# The Effectiveness of ICT-assisted Approach in Learning 3D Linear Algebra

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**Abstract:** *Linear algebra was one of underachieved fields of mathematics for the students of colleges of technology in Japan at the annual INCT achievement tests, which were given to all the students ten months after their learning of linear algebra. The low scores at achievement tests were partially because, in traditional lessons, the procedures of symbolic manipulation were taught without relating to the features of associated graphical objects or how they were used in real world applications. The isolated procedural knowledge did not accompany conceptual understanding and must be easily forgotten just after the term-end examinations. To compensate this situation and deepen the students' conceptual understanding, we redesigned our lesson-plan to be directed from concrete examples towards abstract mathematical ideas, from handling graphic objects and observing their characteristics towards building vector equations and manipulating them symbolically. In the new lessons, several ICT-assisted activities are included to demonstrate the close relation of graphic objects and vector equations interactively. This paper reports how the new lessons are changing our students' learning style from black-box approach to more meaningful one. Their written answers have richer explanations than their former students who learnt in traditional lessons. Their scores in the annual INCT achievement test also increased significantly in 2011.*

## 1. Introduction

In the college of technology the authors working for, low achievement of the students in linear algebra especially in three-dimensional (3D) vectors and matrices had been a serious issue [1]. The annual INCT (Institute of National Colleges of Technology) achievement test showed that our students' average scores were fairly high in algebra, trigonometry, and calculus but lower in 3D linear algebra and matrices.

The students were motivated to learn trigonometry and calculus because they knew that they continued to use those mathematics procedures in engineering subjects, for example, electrical circuit theory and electromagnetism. They also became to understand those mathematics concepts gradually when they continued to use the procedures in the exercises of mathematics and engineering applications.

The opposite was true to linear algebra. They were not supposed to use vector equations and matrices for a while after the learning and seemed to forget the procedures a few years later when they needed them in engineering subjects, for example, automatic control theory. They were not motivated because they could not perceive when they were to use those mathematical procedures and could not feel the reality in them. They did not deepen conceptual understanding, and the learnt procedures were rather quick to disappear. Because the annual INCT tests were given to the students about 10 months after the learning of linear algebra, the lower achievement in them were not avoidable.

The lack of the students' conceptual understanding in vector equations were easily observed in their answer sheets in term-end examinations. The answer sheets of poorly performed students had blank spaces, few drawings, and no sentences. Some of them could not tell if the equation $ax+by+cz=d$ represented a plane or a line in 3D space. Obviously, these students memorize a series of formulas without examining their features or understanding the relations of the formulas with the features of graphic objects. In another words, they had a learning style to memorize procedural knowledge as a black box without caring mathematical concepts even at the time of their learning in linear algebra.

Their learning style must be a natural result of our teaching style in linear algebra. Traditional lessons of linear algebra started from symbolic definitions of vectors, matrices, and their operations, moved to the exercises of manipulating symbolic expressions, but rarely explained the connections between symbolic and graphic representations in detail, and often lacked to introduce real-world applications familiar to the students. The students who seek *cost-effective* learning for scoring enough points to pass the examinations tended to omit conceptual considerations and stuck to blind-memorization of formulas.

To change the situation, we are redesigning our lesson plan [2] that is directed from concrete applications to abstract mathematical ideas, from handling graphic objects and identifying their characteristics toward defining symbolic expressions and manipulating them. The lesson plan is based on the MODEM theory [3] that stresses the importance of identification. Identification is the process where students recognize the tendency, patterns, or rules by observing the experienced phenomena, and gradually structure mathematical knowledge in their minds. In other words, identification is the process of personal discovery for each student even thought the knowledge is well known to the public, and the discovered knowledge is connected to rich personal experience. Such knowledge is usually more structured, and robust, and thus valuable to engineering students.

Identification should be done before production, where they are asked to apply the mathematical knowledge to solve given problems. Production is the popular process in traditional lessons but often misleads our students toward black box approach. Without well-structured knowledge, students can only memorize given procedures and apply them to solve problems in production processes, and they tend to learn only the surface of the knowledge.

