# Focusing Learning on Concepts of Introductory Linear Algebra using *MAPLE* Interactive Tutors

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**Abstract**: *This paper describes the author's experiences on the application of MAPLE's built-in interactive tutors to explore and reinforce fundamental concepts in an introductory Linear Algebra course with students at Zayed University in the United Arab Emirates. Students are allowed to work interactively step-by-step through standard problems, and become engaged in their learning without getting caught up in the arithmetic. The basics of the functionality of interactive tutors for Gaussian Elimination, Gauss-Jordan Elimination, Matrix Inverse, Eigenvalues and Eigenvectors are discussed.*

## 1. Introduction

At Zayed University, an introductory course in Linear Algebra is offered as one of the component courses of a Minor in Applied Mathematics taken by students majoring in a variety of disciplines within the Colleges of Arts & Sciences, Business Sciences, and Information Technology. Students all have laptops and are quite computer literate. The computer algebra system, *MAPLE 13*, is currently being used by mathematics courses in the Department of Mathematics and Statistics. A CAS (Computer Algebra System) such as *MAPLE 13* is able to perform arithmetic manipulations accurately and with speed.

In studying Linear Algebra students encounter a significant amount of numerical computations and it would be advantageous to student learning to make use of available CAS technology ([1]). Students are able to use a CAS to focus on the concepts involved without worrying about tedious calculations and possible arithmetical errors. Research has shown that instructors can use the efficiency gained from using a CAS to focus on conceptual development problem solving and investigations with realistic problems ([2]; [3]). Research also suggests that students can greatly benefit from the accuracy and immediacy of a CAS ([4]). Early classroom experiments of CAS demonstrated that students who used CAS and experienced concept-oriented lessons achieved significantly greater understanding in conceptual knowledge than those students in traditional skills-oriented conditions, without sacrificing the learning of computational skills ([1]; [5]). Later studies have generally confirmed this trend.

Students in our Linear Algebra course learn and use some *MAPLE* syntax and commands to compute a variety of tasks, particularly to reduce the work involved in more complicated computations from real-world application problems. For example, finding the production level matrix for a Leontief input-output model usually requires finding the inverse of a rather involved matrix. However, *MAPLE* additionally has a number of user-friendly built-in interactive tutors which allow students to investigate specific tasks by selecting their own input values and working through a problem in a step-by-step fashion together with immediate feedback at each step. Students can avoid having to use *MAPLE* syntax by using this point-and-click interface allowing them to concentrate more specifically on a given problem. The performance at each step requires the student to make a cognitive decision where the thinking process reinforces the concept

involved. We look here at how these can be applied to some topics in linear algebra which generally involve a considerable amount of numerical manipulation. We shall consider *MAPLE's* interactive tutors for Gaussian Elimination, Gauss-Jordan Elimination, Matrix Inverse, Eigenvalues and Eigenvectors.

## 2. Working with Linear Systems of Equations

Students, in an introductory linear algebra course, deal with systematic procedures of Gaussian Elimination and Gauss-Jordan Elimination to solve linear systems of equations. The first step is to understand how the system is compactly recorded in an augmented matrix form with the coefficients of each variable aligned in columns together with the column of constants from the right side of the equations. Students should see that each equation in the system corresponds to a row in the augmented matrix. The next step is to understand that operations on equations in the linear system have parallel 'elementary row' operations on the corresponding rows of the augmented matrix. In Gaussian Elimination, operations are performed on the equations bringing the system into an 'echelon' form which enables the solutions to be found by back-substitution. The equivalent process in matrix form is to perform elementary row operations on the augmented matrix to bring it to a 'row echelon' form. In Gauss-Jordan Elimination, operations are performed on the equations bringing the system into a 'reduced echelon' form which enables the solutions to be read off directly. Here, the equivalent process in matrix form is to perform elementary row operations on the augmented matrix to bring it to a 'reduced row echelon' form.

