Influence of using KETpic graphics on the development of collegiate students' proof schemes

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Abstract

Results of our questionnaire survey and textbook research in Japan show that the use of graphics in printed class materials and textbooks tends to be reserved. One reason is that many teachers are worried that using specific figures captures students' deductive reasoning about mathematical conceptions and obstructs students' appreciation for generality. In the case of proofs, which are typical mathematical activities related to generality, this seems to have become a severe problem. This paper presents the manner in which geometric models might be effective in students' shaping the conception of proof using some examples. The approach is based on the classification of students' proof schemes proposed by G. Harel. Results show that Harel's classification should be improved. The graphical device used to produce figures described in this paper is **KETpic**, *which is a macro package designed for a computer algebra system to generate high-quality graphical images in high-quality mathematical documents edited using LaTeX.*

1 Introduction

According to the results of our questionnaire survey and textbook research in Japan, the use of graphics in printed class materials and textbooks tends to be reserved ([5]). One reason is that many mathematics teachers might believe that using specific figures can pose an obstacle preventing many students from appreciating the generality aspects of mathematical conceptions. For example, the use of graphics in a "proof" can pose a severe problem because a proof is a typical mathematical activity concerned with generality. Moreover, as reviewed in [1], results of many previous studies have indicated that the complex character of proofs engenders teachers' inadequate approach to them and students' inadequate appreciation for them. Despite these difficulties, many previous reports in the literature have recommended that proofs be taught to all students and in all mathematics courses ([2][4]). To establish a promising framework for teaching proofs, Harel ([1]) classified students' cognitive schemes related to mathematical proofs. Some aspects of this classification are reviewed in section 3.

This paper illustrates how geometric models (or graphical devices to produce their perceptible images) can be effective in students' shaping the conception of proof, with respect to Harel's classification described above. The authors seek to show that using appropriate geometric models can promote the development of students' proof schemes with minimal obstruction of their appreciation for generality.

The graphics described in this paper were produced using KETpic which we have been developing. KETpic is a macro package designed for computer algebra system (CAS) to generate high-quality graphical images (for use with CAS) in high-quality mathematical documents (edited by LaTeX). Our survey results show that more than 70 percent of collegiate mathematics teachers in Japan use LaTeX to edit their teaching materials, and minority of them frequently insert graphics into their materials. Many respondents to our questionnaire reported that they feel inconvenience in using existing devices for inserting graphics into LaTeX documents. It is expected that KETpic will serve as a convenient tool for that purpose.

KETpic is applicable to CASs such as Maple ([9]), Mathematica ([11]), Matlab ([10]), Scilab ([8]), and R ([6]). The corresponding libraries and some interesting examples and documentations are freely downloadable at the URL `http://www.ketpic.com`.

2 Brief Introduction to KETpic

When inserting graphics into LaTeX documents, a graphical editor is usually used to generate artwork. Then the resulting image file (formatted in such as EPS file) is invoked from LaTeX. This method causes some inconveniences such as large file size. To alleviate these inconveniences, we have developed KETpic which generates LaTeX readable code for high-quality mathematical artwork with the aid of CAS.

2.1 Procedure of KETpic drawing

The procedure is presented in Fig. 1.

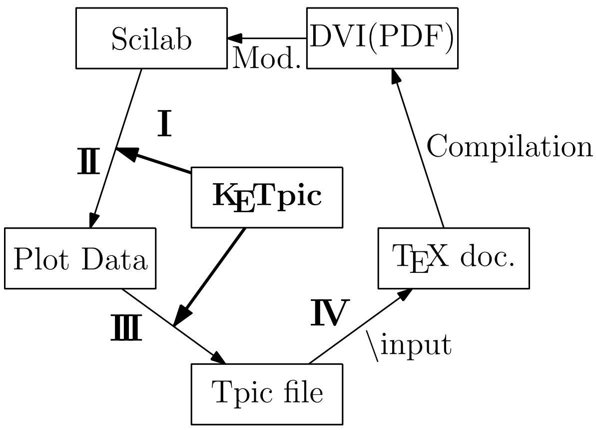

Figure 1 The KETpic cycle.

