Exploring Metric Spaces Visually with *Excel*

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Abstract: *This paper provides a novel approach for an introductory exploration of metric spaces, showing ways to use Excel creatively to discover interesting insights into mathematics. We use diverse examples of metrics to provide insights into the concept of distance, and see how it interacts with the areas of geometry, calculus, and other more familiar parts of mathematics. Along the way, we encounter inequalities and other aspects of classical mathematics, and provide new problem solving and teaching techniques, and a means of discovering patterns by employing spreadsheets to enhance the visual interpretation of a thought-provoking part of mathematics.*

1. Introduction

In recent years many people have contributed to the development of a wide range of creative ways to employ spreadsheets effectively in the teaching of both pure and applied mathematics. A new opportunity for extending the scope of these endeavors arose this year in developing and teaching an on-line class in metric spaces for Divine Word University in Madang, Papua New Guinea.

This paper reflects some of those developments. Since many of the models described here also involve concepts from areas such as calculus, analysis, inequalities, geometry, and linear algebra, our examples not only provide students with tools for metric spaces, but also furnish them with new insights into other subjects in their mathematics and computing science curriculum.

Among the reasons for using *Microsoft Excel* are that it enables us to design mathematical models that closely follow standard ways of presenting and learning mathematics, it is readily available and students are already familiar with its operation, and it provides students with new skills in using a basic and highly valued tool of the workplace.

For this specific class, it was helpful to be able to create interesting, animated spreadsheet models of topics that could be used in live Internet presentations, and to largely avoid a traditional definition-theorem-proof format. The use of *Excel* helped in developing problem solving skills, especially in discovering patterns for the geometry of different metrics. It provided students with an accessible tool for examining various metrics, constructing geometric objects, and initiating new approaches and insights for their studying. During the class students could access *Excel* models and class notes through *Moodle*.

Our present presentation provides a brief overview through selected interactive examples.

2. Foundation Material

Definition: A metric space consists of a set X together with a distance function, $d : X \times X \to \mathbb{R}$, with the following three properties:

i) $d(a,b) \geq 0$ for all $a, b \in X$ with equality if and only if $a = b$
ii) $d(a,b) = d(b,a)$ for all $a, b \in X$
iii) $d(a,b) \leq d(a,c) + d(c,b)$ for all $a, b, c \in X$ (triangle inequality)

The most familiar examples of metric spaces are the real numbers ℝ with the distance function $d(a,b) = |b - a|$, the real Euclidean plane ℝ² with distance function $d\big((x_1,y_1),(x_2,y_2)\big) = \sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2}$, and $\mathbb{R}^n$ for positive integers n with a similar distance function.

However, there many more interesting metric spaces for students to explore in which they will encounter new mathematical concepts and techniques, and for educators to design useful visualizations for teaching. In this presentation we present a few of these that we can implement using a spreadsheet, such as *Microsoft Excel*. In this paper we do not prove that the metrics listed satisfy the conditions for a metric, but refer readers to other sources [7], [8] for these results.

3. Examples of Metrics Spaces

3.1. Euclidean ℝ² Metric: There are many ways to use *Excel* to create Euclidean metric objects in ℝ². For example, in Figure 3.1a we have created an xy-graph of a circle by using the parametric equations $x = x_0 + r\cos t,\ y = y_0 + r\sin t,\ 0 \le t \le 2\pi$, thus locating the set of points whose distance from the point (x_0, y_0) is r. Although *Excel* does not provide a direct way to fill in the interior of a closed curve, we can do this by drawing a series of line segments, as indicated in Figure 3.1b. To fill in the area, we increase the number of segments and the width of the lines. By adjusting the line style for the boundary, in Figure 3.1c we produce the open unit disk (or ball) centered at $(a,b) = (1.4,1.2)$. An outline of the *Excel* worksheet and its equations are shown in Figure 3.2. A fuller description of the methods used can be found in [6].

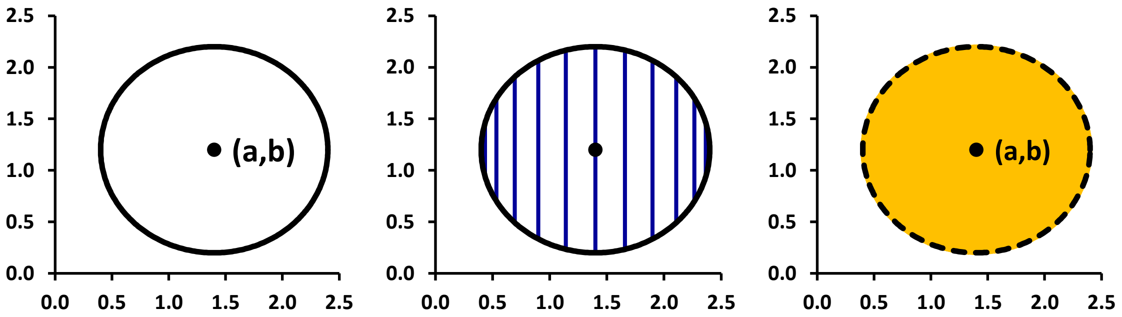


Figure 3.1 Unit Circle and Open Unit Disk in ℝ².

	A	B	C	D	E	F	G	H	I
3		radius:	1	1					
4		center:	1.4	1.2					
5			circle			interior fill		center	
6	deg	t rad	x	y	n	x	y	x	y
7	0	0	2.4	1.2	0	2.4	1.2	1.4	1.2
8	1	0.017	2.4	1.217	360	2.4	1.2		
9	2	0.035	2.399	1.235					
10	3	0.052	2.399	1.252	1	2.4	1.217		
11	4	0.07	2.398	1.27	359	2.4	1.183		
12	5	0.087	2.396	1.287					

	A	B	C	D
3		radius:	1	=C3
4		center:	1.4	1.2
5			circle	
6	deg	t rad	x	y
7	0	=RADIANS(A7)	=C$3*COS($B7)+C$4	=D$3*SIN($B7)+D$4
8	=1+A7	=RADIANS(A8)	=C$3*COS($B8)+C$4	=D$3*SIN($B8)+D$4
9	=1+A8	=RADIANS(A9)	=C$3*COS($B9)+C$4	=D$3*SIN($B9)+D$4
10	=1+A9	=RADIANS(A10)	=C$3*COS($B10)+C$4	=D$3*SIN($B10)+D$4
11	=1+A10	=RADIANS(A11)	=C$3*COS($B11)+C$4	=D$3*SIN($B11)+D$4
12	=1+A11	=RADIANS(A12)	=C$3*COS($B12)+C$4	=D$3*SIN($B12)+D$4

Figure 3.2 *Excel* Output and Formulas for Unit Circle and Disk

Suppose that we wish to find all points in $\mathbb{R}^2$ that are equidistant from two given points. In this example we first use algebra to find the points of intersection of the circles of radius r whose centers are the two given points. We then use those points together with our techniques for generating circles to form the diagram of Figure 3.3. If we then link the cell containing the radius to a scroll bar, we animate the creation process for drawing the perpendicular bisector of the segment between the two points as we increase *r*. We will use this idea for a similar construction with another metric. Additional related examples can be found in [2], [6].