Before students can use these techniques, it is essential that they have a clear conceptual understanding of both row echelon and reduced row echelon forms. This understanding can be reinforced by interacting with the *MAPLE* tutors for Gaussian Elimination and Gauss-Jordan Elimination. Figure 1 below shows how the *MAPLE* tutor allows students to create the augmented matrix from a given linear system after first entering the order of the matrix.
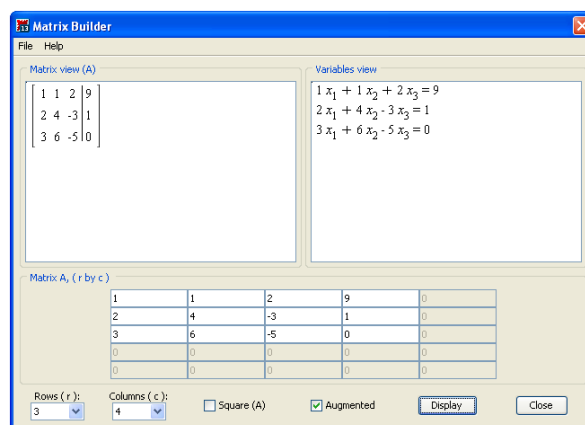


**Figure 2.1** Setting up the Augmented Matrix for Gaussian Elimination

Once the matrix is set up, students have access to the interactive process of performing any one of three elementary row operations on the augmented matrix. See Figure 2 below. If the student is not happy with a particular row operation, it can be 'undone' using the Undo button.
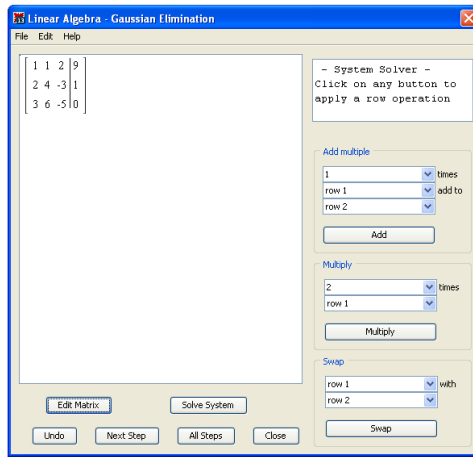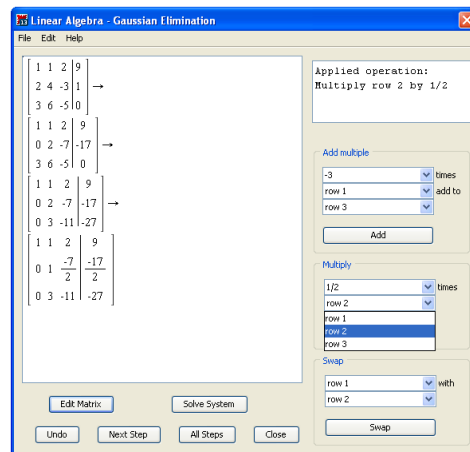
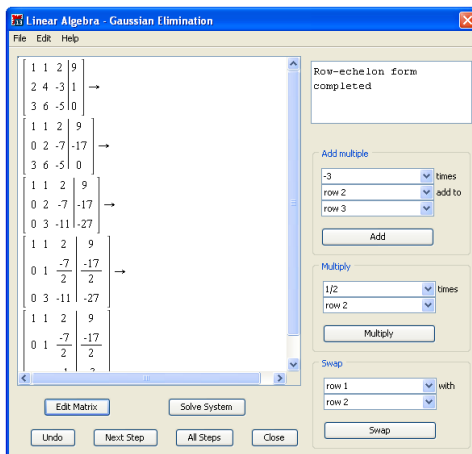**Figure 2.2**  The Gaussian Elimination Interface

On completion of each row operation, the student is shown the resulting row equivalent matrix. Steps to the solution are shown in the Figures 2.3 to 2.8 below.
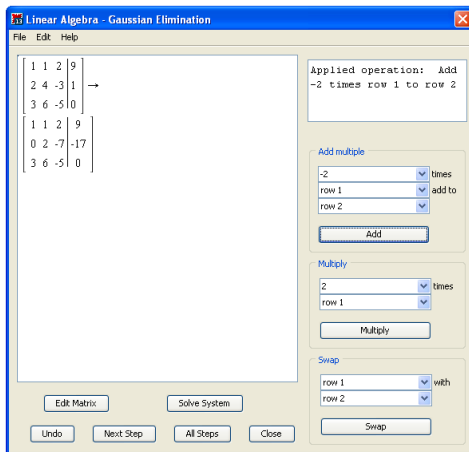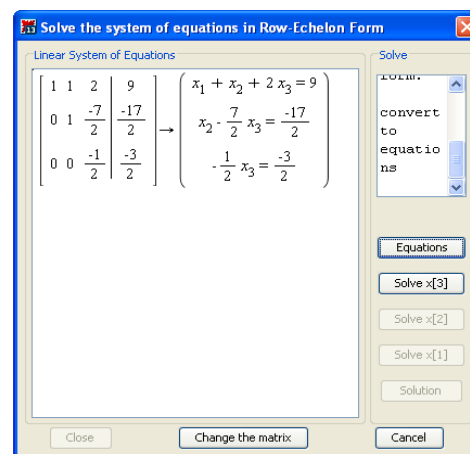