In step **I**, we load the KETpic package into Scilab with the following command lines.

```
Ketlib=lib('<folder name>:/ketpicscil5/');
Ketinit();
```

In step **II**, plot data of the objects we want to draw are generated with Scilab-embedded KETpic command lines, as shown below.

```
Fd=list('Z=cos(X)*sin(Y)+exp(-(X^2+Y^2)/0.3)','X=[-3,3]','Y=[-3,3]'):
S1=Sfbdparadata(Fd);      // Data of ridgelines and edge lines
W1=Wireparadata(S1,Fd,10,10); // Data of wire frames
```

In step **III**, the plot data are formatted into *Tpic* specials codes and stored in *Tpic* file named "fig.tex". Some accessories are also added. The command lines are as follows.

```
Openfile('<folder name>:/fig.tex');
Beginpicture('');
Drwline(S1,2);   // '2' denotes the curve thickness
Drwline(W1);
Endpicture(0);
Closefile();
```

In step **IV**, "fig.tex" is input into LaTeX document using the following LaTeX command lines.

```
\usepackage{ketpic}
\begin{document}
\input{fig}
\end{document}
```

Then, after the compilation, we obtain the following Figure 2.

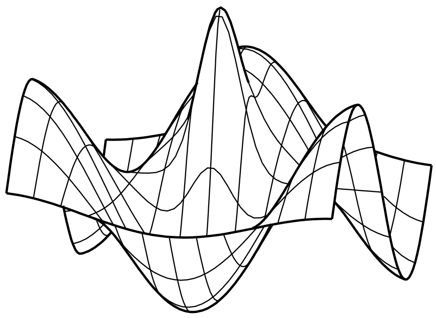

Figure 2 Resulting figure.

2.2 Features of KETpic drawing and comparison to other devices

The KETpic drawing has some remarkable features, as listed below.

1. Manipulations of the figure can be performed easily on the user's demand.
2. Because of using CAS, KETpic drawings are precise in terms of their shape and length.
3. Rich mathematical expressions (with the same quality as that in LaTeX documents) and various accessories (such as hatching, shading, and arrow lines) can be inserted easily.
4. The *Tpic* file size is astonishingly small compared to their image file counterparts. For that reason, they are suitable for fast web-tech-based communication.
5. 3D-graphics can be drawn with precise shape. The function of hidden line (or surface) elimination endows 3D-graphics with rich perspective ([7][11]).

As Figure 2 shows, a surface drawing of KETpic includes some extra features. One feature is that it is not a painting but a monochrome line drawing. Moreover, KETpic represents surfaces using only ridgelines and edge lines. Consequently, the figure quality is maintained when they are copied. Furthermore, it will be easier for students to grasp the global shape of surfaces compared to the case of CAS drawing (see Figure 3).

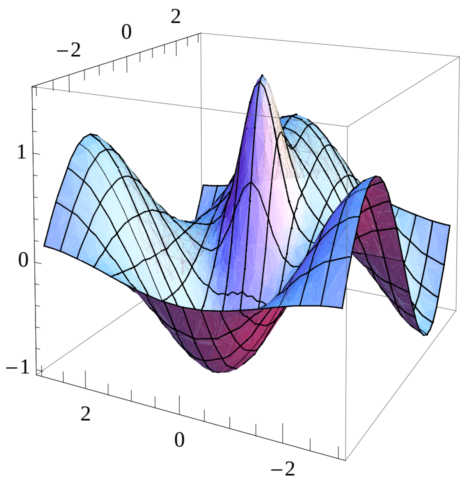

Figure 3 Mathematica counterpart of Fig. 2.