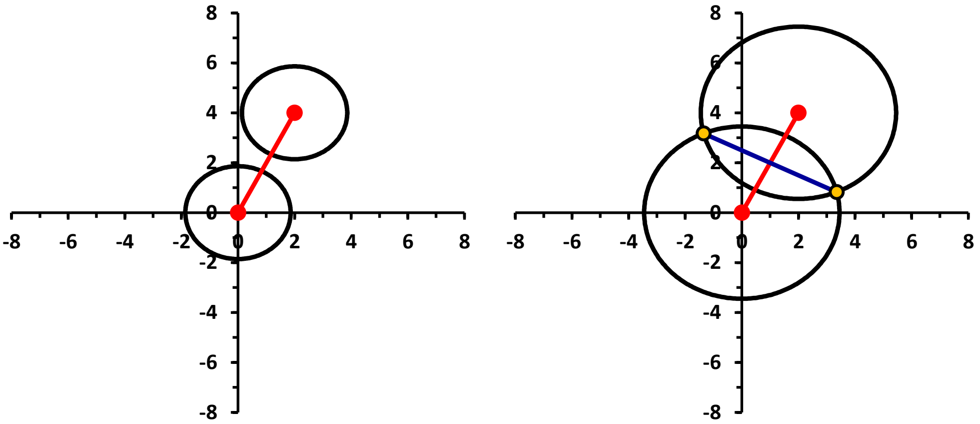


Figure 3.3 Locus of Points Equidistant from Two Given Points (Euclidean)

3.2. Taxicab Metric: Let *X* be the real plane, $\mathbb{R}^2$, but now use the taxicab distance defined by $d\big((x_1, y_1), (x_2, y_2)\big) = |x_2 - x_1| + |y_2 - y_1|$. The name taxicab comes from the fact that this measure of distance is analogous to that used for taxicabs as they traverse the streets of a city having a rectangular street grid, where one can travel only in horizontal or vertical directions. Thus, in Figure 3.4a the distance between the locations $a = (-3,2)$ and $b = (4,-4)$ is $d(a,b) = |4-(-3)| + |-4-2| = 13$. In Figure 3.4a we also see that a trip of that length can be made in a variety of ways, with the intermediate corners interpreted as points lying between a and b. However, in this metric space we are not limited to traveling on lines with integer coordinates. Thus, in Figure 3.4b we see that the distance between $a = (-2.5,3.3)$ and $b = (4.3,-2.6)$ is $d(a,b) = 12.7$.

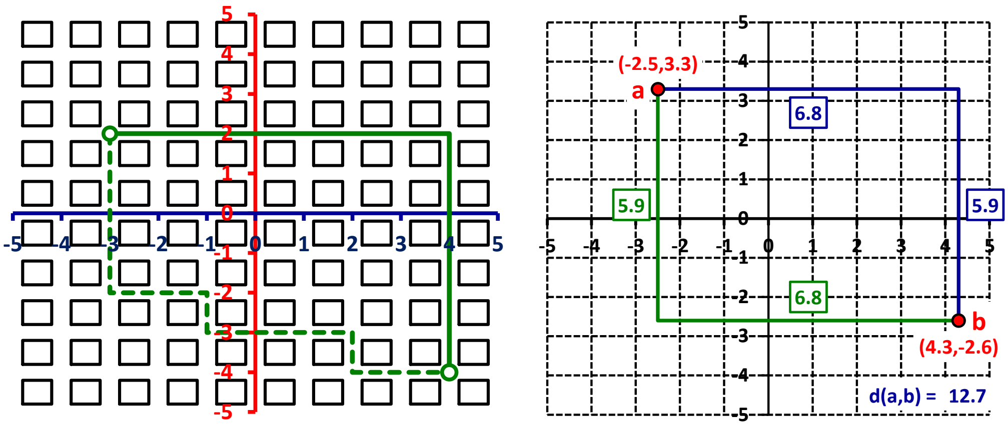
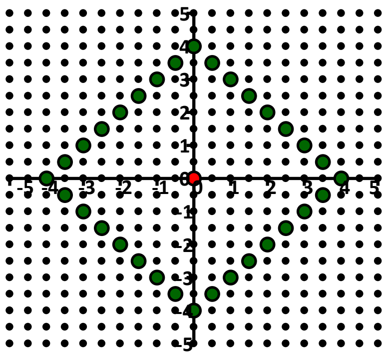


Figure 3.4. Taxicab Distance

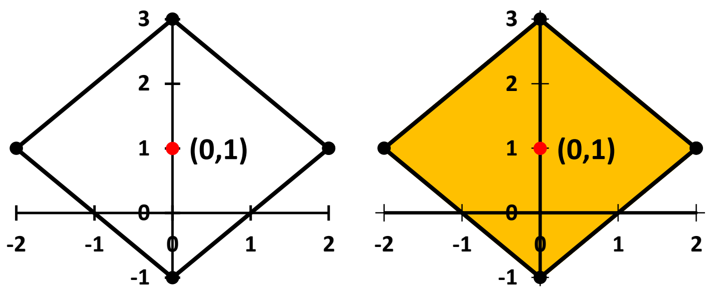
In working with this metric by hand, it is convenient to use graph paper. A spreadsheet provides us with a similar medium through which we can gain insights before creating more advanced illustrations. In Figure 3.5a we show a xy-scatter chart produced from the *Excel* layout in Figure 3.5b to find those points that are distance $r = 4$ from point $(a, b) = (0,0)$. Columns C:D are used to generate a grid with gaps of size 0.5. We start by first entering C4 = D4 = –5, and then the formulas C5: =IF(D4=5,0.5+C4,C4), D5: =IF(D4=5,–5,0.5+D4), and afterwards filling down the two columns. In Column E we compute the distance from each point to the center, then use an =IF function in Columns F:G to reproduce only those (x, y) values from Columns C:D whose distance from (a, b) is r, otherwise it returns NA(). We insert the values in Columns F:G into the chart, showing the points as green markers. We can use a similar technique to generate other structures such as bisectors and other conic sections. With our model we vary the values of the center and radius to see the resulting output and to discover patterns.



