(2n-3) = f(2n)$ for all $n$. Let $a, b, c$ be the sides of an integer-sided triangle with perimeter $2n-3$, labeled so that $a \geq b \geq c$; then $a+b+c = 2n-3$, and $b+c > a$. Let $a' = a+1$, $b' = b+1$, $c' = c+1$; then $a'+b'+c' = 2n$, $a' \geq b' \geq c'$, and $b'+c' > a'$. This yields an integer-sided triangle with perimeter $2n$.

Conversely, given an integer-sided triangle with perimeter $2n$, let its sides be labeled $a, b, c$ so that $a \geq b \geq c$; then $a+b+c = 2n$, and $b+c > a$. Let $a' = a-1$, $b' = b-1$, $c' = c-1$; then $a'+b'+c' = 2n-3$, $a' \geq b' \geq c'$, and $b'+c' \geq a'$. Can equality hold in the relation $b'+c' \geq a'$? No, precisely because $a'+b'+c' = 2n-3$, and $2n-3$ is odd (equality would make the sum $a'+b'+c'$ an even number). So strict inequality holds, $b'+c' > a'$, and thus $a', b', c'$ are the sides of an integer-sided triangle with perimeter $2n-3$, labeled the 'right' way.

It follows that $f(2n-3) = f(2n)$, as stated.

## 2.7 Proof of conjecture 4

We prove that $f(n+24) - 2f(n+12) + f(n) = 6$ through the following sequence of simpler results ('lemmas'). For each $n$, let $I(n)$ denote the number of isosceles integer-sided triangles with perimeter $n$ (note that 'isosceles' includes 'equilateral').

**Claim:** *The number of scalene integer-sided triangles with perimeter $n$ is equal to the number of integer-sided triangles with perimeter $n-6$.*

For example, take $n = 9$. There is just one integer-sided triangle with perimeter 3 (sides $1, 1, 1$) and just one scalene integer-sided triangle with perimeter 9 (sides $2, 3, 4$).

**Proof:** We shall exhibit a one to one correspondence between the set of scalene integer-sided triangles with perimeter $n$ and the set of integer-sided triangles with perimeter $n-6$.

Let $a, b, c$ be the sides of a scalene integer-sided triangle with perimeter $n$, with $a > b > c$; then $a+b+c = n$ and $b+c > a$. Let $a' = a-3$, $b' = b-2$, $c' = c-1$. Then $a'+b'+c' = n-6$, $a' \geq b' \geq c'$ and $b'+c'-a' = b+c-a > 0$; hence $a', b', c'$ are the sides of an integer-sided triangle with perimeter $n-6$. The mapping is clearly reversible, and establishes the desired correspondence.

**Corollary:** From the above it follows that $f(n) = f(n-6) + I(n)$ for all positive integers $n$.

**Claim:** $I(n) =$ *the number of integers strictly between* $n/4$ *and* $n/2$.

**Proof:** Let the sides of the triangle be $a, a, n-2a$. Then we must have $a + a > n - 2a$ (hence $a > n/4$) and $a + (n-2a) > a$ (hence $a < n/2$). So $I(n)$ must equal the number of integers lying strictly between $n/4$ and $n/2$.

**Claim:** $I(n+12) - I(n) = 3$ *for all* $n$.

**Proof:** $I(n+12) =$ the number of integers strictly between $(n+12)/4$ and $(n+12)/2$, i.e., between $3 + n/4$ and $6 + n/2$. This will clearly exceed by 3 the number of integers strictly between $n/4$ and $n/2$, i.e., it will exceed $I(n)$ by 3. Hence $I(n+12) - I(n) = 3$.

**Claim:** $f(n+24) - 2f(n+12) + f(n) = 6$ *for all positive integers* $n$.

**Proof:** Since $f(n) - f(n-6) = I(n)$ for all $n$, it follows that

$$\begin{aligned} f(n+12) - f(n) &= \Big(f(n+12) - f(n+6)\Big) + \Big(f(n+6) - f(n)\Big) \\ &= I(n+12) + I(n+6). \end{aligned}$$

Hence:

$$\begin{aligned} f(n+24) - f(n+12) &= I(n+24) + I(n+18), \\ f(n+12) - f(n) &= I(n+12) + I(n+6). \end{aligned}$$

By subtraction we get:

$$\begin{aligned} f(n+24) - 2f(n+12) + f(n) &= \Big(I(n+24) - I(n+12)\Big) + \Big(I(n+18) - I(n+6)\Big) \\ &= 3 + 3 = 6, \end{aligned}$$

as required.