In the lesson plan, we provided students with the opportunity to handle 3D graphic objects by themselves and to help them to identify the relationships between three representations: graphic objects, verbal explanations, and symbolic expressions. Interactiveness and simultaneous changes of graphic objects and symbolic expressions often help the students to idenfity the close relation between them. We used several software programs to install the interactive environment including web-browser/server, a database management system, a dynamic geometry software, and a computer algebra system.

## 2. New lesson plan

Our new lesson plan consists of seven steps described in Table 1 [2] and takes 5.65 hours of lesson time plus additional outside-class activities. It introduces 3D vectors in a real-world application, moves to handle them with real objects in classroom and with graphic objects in 3D virtual space, and discusses the symbolic representations and their mechanisms at the later steps. It also blends traditional classroom activities with technology-based ones.

Table 1 New lesson plan of 3D linear algebra for engineering students

| Step | | Activity | Lesson time |
|---|---|---|---|
| 1 | Start from actual application | Students recognize the usefulness of linear algebra to solve engineering problems and are motivated to learn. | 0.75h + outside class |
| 2 | Handling real objects in class | Students handle simple real objects and are accustomed to the idea of parallel and perpendicular. | 0.4h |
| 3 | Observing and constructing graphic objects in 3D virtual space | Students observe graphically how 3D objects can be constructed with vectors in interactive webpages developed with dynamic geometry software. They concentrate on the relation of graphic objects, but not on the symbolic representations. | 0.75h+ outside class |
| 4 | Identifying the relation of different representations | Students identify the relation of graphic objects, explanatory sentences, and symbolic expressions by the simultaneously changing representations in the computer screens. | 0.75h + outside class |
| 5 | Deducing formulas | The teacher explains the deducing process of formulas using the chalkboard. | 1.5h |
| 6 | Exercises with pencil-and-paper | Students solve traditional pencil-and-paper problems using more descriptive explanation of the solving processes. | 0.5h * 3 + outside class |
| 7 | Mutual mini-lectures | Students explain their own solutions to the other students in a small group. | outside class |

On the first step, we introduce 3D vectors as the building blocks of constructing a virtual game, which helps engineering students to feel the reality in vectors through playing the game [1]. However, since only a few students involved in this study actually played the virtual game, we do not expect the effect of this step yet.

On the second step, we let the students use real objects: thin wooden sticks and a plastic plate before virtual 3D graphic objects. We set a corner of each student's desk as the origin of 3D space with two sides of the desktop as $x$ and $y$ axes, and a stick fixed to stand vertically from the corner as $z$ axis. In this *real* 3D space, we ask the students to hold the plastic plate at a certain position and angle over their desks using sticks and pieces of clay. They recognize that they need three sticks to hold a plate. We encourage them to find as much facts as possible by themselves in the activity. This step makes a good introduction for the students before handling 3D graphic objects in a virtual space because the experience helps them to imagine similar objects in their minds.

On the third step, the students use the webpage [4] in which interactive graphic objects constructed with a dynamic geometry software Cabri 3D [5] are embedded to observe the relation among graphic objects. For example, an animation of a shrinking and stretching arrow $t\vec{a}$ in Figure 1 shows that any point on a straight line, which is expressed as the endpoint of arrow $\vec{x}$, is located by a combination of a fixed arrow $\vec{b}$ and the stretching/shrinking arrow $t\vec{a}$. The students observe that the point always stays on the line when the animation changes the value $t$ and the length of the arrow $t\vec{a}$. It also stays on the line even if the students change the length and direction of arrows $\vec{a}$ and $\vec{b}$ by dragging their endpoints in the screen, thus changing the direction and position of the straight line. They can observe the relation from different viewpoints by rotating the whole objects by mouse operations.