	A	B	C	D	E	F	G
1		**distance**	**4**				
2			grid points			ball outline	
3	x	y	x	y	dist	x	y
4	0	0	-5	-5	10	#N/A	#N/A
5			-5	-4.5	9.5	#N/A	#N/A
6			-5	-4	9	#N/A	#N/A
7			-5	-3.5	8.5	#N/A	#N/A
76			-3.5	-0.5	4	-3.5	-0.5
77			-3.5	0	3.5	#N/A	#N/A
78			-3.5	0.5	4	-3.5	0.5
79			-3.5	1	4.5	#N/A	#N/A

Figure 3.5. Taxicab Circle Special Layout

Using the observations with above, it is straight-forward to see one way to create taxicab circles for radius r. We simply plot the four points that are located r units horizontally and vertically from the center and connect them with line segments (Figure 3.6). We can also verify our findings algebraically if desired. We fill in the interior to produce a closed ball of radius 2 using a variation of the technique addressed previously.



Figures 3.6 Closed Ball in Taxicab Metric

Finding all points that are equidistant from two given points is more difficult to do in the taxicab metric. A good approach is first to examine the output a grid model. We see what may be surprising output in Figure 3.7a. By studying patterns and realizing that the location of the two points influences the resulting outcome, we can produce an animated version as seen in Figures 3.7b.

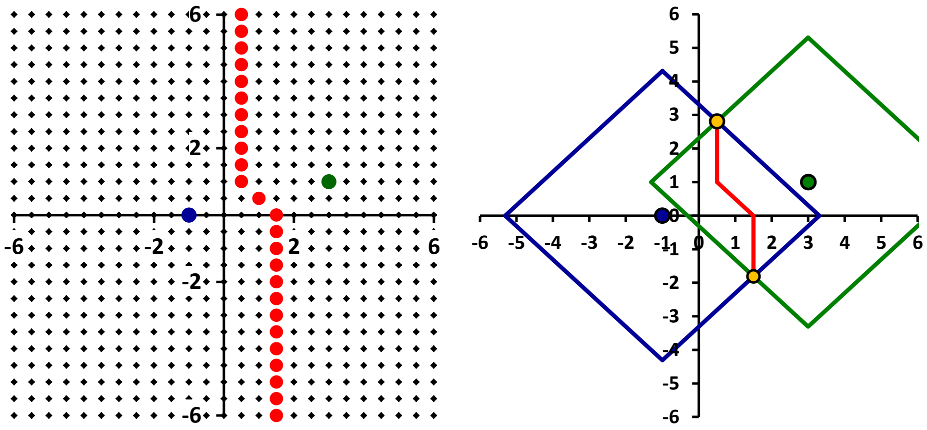


Figure 3.7 Locus of Points Equidistant from Two Given Points (Taxicab)

From our grid model we will find a bigger surprise. Sometimes, there is a whole region of points that are equidistant from the two points, as in Figure 3.8a. After further investigations, we can discover that this happens when the horizontal and vertical distances between the points are equal. We can then use our discoveries to create a more sophisticated animated graph such as in Figure 3.8b. This example is enhanced by using a scroll bar to vary *r*. Consult [5] to find a wealth of other interesting taxicab aspects to pursue. Additional taxicab implementations in *Excel* have been investigated by two DWU colleagues, Maryanne Bagore and Jeffrey Ambelye.

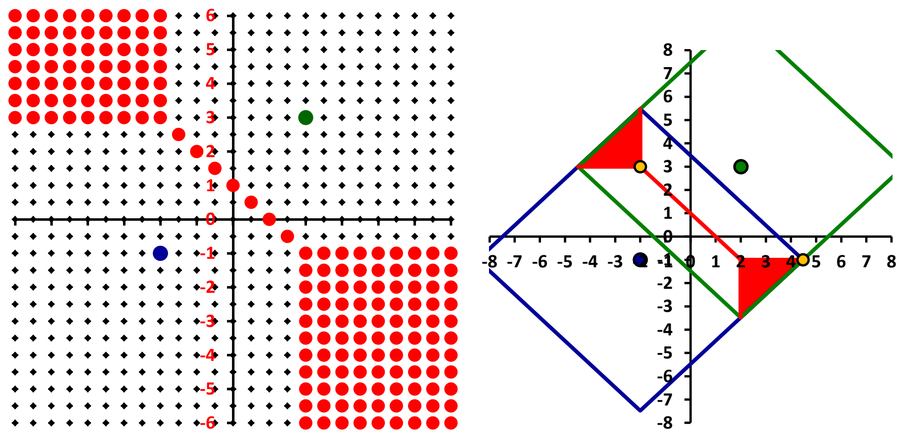


Figure 3.8 Special Case Equidistant Points

3.3. Maximal Metric: For the maximal metric space we again have $X = \mathbb{R}^2$, but the distance now is given by the expression $d\big((x_1, y_1), (x_2, y_2)\big) = \max(|x_2 - x_1|, |y_2 - y_1|)$. We can define a similar metric on $\mathbb{R}^n$ as follows. Let $a = (x_1, x_2, \cdots, x_n)$, $b = (y_1, y_2, \cdots, y_n)$, then $d(a, b) = \max_{i=1,\cdots,n}|x_i - y_i|$. We notice (Figure 3.9) that balls in $\mathbb{R}^2$ are squares with sides parallel to the axes, while in $\mathbb{R}^3$ they are cubes, and in $\mathbb{R}^n$ they are hyper-cubes.