Although students would be able to grasp the local surface structure by seeing Figure 3 on a computer display, the global structure seems to be difficult to view. Moreover, it is extremely difficult to guarantee the quality of this figure when it is copied to mass printed (monochrome) materials. Therefore, such surface drawings will be inadequate for educational purposes.

3 Classification of Proof Schemes by Harel

As described in the Introduction, G. Harel classified students' proof schemes using the classification summarized in the following Table 1. This classification is not of proof content or proof method but of the cognitive stage and intellectual ability in students' mathematical development. Although these schemes are not mutually exclusive, those in the lower rows are regarded as more sophisticated and developed than those in the upper rows. In this section, among these proof schemes, we review only those which are related to the context of this paper.

Table 1 Harel's proof scheme classification

<table>
<tr><td rowspan="3">External Conviction</td><td>Ritual</td><td rowspan="5" colspan="2"></td></tr>
<tr><td>Authoritarian</td></tr>
<tr><td>Symbolic</td></tr>
<tr><td rowspan="2">Empirical</td><td>Inductive</td></tr>
<tr><td>Perceptual</td></tr>
<tr><td rowspan="8">Analytical</td><td rowspan="5">Transformational</td><td>Internalized</td><td rowspan="2"></td></tr>
<tr><td>Interiorized</td></tr>
<tr><td rowspan="3">Restrictive</td><td>Contextual</td></tr>
<tr><td>Generic</td></tr>
<tr><td>Constructive</td></tr>
<tr><td rowspan="3">Axiomatic</td><td>Intuitive Axiomatic</td><td rowspan="3"></td></tr>
<tr><td>Structural</td></tr>
<tr><td>Axiomatizing</td></tr>
</table>

One is the *Empirical* proof scheme in which "conjectures are validated or subverted by appeals to physical facts or sensory experiences". One subcategory is an *Inductive* proof scheme in which "students ascertain and persuade others about the truth of a conjecture by evaluating it quantitatively in specific cases". The other subcategory is a *Perceptual* proof scheme in which "students convict and persuade the truth of a conjecture through perceptual observations which are made by choosing variable objects".

The other is the *Transformational* proof scheme in which "students validate conjectures using logical deductions involving mental operations on objects and anticipations for the operations' results". Such operations are motivated by the need to consider the generality aspects of the conjecture. At its most primitive level (*Internalized* proof scheme), the students translate the given question into well-understood model. In the next level (*Interiorized* proof scheme), the students reflect their reasoning in the internalized stage and formulate their underlying structure as a general method of proof. In the last level (*Restrictive* proof scheme), students are captured by their specific mental images, although their deductive reasoning is potentially applicable to general cases.

Structural proof scheme "by which one thinks of conjectures and theorems as representations of situations from different realizations that are understood to share a common structure characterized by a collection of axioms (or definitions)" is highly sophisticated. In our previous work ([5]), we showed that using an appropriate geometric model can be helpful for students' development into this proof scheme.

4 Using Graphics to Develop Students' Proof Schemes

In this section, we present some examples of using graphics in the processing of proofs. The presumable influence of such graphics use on the development of students' proof schemes will be analyzed.

Example 1. Convexity (the case of linear algebra).

First, we consider the proof of the following proposition.

> Presuming that A is a positive definite symmetric matrix and k is a positive number, then the closed domain $D = \{\mathbf{x} \in \mathrm{R^n} | (\mathbf{x}, \mathrm{A}\mathbf{x}) \leq \mathrm{k}\}$ is convex.

Because a (high-dimensional) $\mathrm{R^n}$ model is used, students can not perceive the actual objects to support their reasoning. Therefore, their deduction should be based not on geometric intuition, but on symbolic transformations. In fact, the usual mode of proof is the following.