**Figure 2.3**



**Figure 2.4**
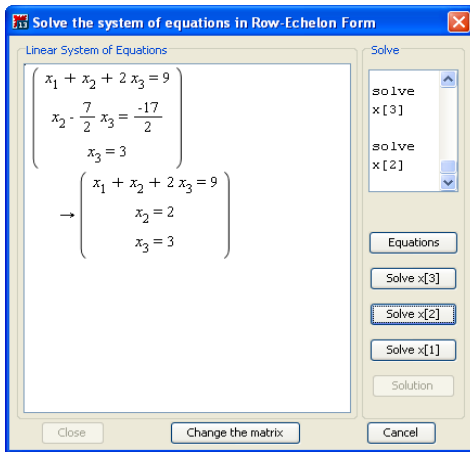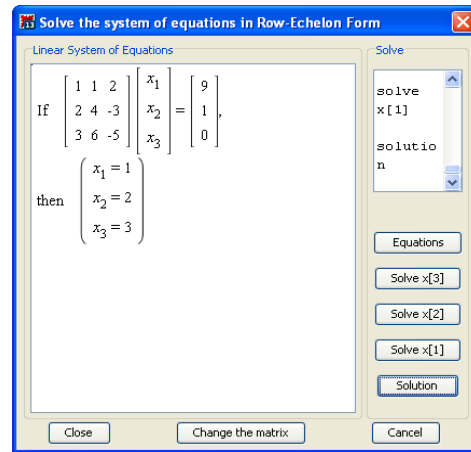


**Figure 2.5**



**Figure 2.6**

**Figure 2.7**                    **Figure 2.8**

Using the Gauss-Jordan interface the student proceeds as with Gaussian elimination continuing on to the reduced row echelon form. Deciding on, and applying, a given row operation is central to process and the student is being continually reinforced at each step. The final steps to the solution are shown in the figures below.
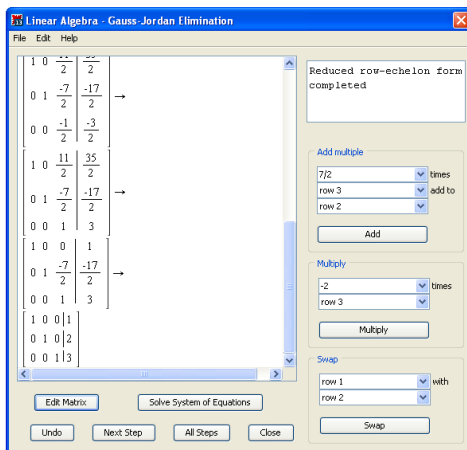


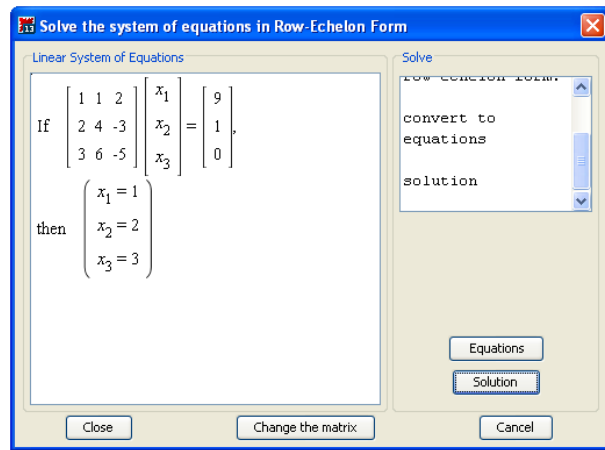**Figure 2.9**                    **Figure 2.10**