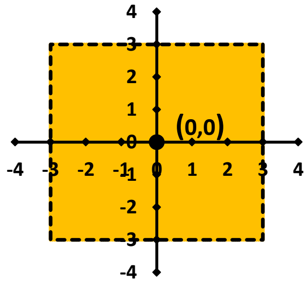


Figure 3.9 Open Ball in Maximal Metric

3.4. l_p Metrics: In $X = \mathbb{R}^2$, for any real number $p \geq 1$, we define the l_p metric by $d\big((x_1, y_1), (x_2, y_2)\big) = (|x_1 - x_2|^p + |y_1 - y_2|^p)^{1/p}$. With $p = 2$, we get the Euclidean metric, while with $p = 1$ we have the taxicab metric. We can define a similar l_n^p metric in $\mathbb{R}^n$ by $d\big((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)\big) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p}$. As we see in Figure 3.10, with *Excel* we can create a ball in $\mathbb{R}^2$ centered at (x_0, y_0) with radius r much as we did for the Euclidean metric, via the parametric equations

$$x = x_0 + r|\cos t|^{2/p}\text{sign}(\cos t),\ y = y_0 + r|\sin t|^{2/p}\text{sign}(\sin t),$$

If we link the cell for the parameter p to a scroll bar in our model, then as we increase p we see what are sometimes called “super ellipses”. We can prove that for $p \geq 1$ our definition satisfies the requirements for a metric. However, for $0 < p < 1$ this is not the case, since the triangle inequality fails (Figure 3.10b). Nonetheless, the curves produced for $0 < p < 1$ have geometric significance. For example, the curve for $p = 2/3$ is an astroid [2]. Figure 3.10c shows unit circles for $p = 1, 2, 4, 1000$. As $p \to \infty$, the resulting metric approaches the l_∞ metric, which is the same as the maximal metric. Members of the l_p family of metrics are used in the field of statistics.

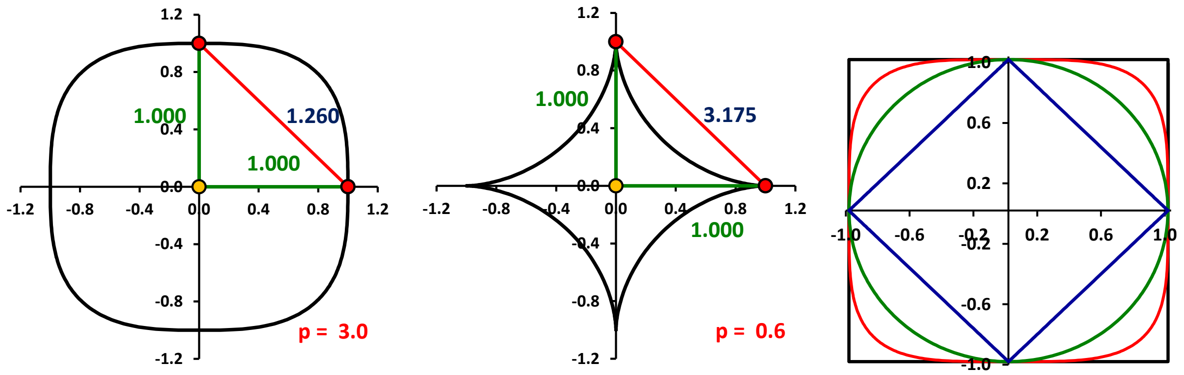


Figure 3.10 Unit Circles in l_p

3.5. British Rail Metric: For this metric on $X = \mathbb{R}^2$, the distance between two points $a = (x_1, y_1)$ and $b = (x_2, y_2)$ is defined by using the Euclidean distance needed to go first from a to the origin and then to b. Thus,

$$d(a, b) = (x_1^2 + y_1^2)^{1/2} + (x_2^2 + y_2^2)^{1/2}$$

Its name is based on the British rail system, where routes usually require traveling via London. In fact, most internal travel in Papua New Guinea on Air Niugini similarly requires travel via Port Moresby. In Figure 3.11, both the airline's route map and its bird of paradise logo in were created in *Excel*, the latter using Euclidean circles almost exclusively.

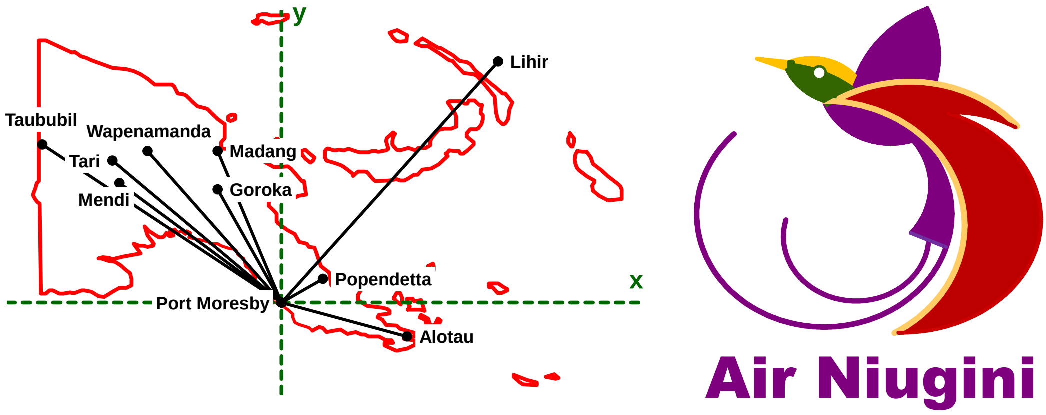


Figure 3.11 Air Niugini Flight Routes

3.6. Great Circle Metric: Here we let X be the points in $\mathbb{R}^3$ on a sphere of radius ρ. The distance between points $a = (x_1, y_1, z_1)$ and $b = (x_2, y_2, z_2)$ is given by the shorter arc length of the great circle that passes through those points and the origin $(0,0,0)$. Using the dot product and related results from calculus or linear algebra, the angle (in radians) between the radii on the great circle to those points is given by $\theta = \cos^{-1}(\vec{a} \cdot \vec{b})/(\|\vec{a}\|\|\vec{b}\|)$, where $\|\vec{v}\|$ denotes the length of a vector and $\cdot$ is the dot product. *Excel*'s =SUMPRODUCT function gives the dot product. Thus, $d(a, b) = \rho\theta$.

We have designed our illustrations in Figure 3.12 as static displays. The first shows distances for coordinates with rectangular coordinates, while the second uses longitude and latitude. From calculus, the relationship between spherical and rectangular coordinate systems is given by $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$, $\rho^2 = x^2 + y^2 + z^2$.

