Dynamic Construction of the Common Perpendiculars in the Three-sphere

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Abstract: *Construction problems are good exercises to understand geometry deeply. Using dynamic geometry software, we can "easily" check whether some conjecture is true or false. In this paper, we introduce a construction of the common perpendiculars to two great circles in the three-dimensional sphere. We will see that the foci of some hyperbola play an important role in the construction.*

1. Introduction

The stereographic projection ([1, page 260], [2, page 74]) is a very important map in mathematics. It maps a sphere minus one point (the north pole N) to the plane containing the equator by projecting along lines through N as in Figure 1.1. This projection preserves angles and maps a circle on the sphere to a circle on the plane. In particular, a great circle on the sphere is projected to a circle passing through two antipodal points (E and $-E$ in Figure 1.1) on the equator. This projection enables us to draw spherical objects on a plane. In this sense, the unit circle is regarded as the equator, and a circle passing through two antipodal points on the unit circle is regarded as a great circle.

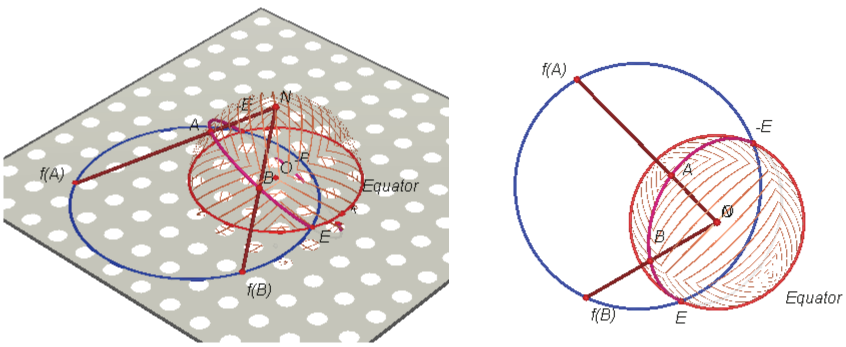

Figure 1.1 The stereographic projection (left) and its top view (right).

In the same way, the stereographic projection from the north pole of the three-dimensional sphere $\mathbf{S}^3$ onto the three-dimensional Euclidean space $\mathbf{E}^3$ enables us to draw spherical objects in $\mathbf{E}^3$. In this sense, the unit sphere is regarded as the equator (geodesic plane) of $\mathbf{S}^3$, and a circle passing through two antipodal points on the unit sphere is regarded as a great circle.

In this paper, we will introduce a construction of the common perpendiculars to two great circles in $\mathbf{S}^3$. We can realize this construction in $\mathbf{E}^3$ as the stereographic projection of $\mathbf{S}^3$. As in Figure 1.2, our start line is a pair of great circles g_1 and g_2 (blue circles passing through two antipodal points on the unit sphere $\mathbf{S}^2$). In fact, there are *two* common perpendiculars in general. This fact is a remarkable difference between $\mathbf{S}^3$ and three-dimensional hyperbolic space $\mathbf{H}^3$ (see, [3]), or $\mathbf{E}^3$. Our goal is to construct the common perpendiculars (green circles passing through two antipodal points on $\mathbf{S}^2$) as in Figure 1.2.

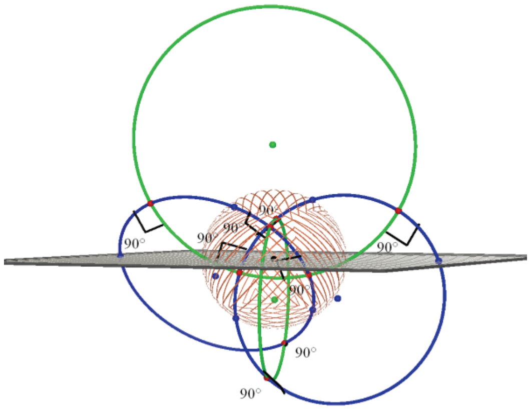

Figure 1.2 Two great circles (blue) and the common perpendiculars (green).

