Exploring space-filling origami

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Abstract

A new tetrahedra origami model was discussed in this paper. Sommerville in 1922 found four kinds of tetrahedron can fill space. This model starts from a rectangle paper, whose shape ratio is 1:√2, to produce one Sommerville's main tetrahedron. It is proved that as long as the original paper is big enough, one can produce by this origami method more and more so-called Sommerville-tetrahedrons without cutting or pasting. Furthermore, these tetrahedrons fill three-dimension space in the same time. Just like Peano curve fills two-dimensional space, the original paper (two dimensional manifold) used in producing tetrahedrons fills the three-dimensional space. This article also introduces some interesting models in the lower number cases.

Introduction

In 1922, D.M.Y Sommerville found Aristotle, a famous philosopher during Ancient Greek Times, made a mistake. Aristotle stated there were two kinds of regular polyhedron, including regular tetrahedra could tile space. In fact, cube is the only regular polyhedron that can fill space. Sommerville in his paper ended this mistake and pointed out that several general tetrahedra could fill space indeed. Resulting from the way of cutting and reassembling a cube, he found 4 kinds of tetrahedra. The most important tetrahedron was the one made up of four triangles whose edge ratios are 2: √3: √3. I will call it Sommerville-tetrahedron within this paper. Another three kinds of tetrahedra were developed by dissection or reassembling of the dissection to this original tetrahedron.

A4 paper is a special kind of rectangle paper whose shape is 1: √2. It is perfect to form a Sommerville tetrahedron by bending its four corners and central line. It is perfect because this origami has nowhere overlapped. Since A4 paper is similar to its halves, it can be anticipated that one sheet of A4 paper can form more than one such tetrahedra. Actually Chang & Liang 2012 [5] has proved that singe sheet of A4 paper can yield n^2 or 2n^2 tetrahedra at one time for any positive integer n. However, instead of tiling space, the tetrahedra have holes between them.

Can we find a way of tetrahedron origami that tile space?

The answer is yes.

Starting from an infinite long strip of rectangular paper, one can first easily origami it into a string of Sommerville tetrahedra. Then by twisting this string, it becomes a triangular prism with infinite length. Finally, this prism fills space by a sequence of U-turns at some proper places.

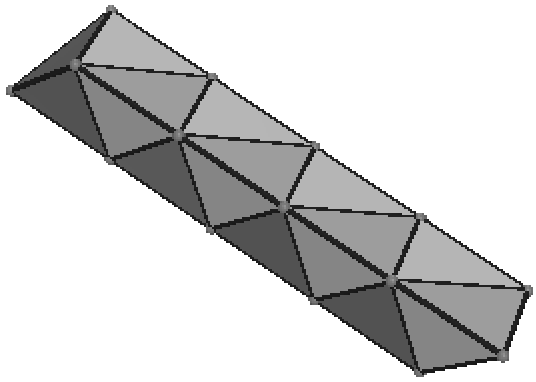

Fig. 1 Twisted string of Sommerville tetrahedra

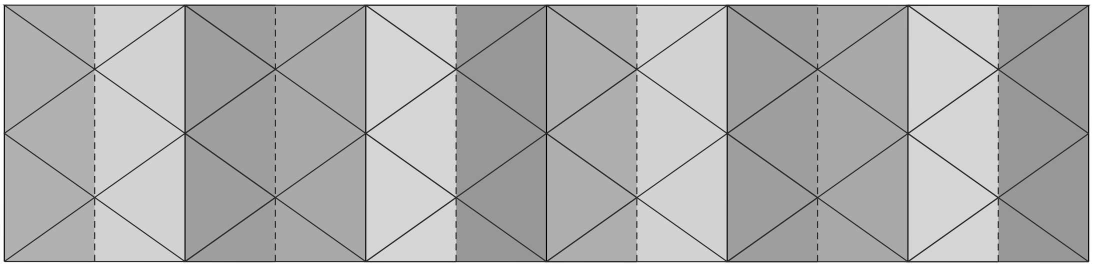

fig. 2 Origional long strip with crease pattern on it

Though this method does work, it is not elegant. Shape of the material changes all the time. In the following paper, a slightly stronger quality constraint is added: the paper used in filling space is forced to have shape ratio of 1: $\sqrt{2}$. So we come to the question bellow:

Problem. Is there a way of A4 tetrahedron origami that tiles space?