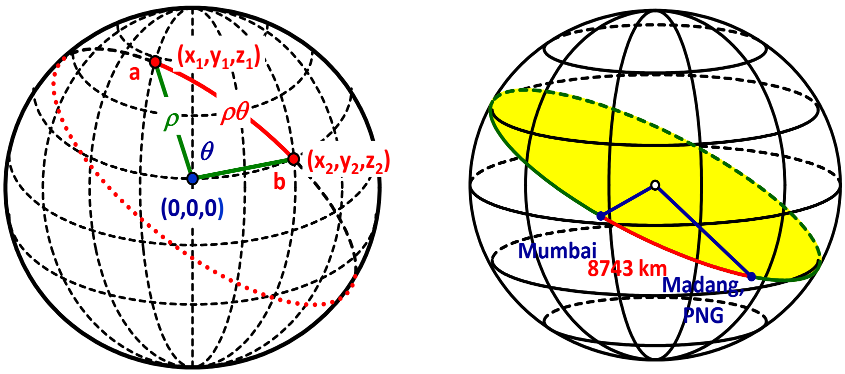


Figure 3.12 Spherical Distances

3.7. Continuous Real Functions – Distance Metric: Let $X = \mathrm{C}[a, b]$ be the set of real-values continuous functions on the interval $a \le x \le b$ with metric $d(f_1, f_2) = \max_{x\in[a,b]}|f_2(x) - f_1(x)|$. With this definition of distance, it can be shown that X is a metric space. The definition says that the distance between two functions is determined at the point, x_0, where the two functions are the farthest apart. This is illustrated by the graph in Figure 3.13b. We use the model of Figure 3.13a for functions in $\mathrm{C}[0,1]$ to find the distance between $f_1(x) = x$ and $f_2(x) = x^n$, where n is a parameter (Cell C3) of the model.

There are at least three ways in which we can either determine or closely estimate this distance between two functions: (a) We can use calculus to compute the derivative of $f_1 - f_2$ to determine the maximum value of $|f_1(x) - f_2(x)|$. We enter the calculated value of x_0 in Cell E3, computing the distance between the functions in Cell E4. (b) We can enter an estimate for the point x_0 at which the maximum distance occurs (Cell E3), enter the distance expression in Cell E4, and then use *Excel*'s solver tool to find the maximum of E4 by varying E3. (c) We can use Column D to compute $|f_1(x) - f_2(x)|$ for each x, and use *Excel*'s =MAX function to find the greatest in Column D. Column E lets us find the corresponding point, x_0. Of course, (c) only gives an approximation, since the actual x_0, as the maximum will seldom be exactly one of the values in Column D.

	A	B	C	D	E
1		$f_1(x) = x$			
2		$f_2(x) = x^n$		n > 1	
3		n =	3	x_{max}	0.58
4				$d(f_1,f_2)$	0.385
5		functions			
6	x	$f_1(x)$	$f_2(x)$	$\lvert f_1{-}f_2\rvert$	x_{max}
7	0.00	0.00	0.00	0	-999
8	0.01	0.01	0.00	0.01	-999
9	0.02	0.02	0.00	0.02	-999
10	0.03	0.03	0.00	0.03	-999
11	0.04	0.04	0.00	0.04	-999

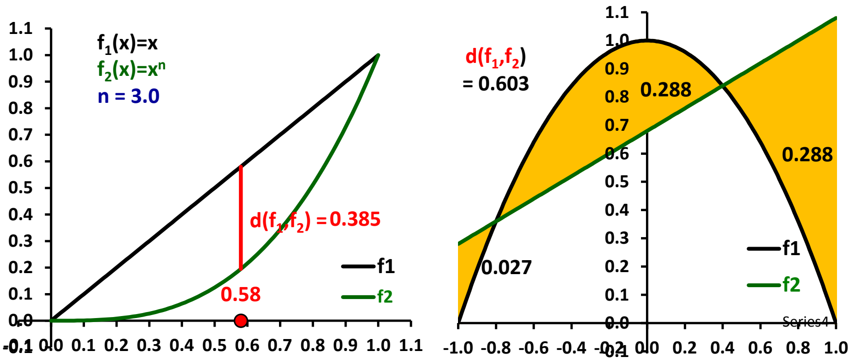


Figure 3.13 Metrics using the Maximum Distance and the Area between Functions

3.8. Continuous Real Functions – Area Metric: Let $x = \mathrm{C}[a, b]$ be the space of continuous functions over the interval $a \le x \le b$, and $d(f_1, f_2) = \int_a^b |f_2(x) - f_1(x)|\,dx$. Thus, the distance between functions f_1 and f_2 is the net area between their graphs. We find this using calculus, although we could enter formulas for the areas of certain functions in advance. In the illustration of Figure 3.13c we have $f_1(x) = 1 - x^2$ and $f_2(x) = 0.4x - 0.68$ as functions in $\mathrm{C}[-1,1]$. Because the curves intersect, we need to compute the integral in segments. We factor $f_2(x) - f_1(x) = x^2 + 0.4x - 0.32 = (x + 0.8)(x - 0.4)$, and evaluate the following integrals to find the area.

$$d(f_1, f_2) = \int_{-1.0}^{-0.8} \big(f_2(x) - f_1(x)\big)\,dx + \int_{-0.8}^{0.4} \big(f_1(x) - f_2(x)\big)\,dx + \int_{0.4}^{1.0} \big(f_2(x) - f_1(x)\big)\,dx \approx 0.60$$

4. Topics from the Study of Metric Spaces

Besides using *Excel* to illustrate and find distances for metric spaces, we can use its models to discover patterns, illustrate topology and geometry properties, and understand the ideas behind proofs of such topics as the continuity of functions, the convergence of sequences, and others through creative spreadsheet techniques.

4.1. Open and Closed Sets: In a metric space X, a set S is open if for each $x_0 \in S$ we can find an open ball $S(x_0, \varepsilon)$ that centered at x_0 so that $S(x_0, \varepsilon) \subseteq S$. In Euclidean $\mathbb{R}$ space, using scroll bars to vary x_0 and ε, in Figure 4.1 we illustrate the definition by showing that the open interval $(0,1)$ is an open set, while $(0, 1]$ is not.

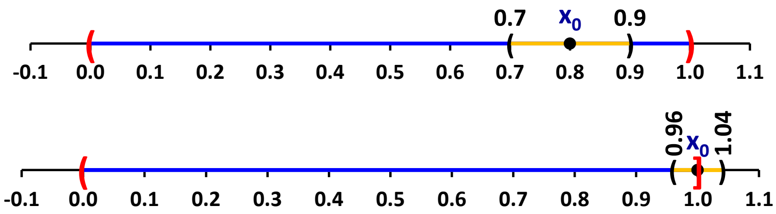


Figure 4.1 Open and Non-open Sets in $\mathbb{R}$

The diagrams in Figure 4.2 come from a similar model for $\mathbb{R}^2$ in which we can vary the metric used in the illustration by clicking a spin button, and vary the radius of an open ball with a scroll bar. We see that the physical size of these balls of the same radius vary with the metric that is used.