For the construction, we will see that the foci of a hyperbola in two-dimensional Euclidean geometry play a very important role. In Section 2, several properties of the common perpendiculars are introduced which lead a way to the construction. In Section 3, we review an important property of the foci of a hyperbola in $\mathbf{E}^2$ which is useful to find the foci of a hyperbola. In Section 4, the construction of the common perpendiculars is introduced. As an appendix, we introduce a simple construction of the tangent lines through a point on the non-focal axis of a hyperbola. This technique is used in a construction in Section 4. All pictures in this paper are drawn by dynamic geometry software *Cabri II plus* and *Cabri 3D*. With this type of software, we can intuitively find out that our constructions are correct.

2. The Properties of the common perpendiculars

2.1 Two common perpendiculars

For the construction of the common perpendiculars, we have to study their several properties. Let us start from a special case; one of the common perpendiculars is a Euclidean line h as in Figure 2.1. In this case, the Euclidean center C_1 of the great circle g_1 lies on h, and also the center C_2 of the great circle g_2 lies on h. Let $h^{\perp}$ be the conjugate great circle of h which is the set of points with distance $\pi/2$ from h. $h^{\perp}$ is given as the intersection of a pair of orthogonal great spheres including h. For example, a pair of orthogonal great spheres are the unit sphere and the plane perpendicular to h through the origin O in this case. Notice that $h^{\perp}$ is another common perpendicular of g_1 and g_2. In this way, there are two common perpendiculars in general.

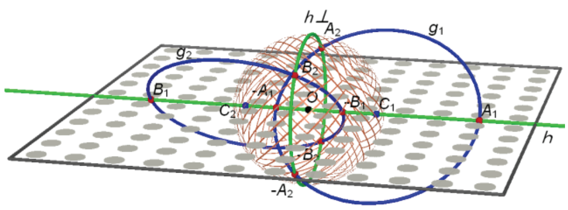

Figure 2.1 Common perpendiculars: h and $h^{\perp}$.

2.2 From two geodesic circles to four geodesic spheres

Figure 2.1 also shows that g_1 is equally divided into four parts by h and $h^{\perp}$ with the original metric in $\mathbf{S}^3$. Let A_1 and $-A_1$ be the intersections of g_1 and h. Let A_2 and $-A_2$ be the intersections of g_1 and $h^{\perp}$. Then the lengths of the four arcs are the same:

$$A_1A_2 = A_2(-A_1) = (-A_1)(-A_2) = (-A_2)A_1 = \pi/2.$$

Let α be the great sphere containing g_1 and h, and also let $\alpha^{\perp}$ be the great sphere containing g_1 and $h^{\perp}$:

$$\alpha = g_1 \cup h, \quad \alpha^{\perp} = g_1 \cup h^{\perp}, \quad g_1 = \alpha \cap \alpha^{\perp}.$$

Then, α and $\alpha^{\perp}$ intersect orthogonally, and with these two great spheres g_2 is equally divided into four parts:

$$\{B_1, -B_1\} = \alpha \cap g_2, \quad \{B_2, -B_2\} = \alpha^{\perp} \cap g_2.$$

In the same way, let β be the great sphere containing g_2 and h, and also let $\beta^{\perp}$ be the great sphere containing g_2 and $h^{\perp}$:

$$\beta = g_2 \cup h, \quad \beta^{\perp} = g_2 \cup h^{\perp}, \quad g_2 = \beta \cap \beta^{\perp}.$$

Then, β and $\beta^{\perp}$ intersect orthogonally, and with these two great spheres g_1 is equally divided into four parts:

$$\{A_1, -A_1\} = \beta \cap g_1, \quad \{A_2, -A_2\} = \beta^{\perp} \cap g_1.$$

Furthermore, notice that $\alpha \perp \beta^{\perp}$, because α contains h and $\beta^{\perp}$ contains $h^{\perp}$. In the same way, $\alpha^{\perp} \perp \beta$.