In the main part of this paper, a so-called "basic form" is first defined by origami 1/3 of a sheet of A4 paper. After discussing several properties of the basic form, a cube-like unit of origami is constructed from 4 basic forms by melting the connected edges between them. Finally through cut-melt method, we prove that space can be filled by these units and they merge together into one huge tube at any stage. A4 shape remained when the tube is cut alone certain line.

Near the end of the paper, two special cases of the above model are showed to have applications in our real life.

1. Basic form

Suppose a sheet of A4 paper is divided into three equal parts along the height, one can construct two mirror-symmetric structures from two of these rectangular papers.

According to the crease patterns on figure 3 and figure 4, it is quite easy to make this origami just to remember the valley folds and mountain folds must be obeyed and that in the end of the origami edge 3 and 4 meet, edge c and d meet.

Definition 1. The above two constructions are called basic forms.

One can easily observe and confirm that any of the two basic forms contains one complete Sommerville tetrahedron and 4 semi Sommerville tetrahedra.

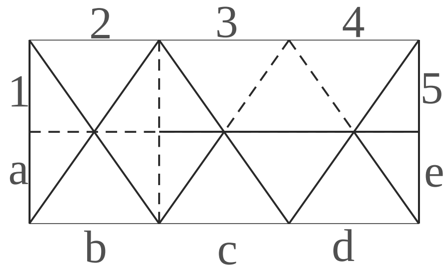

fig. 3 right hand basic form crease patern and the names of edges

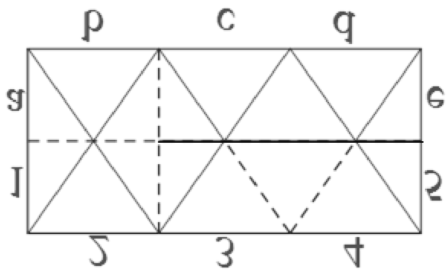

fig. 4 left hand basic form crease pattern (mirror of fig. 3)