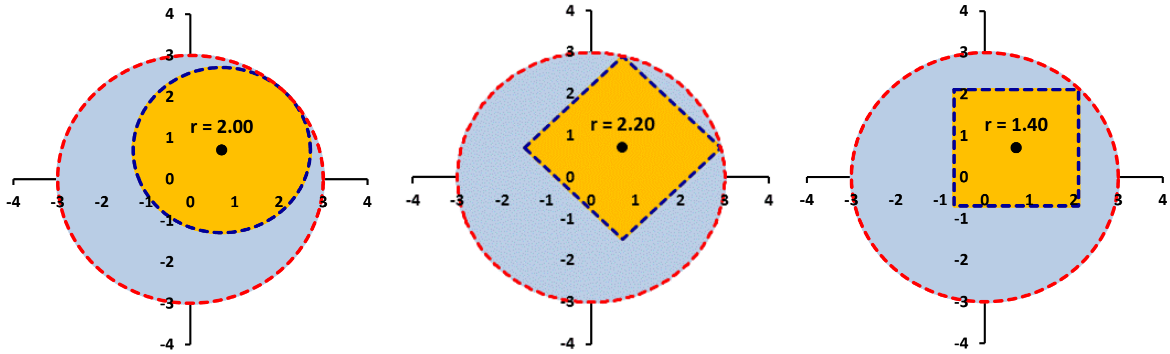


Figure 4.2 Inserting an Open Ball in an Open Set for Different Metrics

4.2. Visualizing Proofs and Definitions: In a class that involves proving results, many students initially have difficulties in learning how to carry out, or even to understand, proofs. In a standard class room instructors typically help students to visualize these aspects by drawing images on a board. Although it takes more time to create similar images on *Excel*, doing so can allow us to create animated drawings that may provide additional insights into these aspects. In Figure 4.3a we see a drawing illustrating a step in the proof that the intersection of open sets is open, while Figure 4.3b illustrates showing how to create an open ball centered at a given point so that it fits inside another open ball.

In Figure 4.4 we illustrate the definition of the diameter of an arbitrary set (here using Euclidean metric), by choosing two arbitrary points on the boundary of a set created by the polar equation $r = \cos^3 x + \sin^3 x$, and then using *Excel*'s solver to maximize the distance between two points, while including constraints that insure points lie on the boundary.

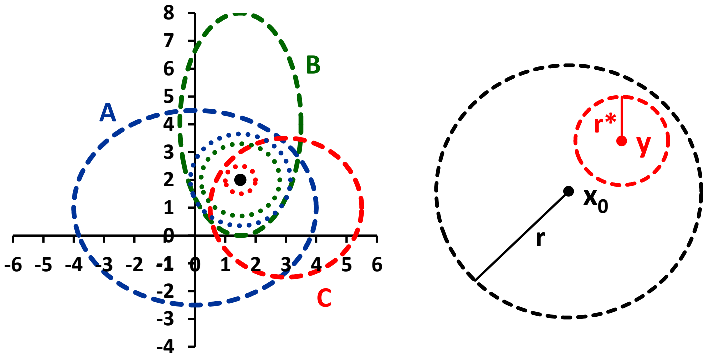


Figure 4.3 Illustrating Proofs About Open Sets

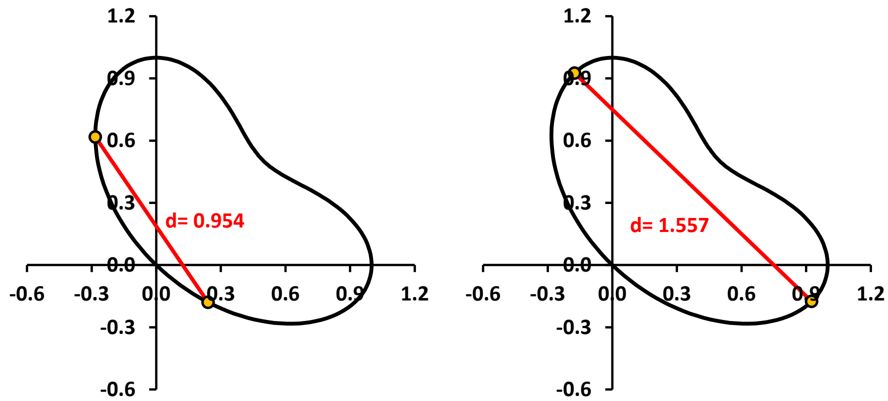


Figure 4.4 Visualizing the Diameter of a Set (before and after Solver)

4.3. Limits, Continuity, and Convergence: One of the basic concepts in an analysis course is proving results involving limits and continuity. In Figure 4.5 we see the output of an interactive model for illustrating that the function $f: \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = (x^2 - y^2, xy)$ is continuous at the point $a = (0.65, 0.50)$.

In Figure 4.5a, we generate grid points throughout the square region, $-1 \leq x \leq 1, -1 \leq y \leq 1$, and produce their images in Figure 4.5b. Next, we plot the points a and $f(a)$ in the respective graphs, and for an $\varepsilon > 0$ produce a green ball of radius ε around $f(b)$ in the right graph. Next we choose a $\delta > 0$, form a red/yellow ball of radius δ around point a in the left graph, and then generate the red/yellow image of that set in the right graph. Finally, we attach a scroll bar to the cell that contains δ, and use it to adjust the value of δ until the image set is contained within the green set.

We next display two versions of traditional ways to visualize a δ-ε proof of the continuity of a function $f: \mathbb{R} \to \mathbb{R}$ at the point $x = 2$, where $f(x) = x^2$. In Figure 4.6, using the concept of limit, given an $\varepsilon > 0$, we see a blue open interval of radius ε about $f(a)$. We then select a $\delta > 0$ to produce a red open interval about a, using a scroll bar to vary δ so that the image of the red interval lies within the blue ε-interval.