> Because A is a positive definite symmetric matrix, the inequality
> $$(\mathbf{x}, A\mathbf{x}) + (\mathbf{y}, A\mathbf{y}) \geq 2(\mathbf{x}, A\mathbf{y}) \quad (*)$$
> holds for any vector $\mathbf{x}$ and $\mathbf{y}$. It suffices to show that
> $$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in D$$
> for any elements $\mathbf{x}, \mathbf{y} \in D$ and any number in $0 \leq \lambda \leq 1$. In other words, we must show that the inequality
> $$(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}, \lambda A\mathbf{x} + (1-\lambda)A\mathbf{y}) \leq k$$
> holds for any $\mathbf{x}$ and $\mathbf{y}$ which satisfy the inequalities $(\mathbf{x}, A\mathbf{x}) \leq k$, and $(\mathbf{y}, A\mathbf{y}) \leq k$. For such $\mathbf{x}$ and $\mathbf{y}$, $(\mathbf{x}, A\mathbf{y}), \leq k$ holds by $(*)$. Therefore
> $$\begin{aligned} LHS &= \lambda^2(\mathbf{x}, A\mathbf{x}) + 2\lambda(1-\lambda)(\mathbf{x}, A\mathbf{y}) + (1-\lambda)^2(\mathbf{y}, A\mathbf{y}) \\ &\leq \lambda^2 k + 2\lambda(1-\lambda)k + (1-\lambda)^2 k \\ &= \{\lambda^2 + 2\lambda(1-\lambda) + (1-\lambda)^2\}k \\ &= \{\lambda + (1-\lambda)\}^2 k = k \end{aligned}$$
> Consequently, the proposition is proved.

The process of deduction presented above is sufficient and applicable to general inner product space. It can therefore be regarded as being in a structural proof scheme. However, this proof seems not to enable students to grasp the relation between surface symbolic transformations and the underlying geometric structure. In other words, the reason why the proposition is true can not be understood easily through this proof. Therefore, especially for students with weak mathematical skills, this proof might serve that in the symbolic proof scheme (which is a primitive stage in students' mathematical development). Based on this consideration, we give the following geometrical proof in which graphics play a crucial role.

Because A is a positive definite symmetric matrix, we can choose an orthonormal basis $\mathbf{v_1}, \cdots, \mathbf{v_n}$ of $\mathrm{R^n}$ composed of the eigenvectors of A corresponding to positive eigenvalues $\lambda_1, \cdots, \lambda_n$. Any $\mathbf{x} \in \mathrm{R^n}$ can be represented uniquely as

$$\mathbf{x} = x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n$$

and the following equalities hold.

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \cdots, A\mathbf{v}_n = \lambda_n\mathbf{v}_n$$

Therefore, because of orthonormality, we conclude that the condition $(\mathbf{x}, A\mathbf{x}) \leq k$ is equivalent to $\lambda_1 x_1^2 + \cdots + \lambda_n x_n^2 \leq k$. Consequently, what must be shown is the following.

If $\lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 \leq k$, and $\lambda_1 b_1^2 + \cdots + \lambda_n b_n^2 \leq k$,
then $\lambda_1(\lambda a_1 + (1-\lambda)b_1)^2 + \cdots + \lambda_n(\lambda a_n + (1-\lambda)b_n)^2 \leq k$

As might be readily apparent, this n-dimensional problem can be reduced to a one-dimensional problem:

If $a_1^2 \leq k, b_1^2 \leq k$, then $(\lambda a_1 + (1-\lambda)b_1)^2 \leq k$.

This inference can be validated directly by observation of Figure 4.