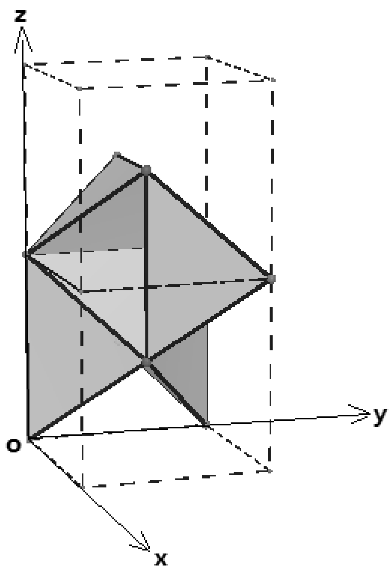

fig. 5 front view of the basic form

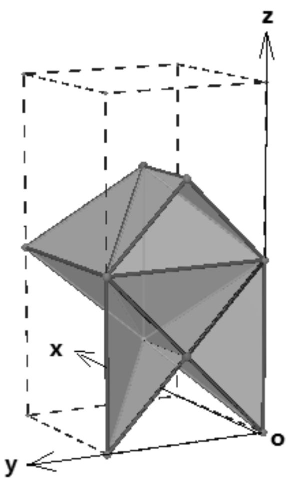

fig. 6 back view of the basic form

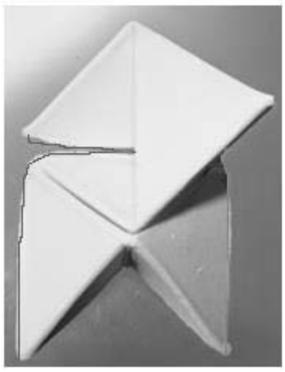
fig. 7 photo of a right hand basic form

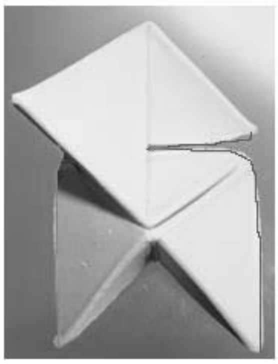
fig. 8 photo of a left hand basic form

After establishing a Descartes coordinate system as in figure 5, we call the side in YOZ plane the "back" side of the basic form. According to the direction of the front tetrahedron we call the basic form in fig. 5 a right hand basic form. One can parallel above discussion and name the left hand basic form according to fig. 4, fig. 6 and fig.8.

2. Growth modes of basic forms

In this section, the following two tasks are focused: 1. any basic form can fill space under reflection, rotation and translation; 2. the result of each step of above transformations can be realized by one sheet of paper.

Proposition **1.** Two different basic forms can join together along their two corresponding "Γ" edges and the combined structure remain developable. （ "developable" means it can be flatten without tearing or stretching.）

Proof. Comparing fig. 7 with the in fig. 3, we can find that the original 2-3-4 edge and b-c-d edge of the rectangle in fig. 3 become two curved Γ-like edge on the three-dimensional structure. That suffices to confirm the assertion.

Following illustrations show two different modes of this edge melting techniques.

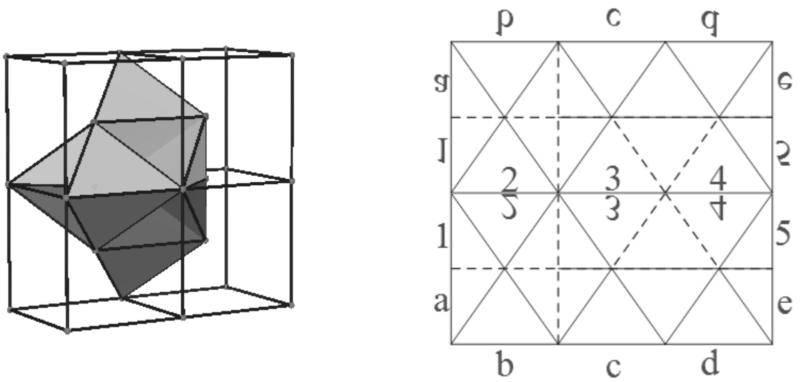

fig. 9 2-3-4 edge melt with 2-3-4 edge

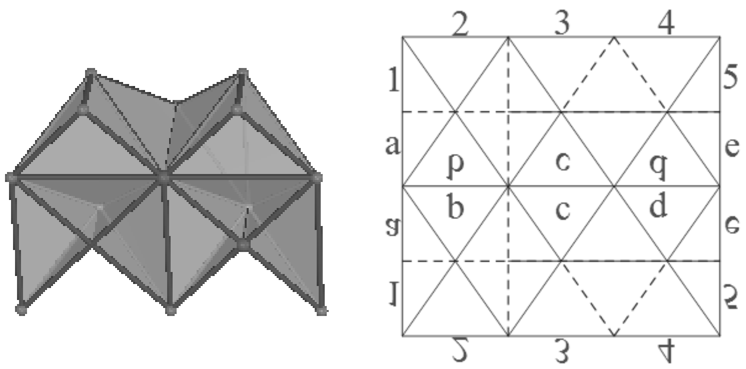

fig. 10 b-c-d edge melt with b-c-d edge

Definition 2. The mode of fig. 9 is called B-B mode (back to back), and mode of fig. 10 is called S-S mode (side by side).

Proposition 2. Two right (left) hand basic forms can join together along their different "Γ" edges and the combined structure remain developable.

Proof. This join procedure equals to melting the 2-3-4 edge of one basic form with b-c-d edge of another. See fig. 11.