Therefore, the construction problem is reduced to find out two pairs of orthogonal great spheres, $\alpha, \alpha^{\perp}, \beta$ and $\beta^{\perp}$ such that

1) $\alpha \cap \alpha^{\perp} = g_1, \quad \beta \cap \beta^{\perp} = g_2,$

2) $\alpha \perp \beta^{\perp}, \quad \alpha^{\perp} \perp \beta,$

and then the common perpendiculars h and $h^{\perp}$ are given as $h = \alpha \cap \beta$ and $h^{\perp} = \alpha^{\perp} \cap \beta^{\perp}$.

2.3 From two spheres to two points

Let us consider a special case that g_2 is a Euclidean line through the origin O in the following argument until Section 4. Let a be the axis of the Euclidean circle g_1, i.e., a is perpendicular to the plane containing g_1 through C_1 which is the center of g_1 as in Figure 2.2. Note that the desired great spheres α and $\alpha^{\perp}$ have their centers on a, and the desired great spheres β and $\beta^{\perp}$ are Euclidean planes containing g_2. If a is parallel to g_2, it is easy to construct the common perpendiculars, so in the following argument let us assume that a is not parallel to g_2.

For a while, let us guess the desired four great spheres $\alpha, \alpha^{\perp}, \beta$ and $\beta^{\perp}$. Let S_1 on a be the center of a sphere α_1 containing g_1. Let H_1 on g_2 be the point such that $S_1H_1 \perp g_2$. Let β_1 be the plane through H_1 perpendicular to S_1H_1. Then, the perpendicular great sphere $\beta_1^{\perp}$ is the plane containing g_2 through S_1. Notice that $\alpha_1 \perp \beta_1^{\perp}$, and the great circle $h_1 = \alpha_1 \cap \beta_1$ is perpendicular to g_2 centered at H_1, however, note that h_1 is not perpendicular to g_1 in general.

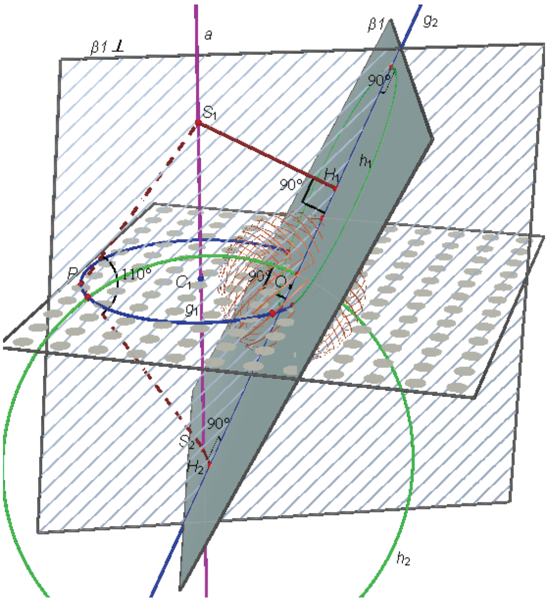

Figure 2.2 Special case.

Let S_2 on a be the intersection of β_1 and a. Let H_2 on g_2 be the point such that $S_2H_2 \perp g_2$. Let α_2 be the great sphere containing g_1 centered at S_2. Notice that $\alpha_2 \perp \beta_1$, and the great circle $h_2 = \alpha_2 \cap \beta_1^{\perp}$ is perpendicular to g_2 centered at H_2, however, note that h_2 is not perpendicular to g_1 in general.

If $\alpha_1 \perp \alpha_2$, or equivalently, $\angle S_1PS_2 = \pi/2$ for any point P on g_1, then $\alpha_1, \alpha_2, \beta_1$ and $\beta_1^{\perp}$ are the desired great spheres and the great circles $h_1 = \alpha_1 \cap \beta_1$ and $h_2 = \alpha_2 \cap \beta_1^{\perp}$ are the desired common perpendiculars of g_1 and g_2. The angle $\angle S_1PS_2$ is not a right angle in general as in Figure 2.2, therefore, in the following argument, let us focus on the construction of the two points S_1 and S_2 on a.