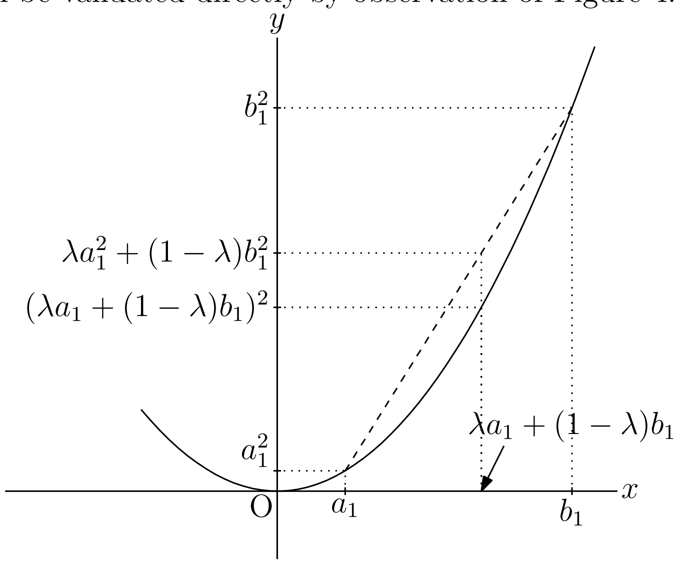

Figure 4 The figure, which is primitive but essential.

Through this example, students will appreciate the importance of considering the basis change and its influence on representation matrix because they translate an invisible problem into a visible one. Consequently, the use of graphics can be expected to motivate the "interiorization" of the students' proof scheme. Furthermore, using the same quality of mathematical expressions in figures as in the text makes it easier for students to compare symbolic expressions and geometric structures.

Example 2. Constrained extremum (The case of multivariable calculus)

Next we consider the proof of the following theorem.

> Under constraint $g(x,y)=0$, presume that the function $z=f(x,y)$ takes its extremum value at (x_0,y_0). Then, at this point, the following equality holds.
> $$\frac{g_x(x_0,y_0)}{g_y(x_0,y_0)}=\frac{f_x(x_0,y_0)}{f_y(x_0,y_0)}$$

The usual proof is not geometric but symbolic, as shown below.

> Using the implicit function theorem, the differential of the curve $g(x,y)=0$ is computed as $\dfrac{dy}{dx}=-\dfrac{g_x}{g_y}$. Function $z=f(x,y)$, restricted to the curve $g(x,y)=0$, can be regarded as a function of only one variable x. Then, by the formula of differential for composite function, its differential with respect to x is computed as
> $$\frac{dz}{dx}=f_x+f_y\frac{dy}{dx}$$
> At the extremum point (x_0,y_0), $\dfrac{dz}{dx}$ must be 0. Therefore the following holds:
> $$f_x-f_y\frac{g_x}{g_y}=0$$
> This leads to the conclusion.

This proof might raise a difficulty similar to that shown for Example 1. Moreover, in our experience, it is not so easy for many students to grasp the situation they must consider. Therefore, we can offer the alternative geometric approach, as described below.

First, we show Figure 5 to support students' cognition using 3D-graphics capability of KETpic. Here we choose a simple example in which

$$g(x,y)=x+y-3,\qquad f(x,y)=\sqrt{xy}+0.5.$$

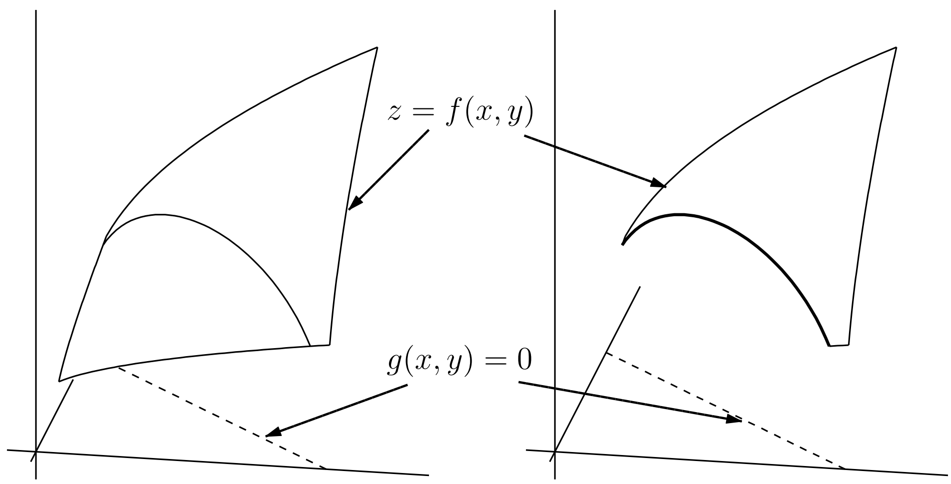

Figure 5 Figure used to support students' cognition.