## 3. Finding the Inverse Matrix

Students learn that a $n \times n$ matrix, $A$, is row equivalent to the identity matrix $I_n$ if and only if $A$ is invertible. They are shown that the finite set of elementary row operations from $A \to I_n$ are the same set of row operations for $I_n \to A^{-1}$, and that these can be done at the same time by applying them to the augmented matrix $\left[ A \mid I_n \right]$. *MAPLE* has an interactive tutor allowing students to investigate and reinforce understanding of this Gauss-Jordan method. The students simply enter the elements of the augmented matrix $\left[ A \mid I_n \right]$ and perform the relevant row operations as shown in the Figures 3.1 and 3.2 below.
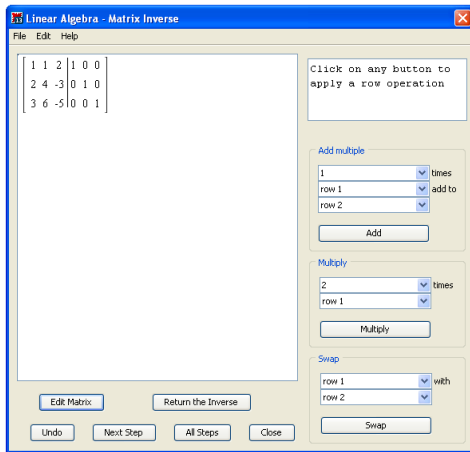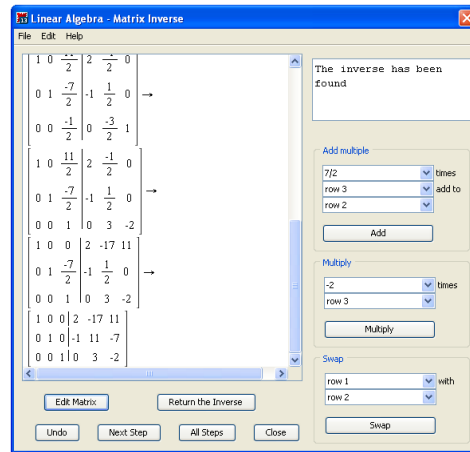
**Figure 3.1**



**Figure 3.2**

With computations minimized, students are able to focus on this Gauss-Jordan method.

# 4. Eigenvalues and Eigenvectors

The concept of the special transformation here, where a vector is transformed by a square matrix *A* into a scalar multiple of itself, is essential to understanding of eigenvalues and their associated eigenvectors. Students should recognize that non-zero solutions to the equation $A\underset{\sim}{x} = \lambda\underset{\sim}{x}$ are solutions of the homogeneous system of equations $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$, and these occur if and only if $\det(A - \lambda I) = 0$, leading to the eigenvalues of *A*. These essential basic concepts are necessary in enabling students to know how to interpret their results. The interactive eigenvalue and eigenvector tutors permit students to focus on these concepts through a step-by-step process.

*MAPLE's Student Linear Algebra* package should be loaded first in order for students to explore eigenvalues and eigenvectors with *MAPLE's* interactive linear algebra tutor. The $n \times n$ transformation matrix has first to be defined in the worksheet before the tutor is activated.
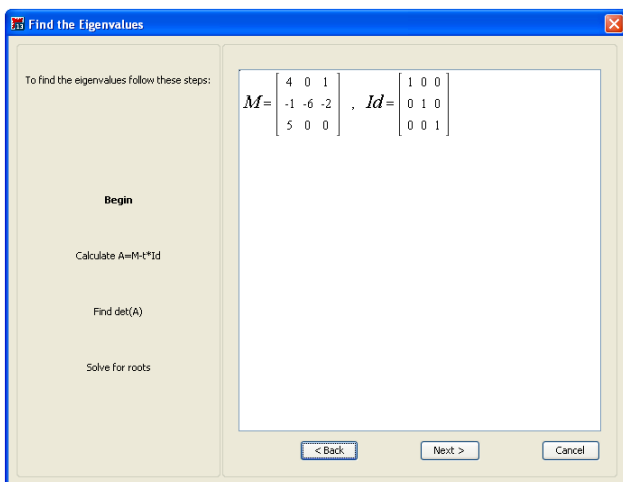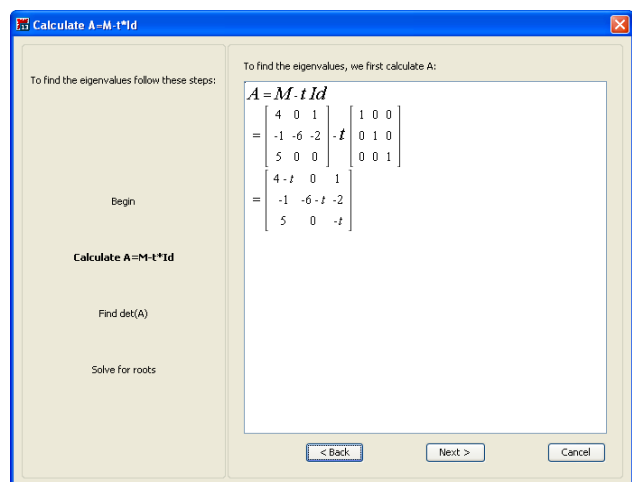