2.4 From a hyperboloid of one sheet to a hyperbola

Let us consider to reduce the construction in three-dimensional Euclidean space to that in two-dimensional Euclidean plane. To do this, let us consider a hyperboloid of one sheet for a while. Let $Q(g_2, a)$ be a hyperboloid of one sheet determined by the axis a and the generator g_2 rotating around a. Then, the Euclidean plane β_1 is the tangent plane of $Q(g_2, a)$ at H_1. To see this, let g_2' be the reflection of g_2 with respect to the plane $a - H_1$ containing a and H_1. Note that g_2' which lies on β_1 is another generator of the double ruled surface $Q(g_2, a)$. Since the tangent plane at H_1 is the plane containing two generators g_2 and g_2', we have proved that the tangent plane at H_1 is the plane β_1.

Let $q(g_2, a)$ be the intersection of $Q(g_2, a)$ and $a - H_1$. $q(g_2, a)$ is a hyperbola on the plane $a - H_1$, and the line S_1H_1 is the normal line of $q(g_2, a)$ at H_1, and the line S_2H_1 is the tangent line of $q(g_2, a)$ at H_1. Now the construction problem is reduced to the next one (Figure 2.3):

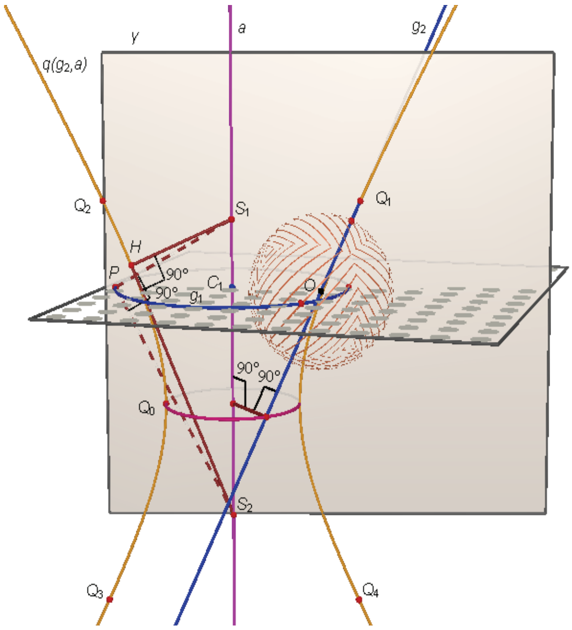

Figure 2.3 Two dimensional construction problem.

Construction problem 1. For the hyperboloid of one sheet $Q(g_2,a)$ and any plane γ containing the axis a, let $q(g_2,a)$ be the intersection of $Q(g_2,a)$ and γ. Let P be one of the intersections of g_1 and γ. Construct two points S_1 and S_2 on a which satisfy the following conditions:

1) $\angle S_1PS_2 = \pi/2$.
2) There exists a point H on $q(g_2,a)$ such that S_1H is normal to $q(g_2,a)$ at H, and S_2H is tangent to $q(g_2,a)$ at H.

In this way, the construction problem of the common perpendiculars in $\mathbf{S}^3$ is completely reduced to a problem in the two-dimensional Euclidean geometry.

3. The Foci of a hyperbola

In this section, let us solve Construction problem 1 proposed in the previous section. The next proposition plays an important role.

Proposition 1 (tangent and perpendicular lines). Let q be a hyperbola with the focal axis a_1 and the non-focal axis a_2 as in Figure 3.1. For any circle c passing through the foci F_1 and F_2, let S_1 and S_2 be the intersections of c and a_2. If an intersection H of q and c is in the same region as S_1 with respect to a_1, then S_1H is the normal line, and S_2H is the tangent line to q at H.