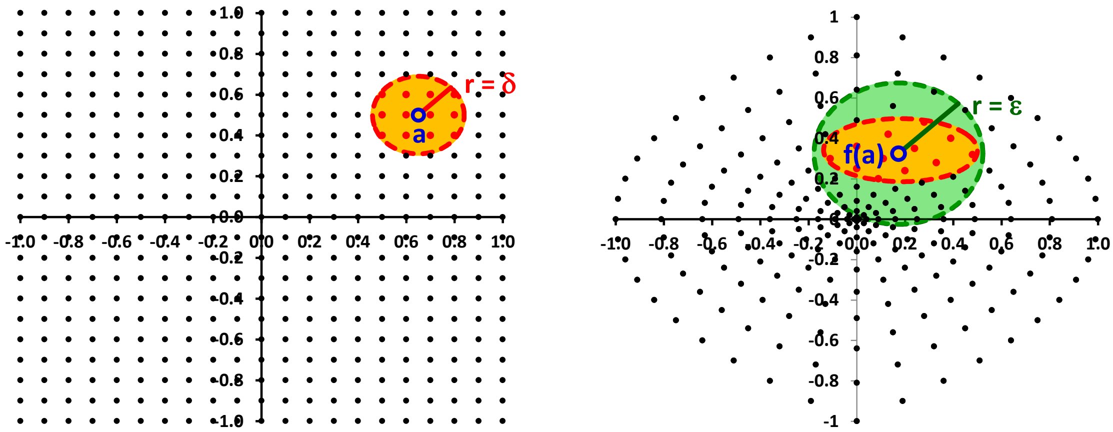


Figure 4.5 Interactive Illustration the Concepts of a δ-ε Proof in $\mathbb{R}^2$

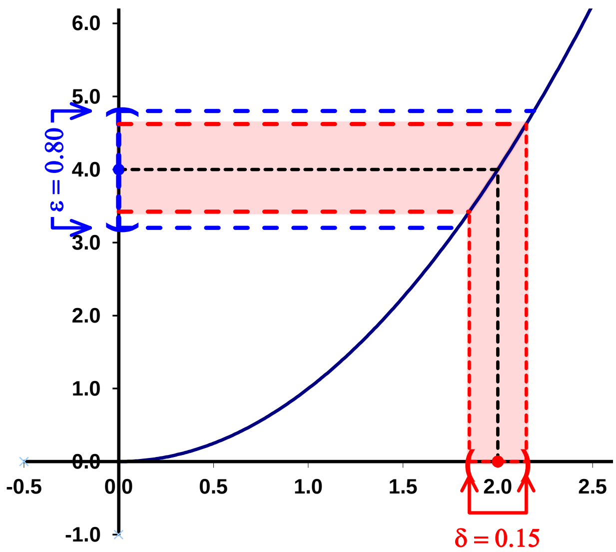


Figure 4.6 Interactive Illustration the Concepts of a Traditional δ-ε Proof in $\mathbb{R}$

In Figure 4.7 a second version of the previous example consists of two intervals, one for the domain and the other for the range. It is a one-dimensional version of the approach that we used in our earlier two-dimensional model. Again we vary the ε- and δ-intervals with scroll bars.

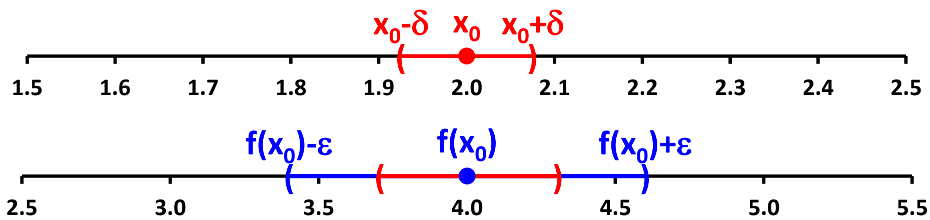


Figure 4.7 Alternate Illustration of a δ-ε Proof in $\mathbb{R}$

We now switch to considering a sequence of continuous functions, $\{f_n\}$, where $f_n(x) = \sin^n(x)$, $0 \le x \le \pi$. As we vary the value of n we see that although $f_n(x) \to 0$ for each $x \ne \pi/2$, the distance $d(f_n, 0)$, where 0 is the function $g(x) = 0$, remains 1. This is an example of a sequence of functions that does not converge uniformly over an interval, *i.e.* $d(f_n, 0)$ does not converge to 0. As in this example, such families may not converge to a continuous function. We could next compare this example to the sequence $\{g_n\}$, with $g_n(x) = \sin t/n,$. Here we would see that $d(g_n, 0) = 1/n \to 0$.

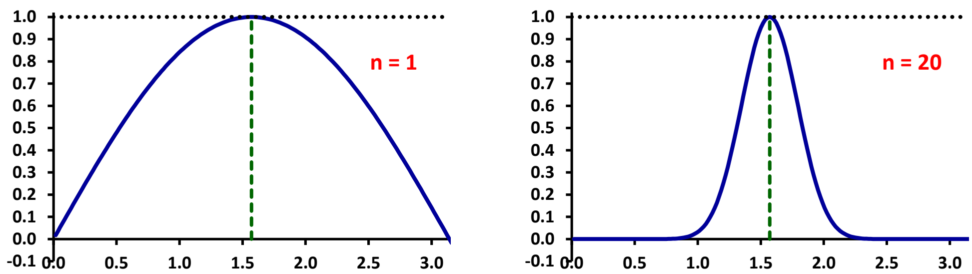


Figure 4.8 Using a Metric to Explore Uniform Convergence

5. Classical Inequalities

By its nature, the study of metric spaces involves using a variety of classical inequalities, especially in verifying the triangle inequality for distances. There are a number of resources available to both the proofs and visualization of inequalities [1], [7], [8]. Here we present two.

First, we show a geometric inspired proof for the 2-variable form of the following inequality:

$$\left(\left(\sum x_i\right)^2 + \left(\sum y_i\right)^2\right)^{1/2} \le \sum (x_i^2 + y_i^2)^{1/2}$$

Consider the triangle of Figure 5.1. The left side of the inequality is equal to the hypotenuse, while the right side of the inequality is the sum of the lengths of the red line segments.