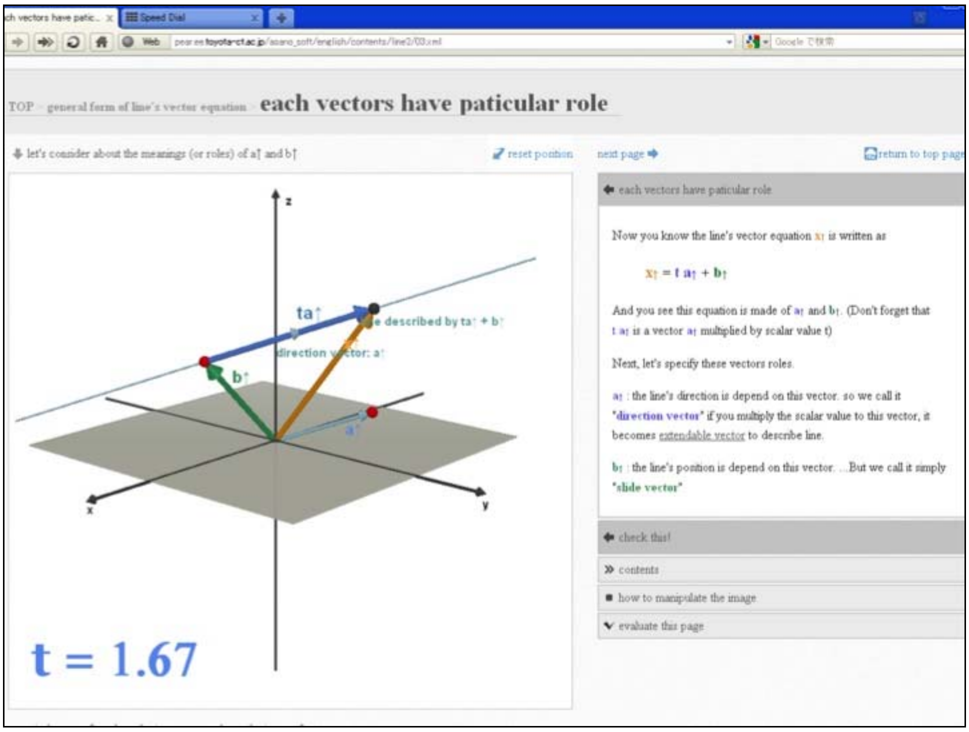


Figure 1 Example webpage embedding interactive graphic objects [4]

In this step, we use arrows to represent vectors, show their symbolic expressions, and add some explanatory texts that tell the construction of symbolic vector equations on the webpage. However, we do not expect the majority of the students to understand the relation between the graphic objects and the vector equations yet in this step. The aim of the step is rather to clarify the fact that simple graphic objects can be defined graphically with a set of arrows. We also expect our students to recognize that there are two types of vectors, position vectors and the other usual vectors during the activities.

On the fourth step, students are expected to identify the relations of graphic objects and their vector equations. A program coded with a computer algebra system MATHEMATICA [6] shows simultaneous change of graphic objects and related symbolic expressions, and let the students recognize what kind of relations they are. In the program, the students cannot move the graphic objects directly. Instead the students use *manipulators* to move the objects. When the students change the direction of a vector with the manipulator on the left side of the screen in Figure 2, the related parameters of the vector equation shown below the screen changes, and the graphic object displayed at the center of the screen also changes the position in the 3D-space simultaneously. Several buttons on the right side of the screen show special cases where the plain is perpendicular to one of the axis, and the direction vectors have zero parameters corresponded to the axis. The simultaneous changes of graphics and parameters are expected to help the students identifying the precise relation between them.