Then we give the following geometric proof.

We must search for the point of extreme height in the thick curve in Figure 5. As Figure 6 shows, at such extreme points, the thick curve is expected to be tangent to a curve on the graph of $z = f(x, y)$ with some constant height.

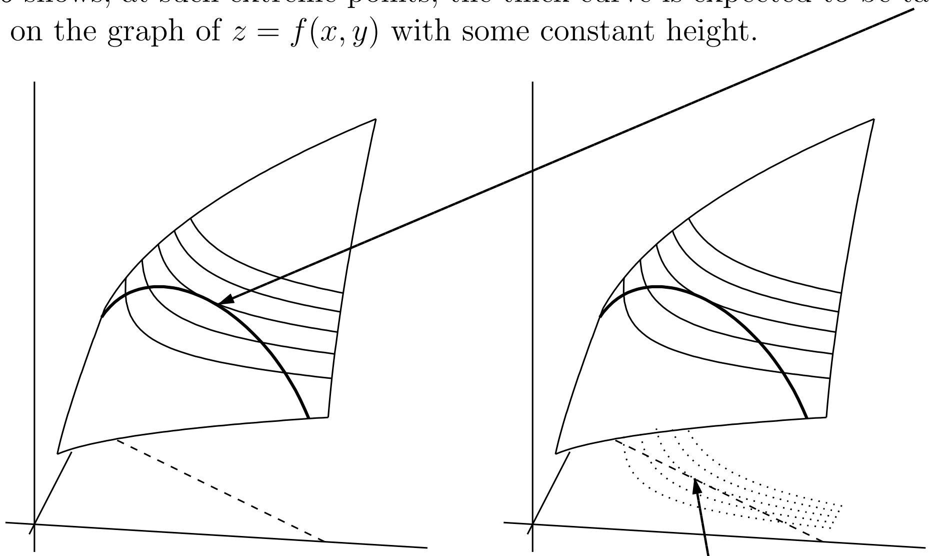

Figure 6 Essential figures.

Because contours of z=f(x,y), f(x,y)=c for some c, is tangent to g(x,y)=0. In other words, the gradient vector $\nabla f(x_0, y_0)$ is parallel to $\nabla g(x_0, y_0)$, or $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ, the proposition is validated.

Because the figures shown above portray only one simple example, the proof presented above might be regarded as a perceptual one. However, using more general examples might prevent students from easily understanding the fundamental geometric structure of the theorem.

Example 3. Stochastic variable change (The case of probability distribution)

Finally, we consider the proof of the following proposition.

Presuming that a smooth function $y = f(x)$ transforms the probability distribution $p(x)$ with respect to a stochastic variable x to the probability distribution $q(y)$ with respect to another variable y, then if $y = f(x)$ has several (local) inverse functions $x_i = \phi_i(y)$, the following equality holds.

$$q(y) = \sum_i p(\phi_i(y)) \frac{d\phi_i(y)}{dy}$$

Because the schematic situation is similar to those presented in the previous examples, we omit the detailed proof and instead give some comments related to the graphics used in the process of proof. First, we remark that the probability distribution ($z = p(x),\ q(y)$) is of different character from the stochastic variables x, y. Therefore, we needed to use the 3D graphics as shown in Fig. 6. Furthermore, we remark that this figure must be drawn so that the following conditions are satisfied.

(1) $z = p(x)$ is the composite of $z = q(y)$ and $y = f(x)$. Therefore, the hatched regions in the figure must have the same "height".

(2) The area of the hatched region in yz-plane (indicated as $q(y)dy$) must be the same as the total area of hatched regions (one of which is indicated as $p(x)dx$) in xz-plane.