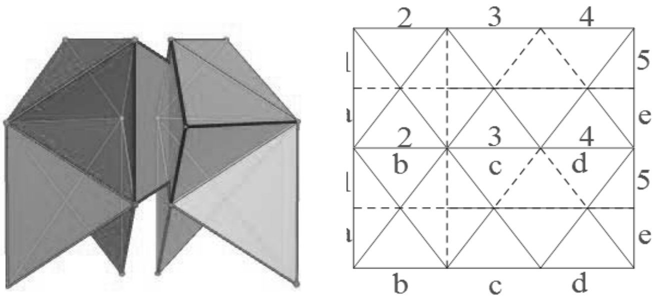

fig. 11 2-3-4 edge melt with b-c-d edge

This mode of growth in fig. 11 is called H-P mode (hand in pocket mode).

Corollary. Four right (left) hand basic forms can join together and the union is developable.

Definition 3 The structure in fig. 12 and its mirror are called *cube-like unit*.

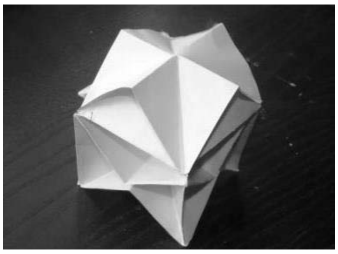

fig. 12 A cube-like unit (left hand)

Proposition 3. Two right (left) hand basic forms can join together along their corresponding short edges on top and bottom and the combined structure remains developable.

Proof. This join procedure equals to melting the 1-a edge of one basic form with 5-e edge of another. This suffices to draw the conclusion. See fig. 13.

This mode of growth is called elevation mode.

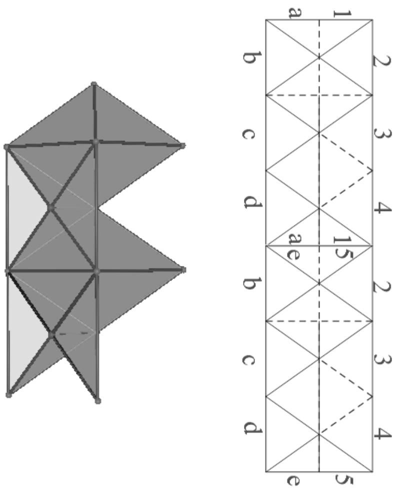

fig. 13 elevation mode

Theorem 1. *Cube-like units* can tile the three dimensional space under translation and reflection and the isolated units can unite as one developable sheet.

Proof. The first part of the theorem is evident. Notice that elevating processing can be done at any stage, it suffices to prove *Cube-like units* tile *XOY*-plane by reflection as one sheet.
In fig 14, two circles stand for a *Cube-like unit* and its mirror symmetry. In this process, both double layer "Γ" edges are cut and melted again in a different manner. {H-P, H-P} changes to {B-B, S-S}. It can be imagined a bigger circle appears.

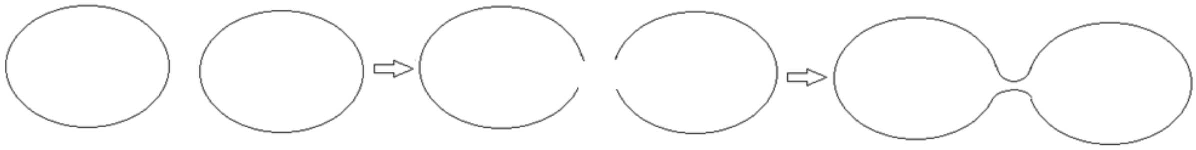

fig. 14 the theory of cut-melt strategy

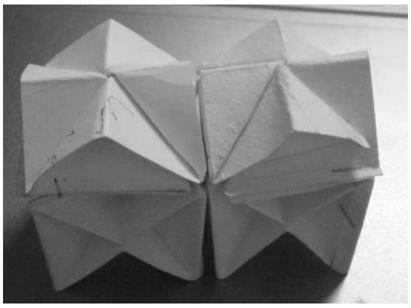

fig. 15 photo of real model

As to more cube units, they can combine together in the same way. The *XOY*-plane can be filled by the pattern shown in fig 16 and fig 17.