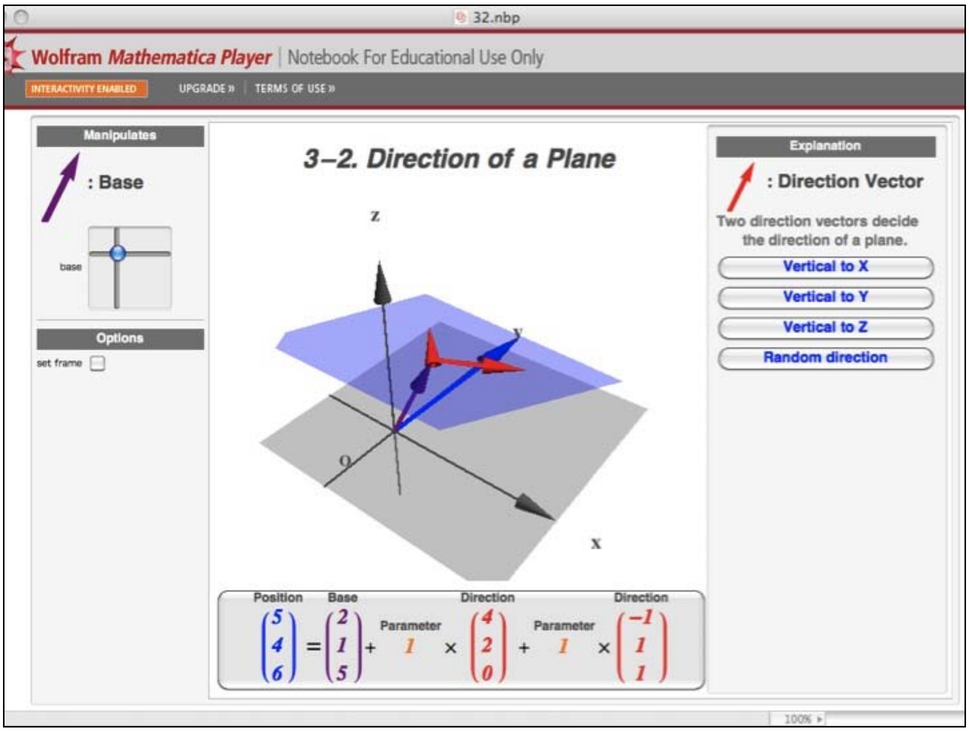



Figure 2 Example of simultaneous changing graphic objects and vector equations

At the step five, after the experience in real and virtual 3D spaces, the students learn how to express vector equations of lines and planes symbolically in traditional lectures. They learn that there are several types of symbolic expressions to express a line or a plane, and two expressions can be rewritten to each other. The lectures especially stress that a feature of a symbolic expression is linked to a certain feature of the graphic object, and the link should be explained verbally. We often reuse the 3D graphics objects already shown to the students during the experiments in virtual space of former steps. The students, who have experience of handling real and virtual graphic objects, seem to accept vector equations easier than the former students learnt with traditional lesson plan.

On the sixth step, we also set up pencil-and-paper exercises after the lectures because it is important for the students to express their own thoughts on papers if they thoroughly understand. In the exercises, they are encouraged to start from a drawing of target objects, add necessary vectors, and then express them in symbolic expressions at the final step. We forbid them to use formulas written in their textbooks as the starting point of their solutions because it hardly explains required mathematical ideas by themselves. If they need symbolic manipulations, they have to write every step of the rewriting process on their paper, too. In this step, we can tell the depth of the students' conceptual understandings from the richness of their description in the answer sheets.

The seventh step has not yet conducted for the students of this study. All the students participated in the class activities, and slower learners were invited to the out-of-class activities in the computer laboratory, where a pair of students used a computer terminal operating the software and discussing the results displayed in the screen. Teaching assistants resided in the laboratory and were ready to help if any student had difficulty operating the machine, but it seldom happened. When the students did not need help, the assistants observed the students' activities and noted their behaviors.

## 3. Preliminary result

We have been changing our lessons gradually for recent five years as shown in Table 2. The group of students who received the 2007 INCT test is the reference among experimental groups. They had only step 6: pencil-and-paper exercises of the lesson plan and did not have any activities of steps 3 and 4, which are supported by information and communication technologies (ICT).

Table 2 Learning activities done by the experimental groups

| Year of INCT test | Average at term-end test [2] | No. of students | Step 1 | Step 2 | Step 3 | Step 4 | Step 5 | Step 6 | Step 7 |
|---|---|---|---|---|---|---|---|---|---|
| 2007 (reference) | 6.4 | 45 | | | | | | ○ | |
| 2008 | 7.7 | 48 | | | | | ○ | ○ | |
| 2009 | 7.2 | 42 | | | | | ○ | ○ | |
| 2010 | 7.5 | 48 | | ○ | △ | | ○ | ○ | |
| 2011 | 9.2 | 41 | | ○ | ◇ | □ | ○ | ○ | |

○: for all the students in class, △: extra work outside the class for 1/3 of the students,

◇: extra work outside the class for 1/3 of the students with *improved* interface (right picture of Figure 4),

□: pilot activities outside the class for some students who had difficulty in pencil-and-paper exercises

The lessons for experimental groups who participated in the 2008 and 2009 INCT tests additionaly included step 5: deducing formulas. The lessons for experimental groups who participated in the 2010 and 2011 INCT tests also included step 2: real objects in classroom. One third of the students in the 2010 and 2011experimental groups, who were slow learners in these two groups, did the extra work described as step 3: observing and handling graphic objects in 3D virtual space using our webpage outside class. The interface of the webpage was improved for the 2011 group to have more sophisticated graphics and structured expanatory sentences rather than simple graphics and lengthy explanatory sentences. Some of the slow learners in the 2011 group attended additional session of step 4: identifying and producing the relation of graphic and symbolic representations.