As this property is now established, the formulas found earlier for $f$ are valid.

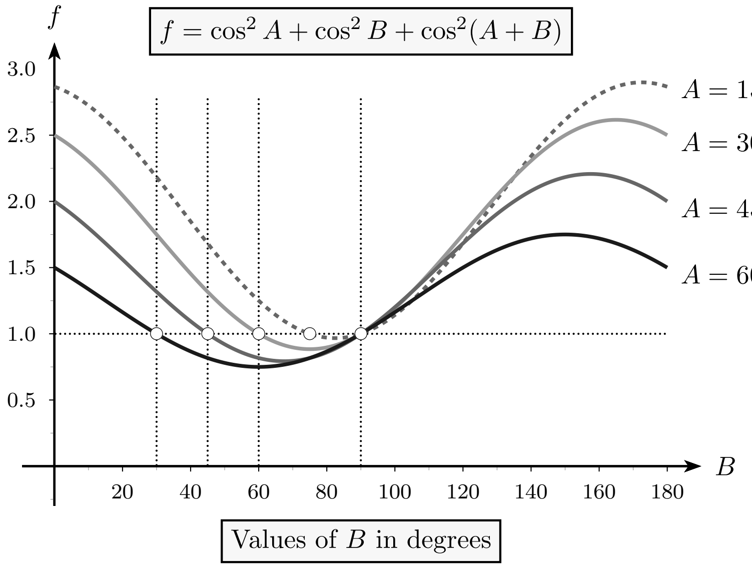


Figure 3: *Variation of* $f(A, B, C)$

# 3 Characterization of a right-angled triangle

In this section we explore a surprising characterization of an important class of triangles. The story we present deals with a natural looking question:

*To explore the variation of the quantity*

$$f(A, B, C) := \cos^2 A + \cos^2 B + \cos^2 C$$

*over the space of all possible triangles* $ABC$.

The question is not only about the extreme values that $f$ can take but also about the 'in-between' values. It seems natural to do the exploration using *GeoGebra*. What do we find? Something curious!

## The extreme values

- Guessing the 'upper end' is easy: if $A \to 0$, $B \to 0$ and $C \to \pi$, then $f(A, B, C) \to 3$. This is clearly the supremum. It is achieved only by a degenerate triangle with angles 0, 0, $\pi$.

- The infimum is achieved by an equilateral triangle, with $f$-value 3/4. To see why, consider the function $\phi(x) := \cos^2 x + \cos^2(k - x)$ where $k$ is a constant, $0 < k < \pi$. What is the infimum of $\phi$ for $0 \leq x \leq k$? The use of derivatives shows that if $k < \pi/2$, then $x = k/2$ yields a global minimum for $\phi$, and if $\pi/2 < k < \pi$, then $x = 0$ yields a global minimum for $\phi$. Hence $\phi(x) \geq 2\cos^2 k/2 = 1 + \cos k$ if $k < \pi/2$, and $\phi(x) \geq 1 + \cos^2 k$ if $\pi/2 \leq k < \pi$. It follows that $f(A, B, C) \geq 1 - \cos C + \cos^2 C$ if $C < \pi/2$, and $f(A, B, C) \geq 1 + 2\cos^2 C$ if $C > \pi/2$. Using derivatives once again we see that $f(A, B, C) \geq 3/4$ if $C < \pi/2$, and

$f(A,B,C) \geq 1$ if $C \geq \pi/2$. It follows that $f(A,B,C) \geq 3/4$ in all cases. As the value of $3/4$ is actually achieved by an equilateral triangle, it follows that this is the global infimum.

## An observation

It appears as though the following is true; if so it yields a surprising characterization of a right-angled triangle:

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 \text{ precisely when the triangle is right-angled!}$$

To show that "If the triangle is right-angled, then $\cos^2 A + \cos^2 B + \cos^2 C = 1$" is trivial. But why should the converse be true? This is far from clear.

### Remark

What makes the problem challenging is that 1 is not an extreme value for $\cos^2 A + \cos^2 B + \cos^2 C$. Some common methods of proof fail precisely for this reason.