**Figure 4.1**



**Figure 4.2**

The first step computes the matrix is $A - \lambda I$ (Figure 4.2: *MAPLE* uses $M - t.Id$). The determinant of $M - t.Id$, the characteristic polynomial, is computed in the next step (Figure 4.3), and finally the characteristic equation is factored and solved for the eigenvalues (Figure 4.4). The eigenvalues are $t_1 = 5$, $t_2 = -6$, $t_3 = -1$. By working through these steps, the students come to a clearer understanding of the sequential process leading to the solution of the characteristic equation.
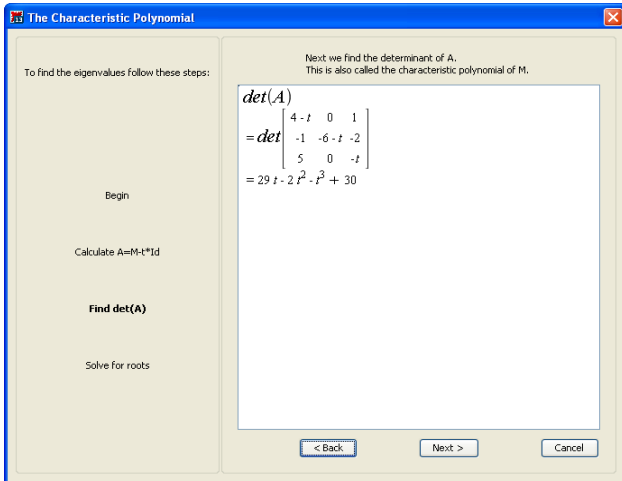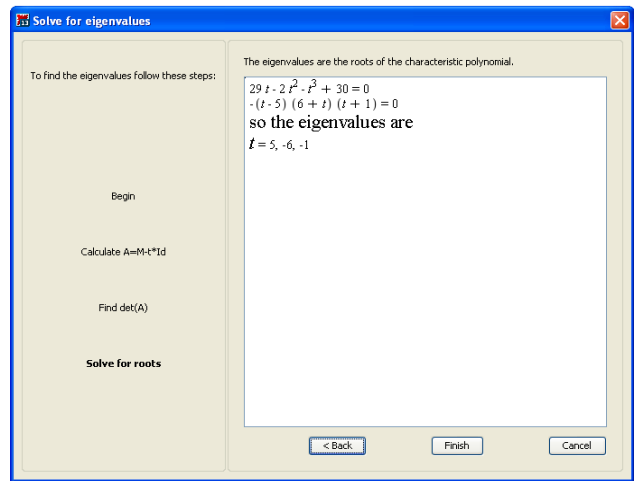


**Figure 4.3**



**Figure 4.4**

We can continue and find the eigenvectors corresponding to each eigenvalue. *MAPLE* gives the following interface as shown in Figure 4.5.
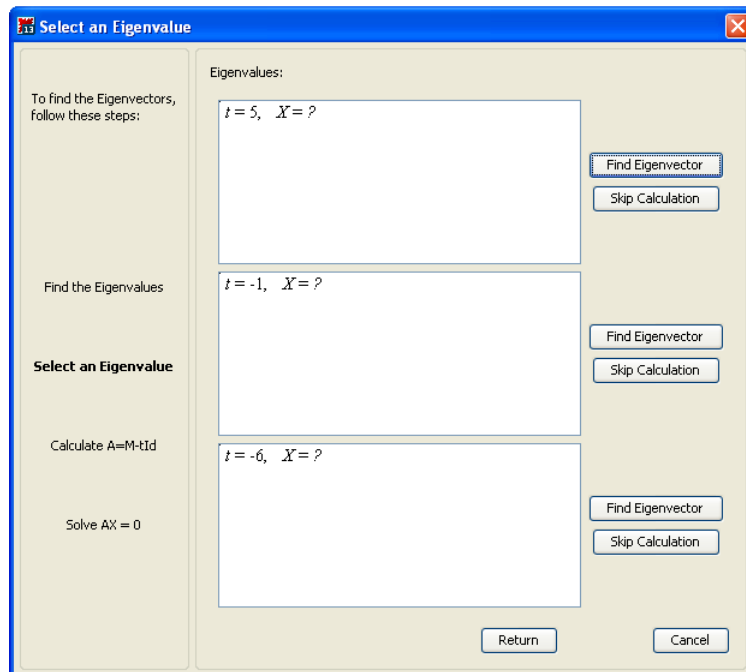