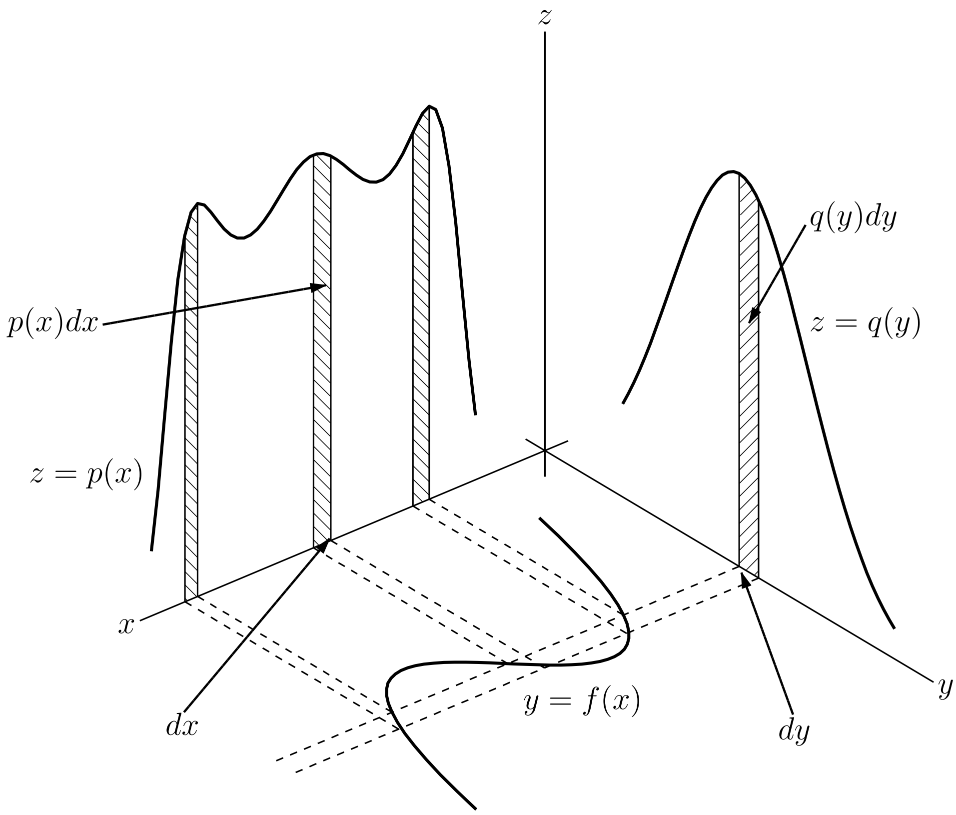

Figure 7 Stochastic variable change.

As shown in the previous examples, the specific choice of simple functions as $q(y)$ and $f(x)$ seemed to obstruct students' development beyond perceptual proof scheme. Therefore, we used a similar figure which had been drawn by hand, in which the shape of graphs was complicated. Nevertheless, it was not easy for us to draw figure in which the conditions described above are satisfied. Moreover, lack of precision raised some confusion in some students' reasoning. Use of the KETpic capability for generating precise graphics enabled us to provide appropriate figures for students. Moreover, by presenting this figure in printed material, students were able to use it to reflect repeatedly upon the meaning of the proposition.

5 Concluding Remarks

The examples presented in this paper indicated that the students' ability to access the structure of proof content and the students' appreciation for the generality of proof method present a tradeoff relation. Although the use of simple figures (or geometric models) helped students to grasp the structure of proof content, it might prevent students from developing their proof schemes beyond an empirical level.

In that sense, some improvement (or reorganization) is necessary in Harel's classification of proof schemes from the perspective of graphics use. The KETpic scheme is expected to provide appropriate environments for this improvement because it enables both the intensive use of mathematical expressions (by virtue of the use of LaTeX) and the flexible use of mathematical artwork (from the use of CAS).

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