Proof. Since the arc F_1S_2 is the same as the arc F_2S_2 in length, $\angle F_1HS_2 = \angle F_2HS_2$. Hence, S_2H is the angular bisector of the angle $\angle F_1HF_2$ which implies that S_2H is the tangent line to q at H (see, [2, page 95]). In addition, since $\angle S_1HS_2 = \pi/2$, S_1H is the normal line to q at H. This completes the proof. ■

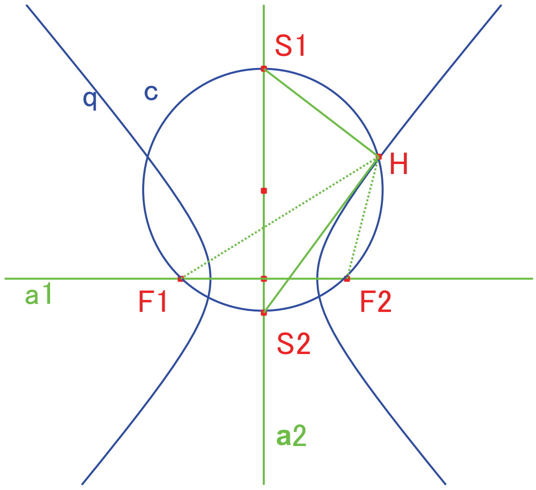

Figure 3.1 Tangent and normal lines.

Proposition 3.1 introduces the following construction of the foci of a hyperbola with axes which is used in the next section.

Construction 1 (foci of a hyperbola).

0. (Input) Hyperbola q with the focal axis a_1 and the non-focal axis a_2.
1. Any point S_1 on a_2.
2. Point H on q such that S_1H is tangent to q at H (by Construction A in Appendix).
3. Point S_2 on a_2 such that $S_1H \perp S_2H$.
4. Circle c such that S_1S_2 is a diameter of c.
5. (Output) Foci F_1 and F_2, the intersections of c and a_1.

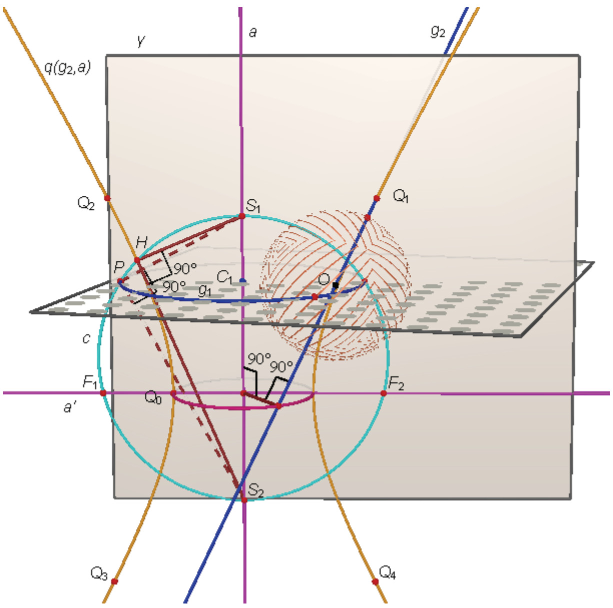

Figure 3.2 Solution of Construction problem 1.

Now we are ready to solve Construction problem 1 as follows (Figure 3.2):

Construction 2 (Solution of Construction problem 1).

0. (Input) Hyperbola $q(g_2,a)$ with the non-focal axis a and the focal axis a', and the fixed point P.
1. Foci F_1 and F_2 of $q(g_2,a)$ on a' (by Construction 1).
2. Circle c through F_1, F_2 and P.
3. (Output) Points S_1 and S_2, the intersections of c and a.

4. Construction of the common perpendiculars in S^3

Combining Constructions 1, 2 and A, we can construct the common perpendiculars in the special case (Figure 4.1):

Construction 3 (Common perpendiculars in the special case).