**Figure 4.5** Interface for finding eigenvectors

Clicking the 'Find Eigenvector' buttons in turn lead to each of the eigenvectors. The steps for the first eigenvector are shown below. Figure 4.6 shows the evaluation of the matrix $M - t.Id$.
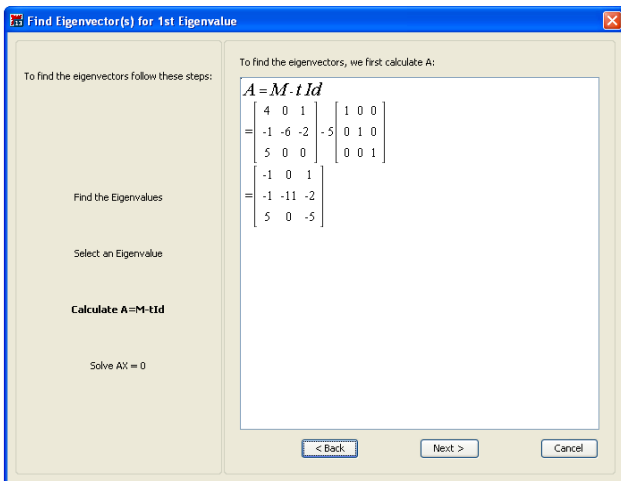


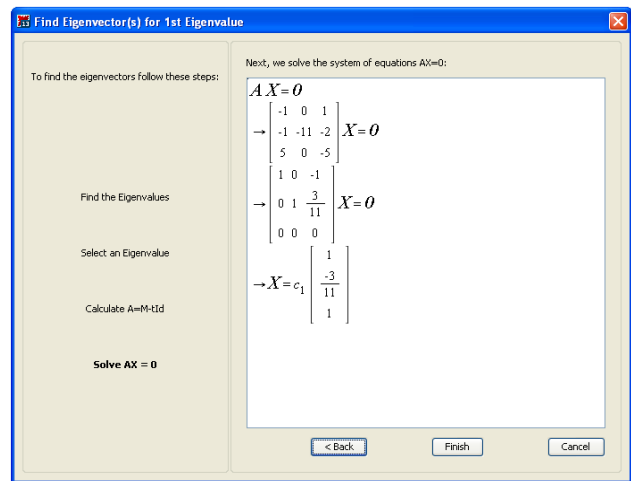**Figure 4.6**                                    **Figure 4.7**

Figure 4.7 shows solution to the homogeneous equation leading to the eigenspace of $M$ corresponding to the eigenvalue $t_1 = 5$. Students are able to see the breakdown of the process at each stage. Figure 4.8 shows the eigenvectors for the remaining eigenvalues which are easily computed in the same way.
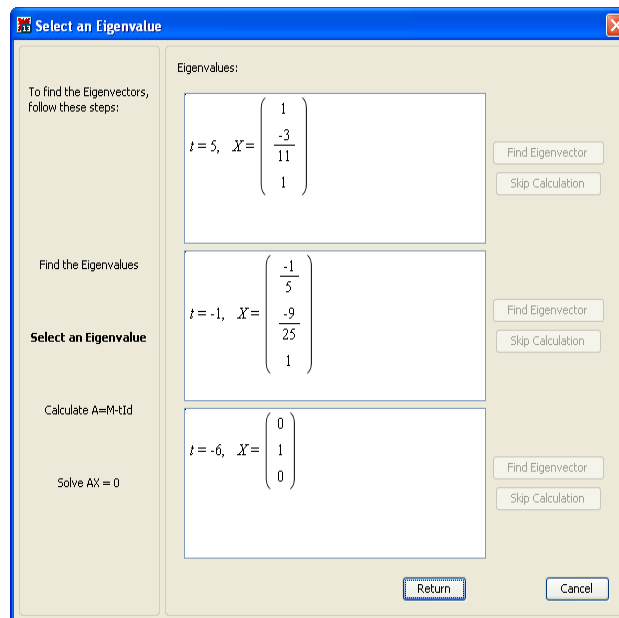


**Figure 4.8**

While students are already familiar with the *MAPLE* syntax commands for finding characteristic polynomials, eigenvalues and eigenvectors, the role of the interactive tutor is to provide students with the further opportunity to experiment and practice with the basic algorithms, and develop their conceptual understanding of eigenvalues and eigenvectors.