At the term-end examinations, the 2008 and 2009 groups had higher average scores than the 2007 group to the same problem of asking the distance between a plain expressed with an equation $ax + by + cz = d$ and the origin (Table 2). The 2010 group had a fewer students who scored zero point at the test, and the 2011 group increased the average score a little more. The 2011 group's answer sheets at the term-end examination were fairly richer in description than the ones of the 2007 group. There were less blank sheets, more drawings and graphs, and more sentences. The interviews to the students of the 2011 group showed that the activities of step 3 and 4 helped some of them recognize the importance of connecting various representations, especially graphic representation to symbolic one, for the first time [2].

We measured the long-lasting effect of steps 2 to 5 of the lesson plan by the students' scores at the INCT tests (Figure 3). The tests have been conducted annually since 2007 for all the third-year students in national colleges of technology. More than 7,000 students participated in each test. The target field of this paper was "3D vectors and matrices" section of the test, and the test was conducted 10 months after the teaching and learning of 3D vectors and matrices.

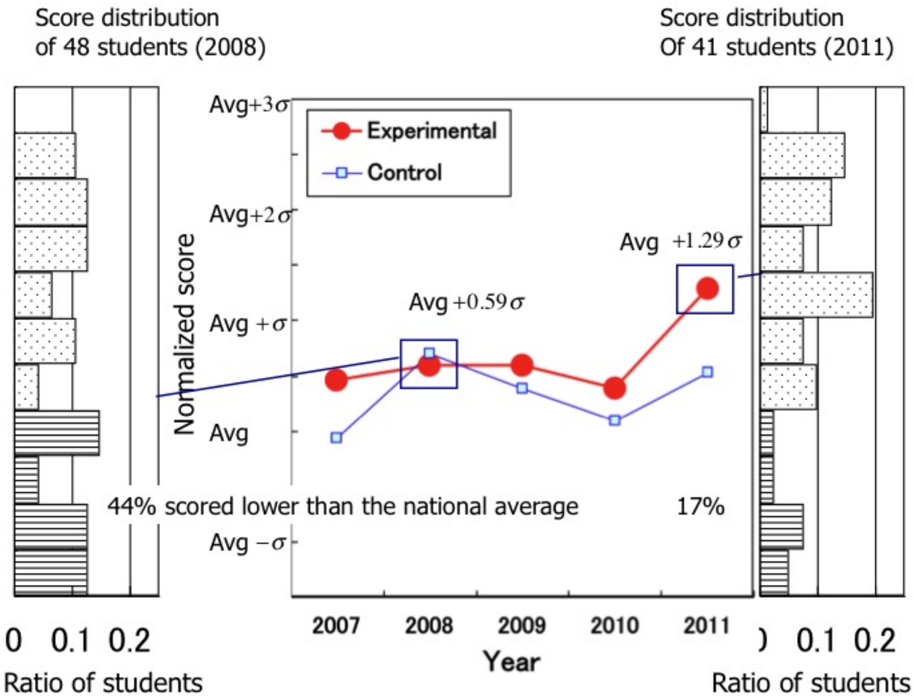



Figure 3 Students' scores of "3D vectors and matrices" in the annual INCT tests

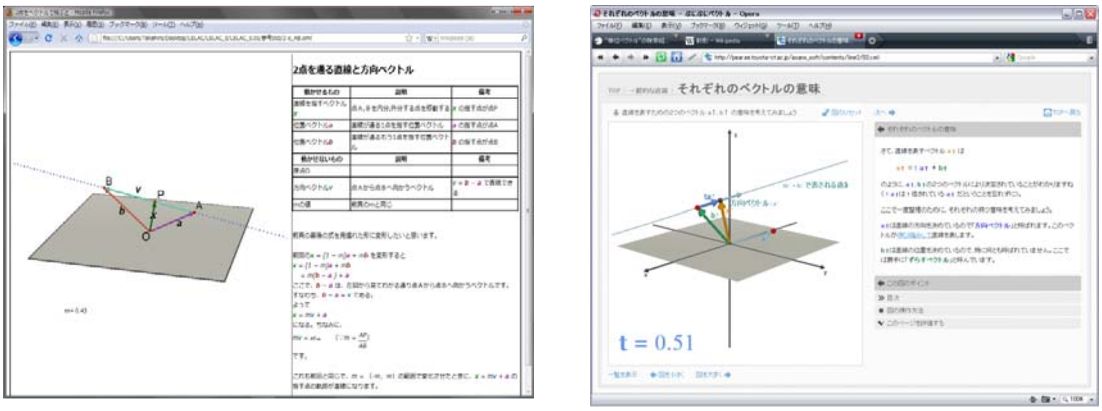
Because the average scores of all the students in the INCT tests fluctuated year to year, we normalized the scores by the average *Avg* and standard deviation $\sigma$ of all the students who participated in the test. Vertical axis of Figure 3 were normalized scores to the national average score of each year, and group averages were expressed using the standard deviation $\sigma$, for example, $Avg + \sigma$ or $Avg + 2\sigma$. The normalized scores showed the relative performance of students' groups relative to the national average in the INCT test of that year.

We compared two groups of about 40 students as the experimental group who received our new lessons and the control group who participated only in traditional lessons. The both groups belonged to the same college, and they had quite similar learning history other than the new lessons.

According to Figure 3, the average score of the 2008 experimental group was not significantly higher than the 2007 reference. It was even lower than the average score of 2008 control group, although the difference was not statistically significant. The average scores of the 2009 and 2010 experimental groups were not significantly higher than the one of the 2007 reference either. They were a little higher than the average scores of the control groups of the same years, but the differences were not statistically significant. The first average score, which was significantly higher than the one of the control group, was of 2011. The score distributions in Figure 3 showed the cause of higher average score of the 2008 experimental group. There were only 17% of the students in the 2011 experimental group who scored lower than the national average. It was a drastic decrease from 44% of the students in the 2008 experimental group. There were fewer students who could not answer to any problems in "3D vectors and matrices" field in the 2011 experimental group.