0. (Input) Circle g_1 centered at C_1, and Euclidean line g_2.
1. Line a, the axis of g_1.
2. Any point P_1 on g_1.
3. Plane γ_1 containing a and P_1.
4. Common perpendicular ℓ of two lines a and g_2.
5. Point P_2, the intersection of a and ℓ.
6. Point P_3, the intersection of g_2 and ℓ.
7. Plane γ_2 perpendicular to a through P_2.
8. Circle c_1 on γ_2 centered at P_2 through P_3.
9. Point Q_0, one of the intersections of c_1 and γ_1.
10. Focal axis a' through P_2 and Q_0.
11. Point Q_1, the intersection of g_2 and γ_1.
12. Points Q_2, Q_3 and Q_4, the symmetric points of Q_1 with respect to a, P_2 and a'.
13. Hyperbola $q(g_2,a)$ through Q_0, Q_1, Q_2, Q_3 and Q_4 on γ_1.
14. Foci F_1 and F_2 of $q(g_2,a)$ on a' (by Construction 1).
15. Circle c_2 on γ_1 through F_1, F_2 and P.
16. Points S_1 and S_2, the intersections of c_2 and a.
17. Sphere α centered at S_1 through P.
18. Sphere $\alpha^{\perp}$ centered at S_2 through P.
19. Plane β containing g_2 through S_2.
20. Plane $\beta^{\perp}$ containing g_2 through S_1.
21. (Output 1) Common perpendicular h, the intersection of α and β.
22. (Output 2) Common perpendicular $h^{\perp}$, the intersection of $\alpha^{\perp}$ and $\beta^{\perp}$.

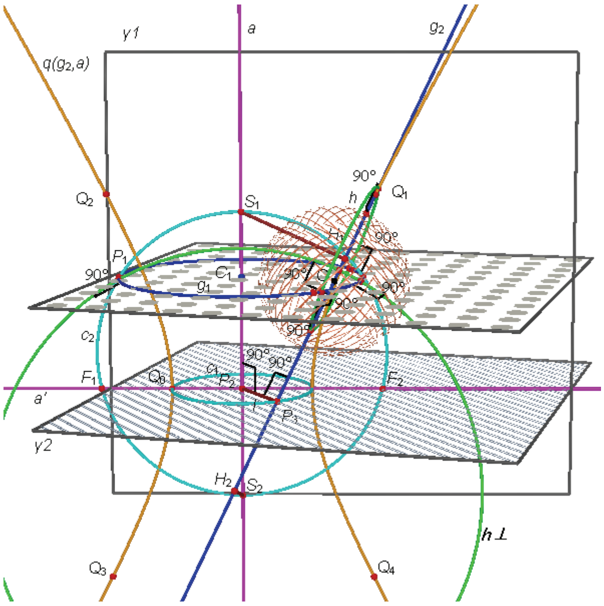

Figure 4.1 Construction of the common perpendiculars (special case).

Finally, let us introduce a construction of the general case in which both great circles g_1 and g_2 are Euclidean circles (Figure 4.2).

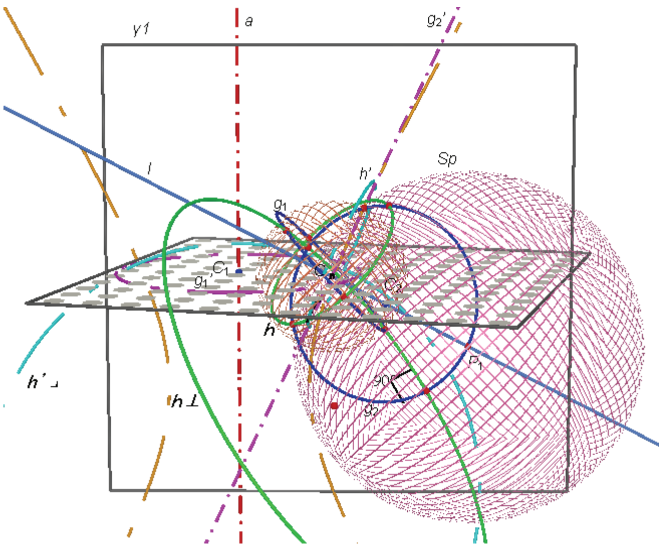

Figure 4.2 Construction of the common perpendiculars (general case).