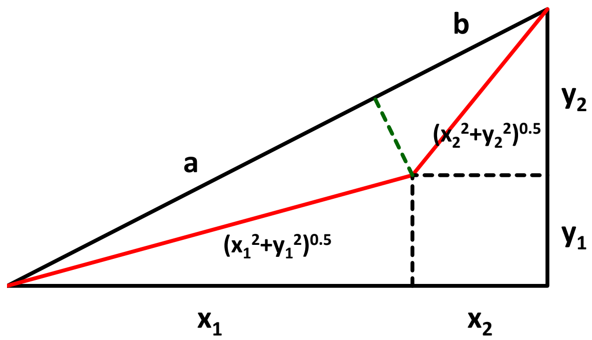


Figure 5.1 Geometric Interpretation of an Inequality

Next we look at Hölder's Inequality. As a preliminary step, we illustrate Young's Inequality [9]: Let f and g be strictly increasing functions on $[0, \infty)$, with $g = f^{-1}$, and $f(0) = g(0) = 0$. Then for $a, b \ge 0$,

$$ab \le \int_0^a f(x)\,dx + \int_0^b g(y)\,dy$$

In the Figure 5.2, the curve represents $y = f(x)$ relative to the x-axis, and $x = f^{-1}(y)$ relative to the y-axis. The yellow area gives the first integral, and the purple area is the second integral. In both cases shown, the area of the rectangle, ab, does not exceed the sum of the areas under the curves. We have used the functions $f(x) = x^2$ and $g(y) = y^{1/2}$ in the illustration.

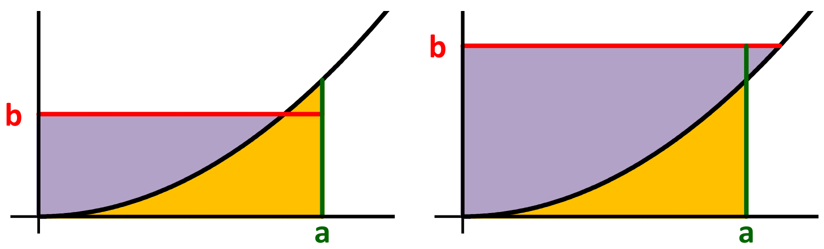


Figure 5.2 Young's Inequality

Then, if $p > 1$, let $f(x) = x^{p-1}$ and $g(y) = y^{1/(p-1)}$. Next, using Young's inequality with $q = p/(p-1)$, we have $(1/p) + (1/q) = 1$ and

$$ab \le \int_0^a x^{p-1}\,dx + \int_0^b y^{1/p-1}\,dy = \frac{a^p}{p} + \frac{p-1}{p} b^{p/(p-1)} = \frac{a^p}{p} + \frac{b^q}{q}$$

with equality when $a^p = b^q$.

Now let

$$A = \left(\sum_{i=1}^{n} |a_i|^p\right)^{1/p} \text{ and } B = \left(\sum_{i=1}^{n} |b_i|^q\right)^{1/q}$$

With $a = |a_i|/A$, $b = |b_i|/B$ and $(1/p) + (1/q) = 1, p > 1, q > 1,$

$$\frac{|a_i|}{A}\frac{|b_i|}{B} \le \frac{(|a_i|/A)^p}{p} + \frac{(|b_i|/B)^q}{q}$$

so that summing these terms

$$\frac{1}{AB}\sum_{i=1}^{n} |a_i b_i| \le \frac{1}{pA^p}\sum_{i=1}^{n} |a_i|^p + \frac{1}{qB^q}\sum_{i=1}^{n} |b_i|^q = \frac{1}{p} + \frac{1}{q} = 1$$

Multiplying by AB then gives Hölder's inequality

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q}$$

6. Some Related Projects

There is a large range of other metrics to examine [4]. For projects, readers can consult the references for both metric spaces and other topics. Also, books on operations research, calculus, and geometry can provide other distance-related topics. We indicate some of these examples below.

First, the diagram of Figure 6.1a illustrates an algorithm to find the minimum distance from an initial node, *S*, to additional nodes in the tree. One can design an interactive diagram, either with user-entered selections at each stage of an algorithm, or completely computed by the spreadsheet.

Figure 6.1b shows an animated *Excel* construction of drawing an Euclidean circle by using a compass. Similar constructions for a variety of physical devices can be created.

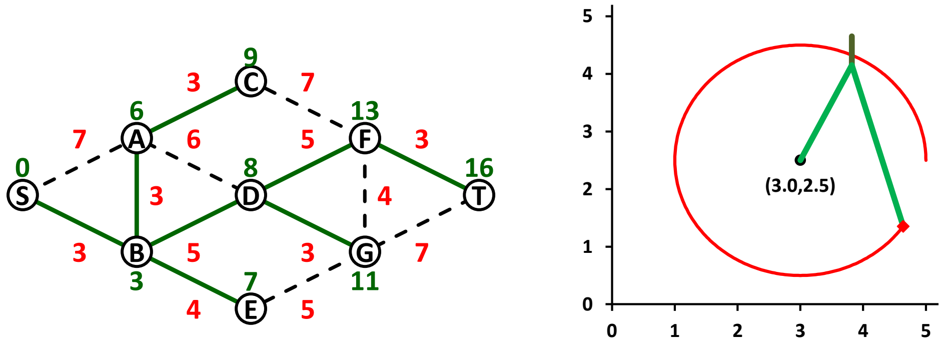


Figure 6.1 A Minimal Distance Algorithm and a Geometric Construction

We can also draw an extensive variety of curves from polar and parametric equations. Figure 6.2a shows the curve $r = \cos(t/5)$. Techniques for creating curves are discussed in [2], [6]. Those approaches are further exploited in pictorial alphabet books for the three national languages of Papua New Guinea [3]. Figure 6.2b shows a carrot whose parametric equations are $x = 0.3 \sin t + 0.1 \sin 2t, y = \cos t - 0.2 \cos 2t$. We also create an image of the flag of India in Figure 6.2c.

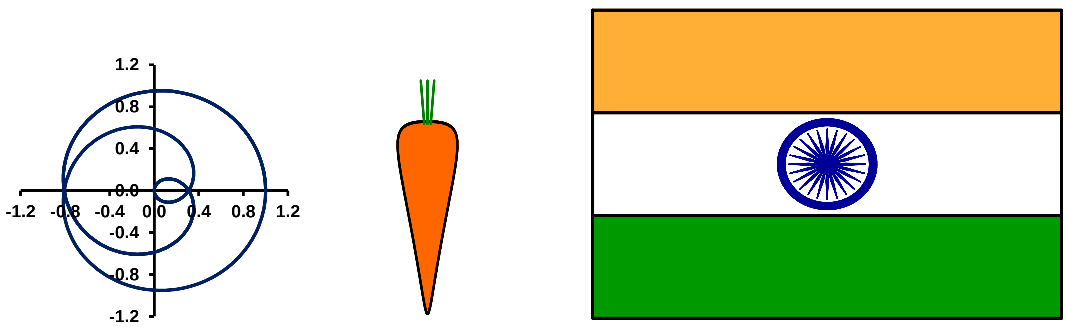


Figure 6.2 Additional Mathematics Creations using *Excel*

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