## 4. Discussion

The result of the INCT tests in 2008 and 2009 suggests that the knowledge learnt through deducing formulas and pencil-and-paper exercises had relatively shorter lifetime. Although those activities increased the students' average scores in the term-end examinations as shown in Table 2, they did not affect the average scores much in the INCT tests, which took place 10 months later the term-end examinations. It is possible that the procedural knowledge could not survive the 10-months period if it was not understood conceptually by closely linked to graphical representations.




Original interface used for 2010 group    Improved interface used for 2011 group

Figure 4 Example webpage for observing graphic objects in 3D virtual space

The result of the 2010 and 2011 INCT tests conflict each other. In both years, the students of experimental groups participated in the activities of steps 2 and 3 in the lesson plan. However, the students of the 2010 group scored not so highly at the INCT test, and the students of the 2011 group scored significantly higher. The difference might be caused by the quality of activities in step 3, which use the interactive webpage (Figure 4). Both pages of Figure 4 had been embedded with Cabri-3D graphic objects that could be modified interactively and explanatory sentences. However, the pages for the 2011 group had shorter but more structured explanatory sentences, and their graphic objects had more sophisticated interactions.

For example, the arrow $\vec{x}$, which represents any point on the line, in the 2011 page (right picture of Figure 4) is easier to distinct from other arrows $\vec{a}$ and $\vec{b}$ than in the 2010 page (left picture of Figure 4). In the 2010 page, the endpoint of $\vec{x}$ can be moved directly with a mouse operation although its movement is restricted to sliding along the line and the movement does not change the position or direction of the line. However, the arrow $\vec{x}$ looks quite similar to the other arrows $\vec{a}$ and $\vec{b}$, which do change the position or direction of the line. Careless students may not recognize the difference between them. In the 2011 page, the arrow $\vec{x}$ cannot be moved directly. Its position can be moved only by changing the arrows $\vec{a}$ and $\vec{b}$, which change the position or direction of the line, or the value of parameter $t$, which slides the location of the endpoint of $\vec{x}$ along the line. Passive nature of the arrow $\vec{x}$ is more obvious to the students in the 2011 page.