(Example: Suppose we want to determine the conditions under which real numbers $x, y, z$ satisfy the relation $x^2 + y^2 + z^2 = xy + yz + zx$. Here in studying the quantity $x^2 + y^2 + z^2 - xy - yz - zx$, we discover that $x^2 + y^2 + z^2 - xy - yz - zx \geq 0$, with equality just when $x = y = z$; this follows when we write $x^2 + y^2 + z^2 - xy - yz - zx = \left[(x-y)^2 + (y-z)^2 + (z-x)^2\right]/2$. But this approach does not work here because (as noted) 1 is not an extreme value for $\cos^2 A + \cos^2 B + \cos^2 C$.)

## A CAS proof

The relation $\cos^2 A + \cos^2 B + \cos^2 C = 1$ leads (via the cosine formula) to:

$$\left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 + \left(\frac{c^2 + a^2 - b^2}{2ca}\right)^2 + \left(\frac{a^2 + b^2 - c^2}{2ab}\right)^2 = 1.$$

This leads in turn to:

$$a^2\left(b^2 + c^2 - a^2\right)^2 + b^2\left(c^2 + a^2 - b^2\right)^2 + c^2\left(a^2 + b^2 - c^2\right)^2 = 4a^2b^2c^2.$$

Now we must factorize 'left side minus right side'. This is rather easy to do using a CAS! Here's what *Derive* and *Mathematica* tell us:

$$\begin{aligned} & a^2\left(b^2 + c^2 - a^2\right)^2 + b^2\left(c^2 + a^2 - b^2\right)^2 + c^2\left(a^2 + b^2 - c^2\right)^2 - 4a^2b^2c^2 \\ & = -\left(a^2 + b^2 - c^2\right) \cdot \left(b^2 + c^2 - a^2\right) \cdot \left(c^2 + a^2 - b^2\right). \end{aligned}$$

And that finishes the problem most decisively, for the last expression is 0 just when the sum of the squares of some two sides equals the square of the third side, i.e., when the triangle is right-angled.

### A paper-and-pencil proof

Write $x = a^2$, $y = b^2$, $z = c^2$. Then from

$$a^2\left(b^2+c^2-a^2\right)^2+b^2\left(c^2+a^2-b^2\right)^2+c^2\left(a^2+b^2-c^2\right)^2=4a^2b^2c^2$$

we get:

$$x(y+z-x)^2+y(z+x-y)^2+z(x+y-z)^2-4xyz=0.$$

As a trial let us put $y + z - x = 0$, i.e., $x = y + z$. On the left side we get:

$$y(2z)^2+z(2y)^2-4(y+z)yz=y\left(4z^2\right)+z\left(4y^2\right)-4yz(y+z)=0,$$

identically. Hence $y + z - x$ is a factor of the expression. By symmetry, $z + x - y$ and $x + y - z$ too are factors.

Hence the equality $\cos^2 A + \cos^2 B + \cos^2 C = 1$ leads to:

$$(y+z-x)\cdot(z+x-y)\cdot(x+y-z)=0,$$

i.e.,

$$\left(b^2+c^2-a^2\right)\cdot\left(c^2+a^2-b^2\right)\cdot\left(a^2+b^2-c^2\right)=0.$$

The proof concludes as earlier.

## 4 Further remarks

Each of the explorations used computers in a basic and important way. However further remarks can be made in the individual cases.

**Two-term iteration** The ability of a computer to rapidly generate the terms of a recursively defined sequence is put to good use. The recursive rule here is easy to program. Without a good data-base of results it is difficult to anticipate what the iteration is 'doing', but the computer makes this much easier. Once the result is known, the proof is not hard to find.

**Counting triangles by perimeter** The ability of a computer to do a large number of operations and to sort data in well-defined ways at great speed is used most effectively. It is difficult to generate by hand values of $f(n)$ beyond, say, $n = 25$. With the computer we are able with ease to go up to $n = 200$, and then to display various subsequences of the main sequence. Finding by eye alone a parity based pattern is not difficult; nor one based on modulo 3 categories. But finding a pattern based on modulo 12 categories is difficult to spot! Here we are able to find a pattern and then prove it using 'old-fashioned' mathematical reasoning.

**Right-angled triangle** This is a problem for which intuition does not easily yield an answer. Exploration using `GeoGebra` strongly suggests a particular answer, and analysis then confirms the answer. Use of a high-powered CAS brings forth a quick answer, but it is possible to devise a paper-and-pencil proof which would satisfy a purist.

## References


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