# 5. Applications

**Example 1**

Consider the following network flow problem [5]: *Find the general traffic pattern in the freeway network shown in the figure when the road whose flow is $x_5$ is closed. (Flow rates are in cars/minute.)*
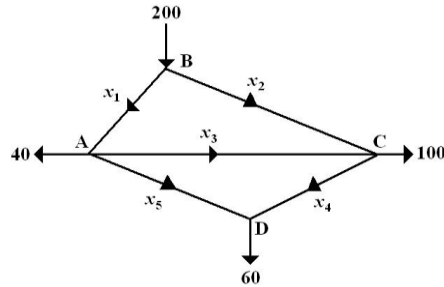


**Figure 5.1**

The assumption here is that the total flow into a junction equals the total flow out of the junction. Students should be able to determine the following system of equations , given that that $x_5 = 0$:

$$A : \quad x_1 \quad - x_3 \quad = 40$$
$$B : \quad x_1 + x_2 \quad = 200$$
$$C : \quad x_2 + x_3 + x_4 = 100$$
$$D : \quad x_4 = 60$$

*MAPLE's* interactive tutors for Gaussian Elimination can be used here to quickly find the general solution to the system, while providing students with row reduction practice and reinforcement. Figure 5.2 shows the reduced system of equations. Figure 5.5 shows the solutions with free variables. Since the solutions are non-negative, students should see that the values for $t$ are in the range $0 \le t \le 160$.
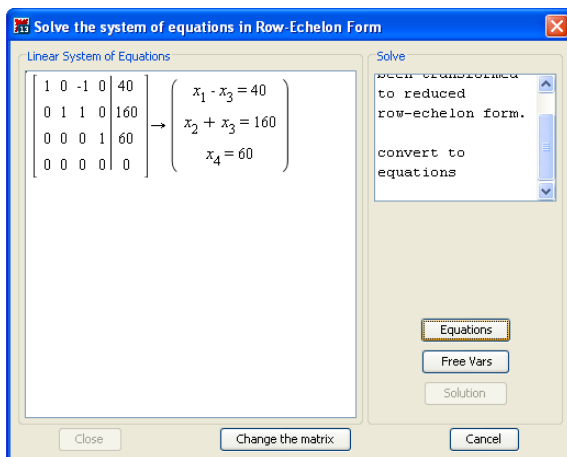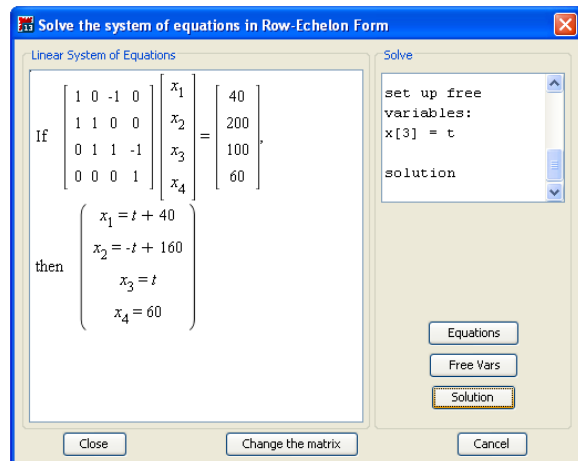


**Figure 5.2**



**Figure 5.3**

**Example 2**

Consider the following Leontief input-output table showing an economy consisting of three sectors showing the inputs consumed per unit of output [5].