The 2010 page depends more deeply on explanatory sentences. They are quite similar to the description of textbooks in explaining the feature of graphics in detail but they are lengthy and need some time to grasp what they tell. Very few students read the sentences through. The 2011 page depends more explanation on the graphics. The sentences are shorter and work auxiliary. If the students grasp the graphical relation between the arrows, the sentences are not necessarily read.

The decreased number of low performers in the 2011 experimental group at INCT test implies that the students learnt more meaningfully and the knowledge lasted longer by adding ICT-assisted activities to the lessons when the activities were well organized. Although we do not think that the improved students' performance is the direct result of the ICT-assisted activities. It is more likely that the students learnt symbolic procedures more meaningfully by connecting them with related graphic features because ICT-assisted activities gave the students a big picture before the learning of detailed symbolic manipulations. The ICT-assisted activities are not worth by themselves, but they enrich the traditional learning and paper-and-pencil exercises of steps 5 and 6, and the combined learning must deepen the students' conceptual understandings.

Additionally, the improved scores at INCT test has been observed only once in 2011 and may be caused by other temporal causes. We have to confirm the effectiveness of adding ICT-assisted activities further in the 2012 and later INCT tests. And we also intend to examine the effectiveness of adding step 1: actual application as the introduction to the lessons in the 2013 and later INCT tests.

The last point we would like to discuss is that students' experience in concrete examples might be more important than we thought in learning abstract mathematical concepts. This paper discussed the importance of students' experience in the learning of 3D linear algebra, and Viholainen [7] points the close relation between formal (abstract) reasoning and informal (concrete) interpretation of Finnish mathematics teacher students in derivative. There might be more fields other than 3D linear algebra where ICT-assisted students' activities is beneficial to add before the traditional lessons.

## 5. Conclusion

The effectiveness of adding ICT-assisted activities is evaluated in this paper. The students' description became richer in the answer sheets of their term-end examination when some of the students had experienced ICT-assisted experience of handling 3D graphic objects and relating them with symbolic expressions before the traditional lessons. The students' scores in the 3D vectors and matrices part increased significantly at the 2011 INCT achievement test. These results imply that the students' procedural knowledge remained after a blank period of 10 months after the learning when they learnt 3D vector equations with the close relation to the features of graphic objects and confirmed the strong connection between them, and thus deepened conceptual understanding. With the help of ICT, the students could have a clear image of 3D vector equations and their relation to graphic objects before they learn the procedural knowledge of symbolic manipulations and exercise with pencil and paper, thus have more meaningful learning.

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**References**

[1] H. Nishizawa & T. Yoshioka, A Proposal to Teach 3D Vector Operations in a Role-Playing Game, *Proceedings of the 13th Asian Technology Conference in Mathematics*, 2008, 364-369.

[2] H. Nishizawa, B. Zgraggen, & T. Yoshioka, A system for helping concept-building of 3D linear algebra by connecting graphic, symbolic, and verbal representations, *The Electronic Journal of Mathematics and Technology*, 2,1, 2009, 54-62.

[3] L. Haapasalo, The conflict between conceptual and procedural knowledge: Should we need to understand in order to be able to do, or vice versa?, *Proceedings on the IXX Symposium of the Finnish Mathematics and Science Education Research Association, University of Joensuu, Bulletins of the Faculty of Education 86*, Joensuu, 2003, 1-20.

[4] Hypermedia Laboratory of Toyota National College of Technology, Punipuni Vector [Interactive Web-page]. Available at http://pear.ee.toyota-ct.ac.jp/asano_soft/english/ on May 25, 2011.

[5] Cabrilog Sas., Cabri: maths software for students [Web page]. Retrieved from http://www.cabri.com/ on May 25, 2011.

[6] Wolfram Research, Mathematica [Web page]. Retrieved from http://www.wolfram.com/ on May 25, 2011.

[7] A. Viholainen, Finnish mathematic teacher students' informal and formal arguing skills in the case of derivative, *Nordic Studies in Mathematics Education*, 13, 2, 2008, 71-92.