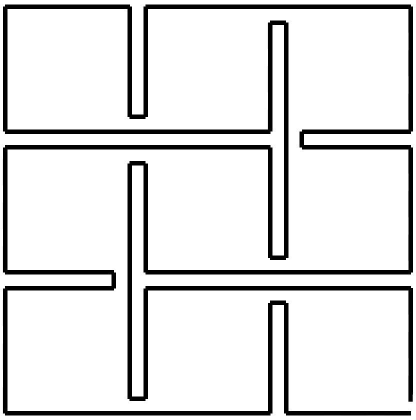

fig. 16 nine squares linked together

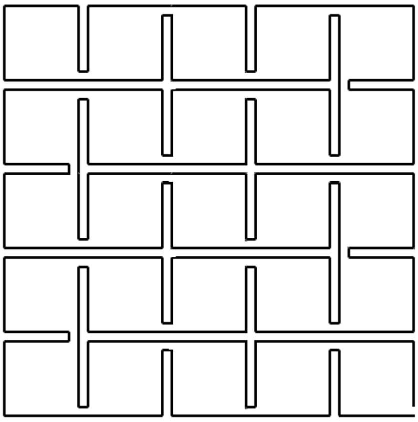

fig. 17 twenty five squares linked together

Theorem 2 The whole Euclidean space can be tiled by Sommerville tetrahedra made from one sheet of infinite A4-like paper.

Proof. Just by simple calculating, we know that to make one cube-like unit needs a rectangle of $4\sqrt{2} \times 3$. Now enlarge its length by 36 times and the height 48 times, this rectangle will become A4-like. And we know this large A4-like paper can make a $6\times 6 \times 48$ box. When subsequently apply reflection and translating from its sides and top respectively, the paper shape remains unchanged. The ever-enlarging box has length 6×2^m, width 6×2^n and height $48\times 2^{m+n}$.

3. Two other space-filling models by origami

Eminent mathematical artist M.C. Escher has a piece of art work named *Waterfall*. People seldom noticed in that paint there is a tower which can fill space. Actually, Sommerville tetrahedron can be arranged to make that tower. Even more surprisingly, one can also build such a tower by origami. The structure in fig.18 spent 6 pieces of A4-like paper.

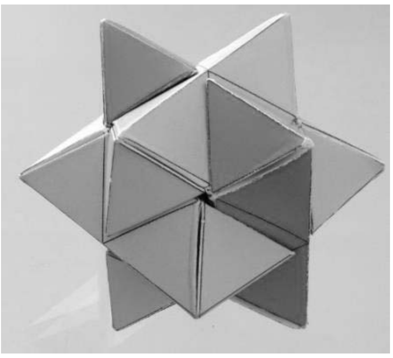

fig. 18 M.C. Escher' tower made by 6 equal parts

Kaleidocycles are three-dimensional rings made from chains of 2*n* tetrahedra attached at edges. The tetrahedra in a ordinarily Kaleidocycle are not that of Sommerville's and *n* is seldom lager than 4.

However, when we arrange 24 Sommerville-tetrahedra within a Kaleidocycle, it shows us amazing behaviors. The most conspicuous thing is that it can form a *rhombic dodecahedron*.

fig. 19 Kaleidocycle of 24 Sommerville tetrahedra

fig. 20 rhombic dodecahedron formed from a Kaleidocycle of 24 Sommerville tetrahedra

As we all know both Escher tower and rhombic dodecahedron can fill space, these two origami give us more direct thinking materials about space filling.

4. Conclusions

As has been discussed, through origami method one can build as much as he wants Sommerville tetrahedra from a sheet of A4 paper. And if the paper material be allowed to enlarge in two directions, the resulting tetrahedra can fill the whole three-dimensional space. Though it may mainly be of theoretically interest, it may have certain potential applications in material industry.

Acknowledgements

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