### Inputs Consumed Per Unit of Sector Output

| | | Output (per $1:00) | | |
| --- | --- | --- | --- | --- |
| | | Manufacturing | Agriculture | Services |
| Input (Intermediate demands) | Manufacturing | 0.50 | 0.40 | 0.20 |
| | Agriculture | 0.20 | 0.30 | 0.10 |
| | Services | 0.10 | 0.10 | 0.30 |

Problem: *Suppose that the final demands for manufacturing, agriculture and services are respectively 50 units, 30 units, and 20 units, where the units are measured in millions of dollars. Find the production level for each sector to satisfy the final demands.*

In working with Leontief input-output models, students have to solve the equation $X=CX+D$, where $X$ is the production level matrix. $C$ the consumption matrix, and $D$ the final demand matrix. The solution requires finding $X$, the levels of output of the sectors, and this is given by $X = (I - C)^{-1}D$ where $I$ is the identity matrix. If $(I - C)$ is invertible, $(I - C)^{-1}$ can be found and multiplication by $D$ gives the production levels for each sector. *MAPLE* will readily compute the inverse of a given matrix where it exists. However, students can also use the interactive tutor to find the solution for $X$ by solving the system $(I-C)X = D$ using Gauss-Jordan Elimination.

$$\text{Here } C = \begin{bmatrix} 0.50 & 0.40 & 0.20 \\ 0.20 & 0.30 & 0.10 \\ 0.10 & 0.10 & 0.30 \end{bmatrix}, \ (I-C) = \begin{bmatrix} 0.50 & -0.40 & -0.20 \\ -0.20 & 0.70 & -0.10 \\ -0.10 & -0.10 & 0.70 \end{bmatrix} \text{ and } D = \begin{bmatrix} 50 \\ 30 \\ 20 \end{bmatrix}$$

Figures 5.4 and 5.5 below show some of the interactive steps to solve the system $(I-C)X = D$. The tutor uses fractional form of the decimals.
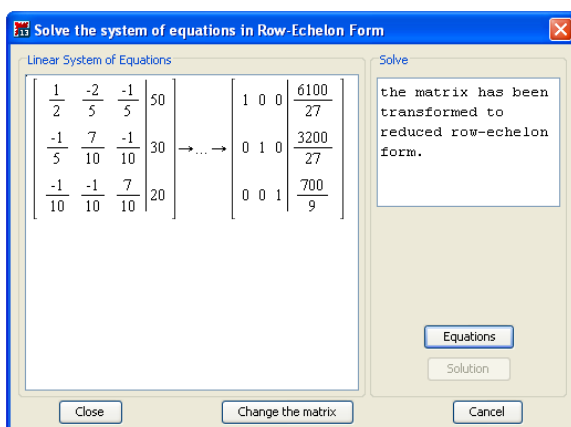


**Figure 5.4**
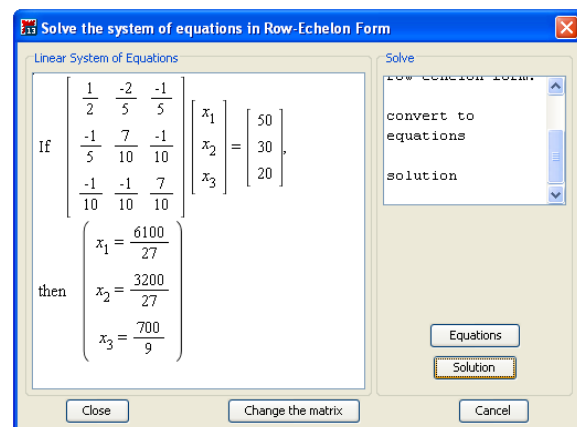


**Figure 5.5**

The output production levels for Manufacturing, Agriculture and Services are $225.93 million, $118.52 million, and $77.78 million, respectively.

# 6. Conclusion

This paper describes one aspect of the application of technology in providing an environment for interactive exploration of mathematical structures in Linear Algebra. The use of interactive tutors, such as *MAPLE's*, provides students with the opportunity to review and practice mathematical procedures involved in solving problems. By working through problems in a step-by-step process, they are making decisions in applying rules as well as obtaining instant feedback on their thinking. The speed of computations enables students to focus their attention on the meaning of the computations and can positively impact the students' understanding of the concepts involved. Students in the author's Linear Algebra class have found the interactive tutors helpful and they have also used them to check for arithmetical errors their own hand-worked solutions, such as with row-reduction computations.

Teaching an introductory Linear Algebra course in the English medium to students whose first language is not English can also be a further challenge. Students may have difficulty with particular topics and experience difficulties in grasping some concepts. Some of the abstractions are not obvious and will take time and effort to learn thoroughly. The use of technology can help to remove some of the language barriers and bring some variation and enrichment to course delivery. Our students have certainly found working with *MAPLE* rewarding.

# References

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