Construction 4 (Common perpendiculars in the general case).

0. (Input) Circle g_1, and circle g_2 with the center C_2.
1. Line ℓ through O and C_2.
2. Point P_1, one of the intersections of ℓ and g_2.
3. Sphere Sp centered at P_1 through the intersections of g_2 and the unit sphere.
4. Line g_2', the inversion of g_2 with respect to Sp.

5. Circle g_1', the inversion of g_1 with respect to Sp.
6. Common perpendiculars h' and $h'^{\perp}$ of g_1' and g_2' (by Construction 3).
7. (Output) Common perpendiculars h and $h^{\perp}$, the inversions of h' and $h'^{\perp}$ with respect to Sp.

5. Conclusion

In this paper, we have introduced a construction of the common perpendicular in $\mathbf{S}^3$. There are two common perpendiculars, hence the construction is more complicated. We started from the properties of the common perpendiculars. Through the following steps;

1) two pairs of orthogonal great spheres $\alpha, \alpha^{\perp}, \beta$ and $\beta^{\perp}$,
2) two points S_1 and S_2,
3) hyperboloid of one sheet $Q(g_2, a)$,
4) hyperbola $q(g_2, a)$,

we finally reduced the construction problem in $\mathbf{E}^3$ to a construction problem in $\mathbf{E}^2$, that is, Construction problem 1 in Subsection 2.4. Construction problem 1 was solved by using a simple property of the foci of a hyperbola. Dynamic geometry software supported the author throughout this research. We can say that it is very hard to do this kind of researches without dynamic geometry software. We can also say that three-dimensional dynamic geometry software is useful for the study of $\mathbf{S}^3$ as well as that of $\mathbf{E}^3$ and $\mathbf{H}^3$.

6. Appendix

In this appendix, let us introduce a simple construction of the tangent lines through a point on the non-focal axis of a hyperbola. Let q be a hyperbola with the non-focal axis a (foci lies on the focal axis) as in Figure 6.1. For any point A on a, the construction of the tangent lines to the hyperbola through A is as follows:

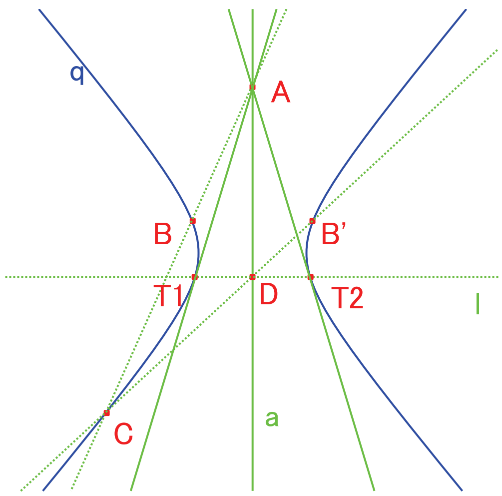

Figure 6.1 Tangent lines to a hyperbola.

Construction A(tangent lines to a hyperbola).

0. (Input) Point A on a.

1. Points B and C on q such that A, B and C are collinear.
2. Point B', the reflection of B with respect to a.
3. Point D, the intersection of line $B'C$ and a.
4. Line ℓ perpendicular to a at D.
5. Points T_1 and T_2, the intersections of ℓ and q.
6. (Output) Tangent lines AT_1 and AT_2.

For more general construction of the tangents to a smooth conic, see [2, page 150].

References

[1] Berger, M. (1987). *Geometry II.* Berlin Heidelberg, Germany: Springer-Verlag.
[2] Jennings, G. (1994). *Modern Geometry with Applications.* Springer-Verlag New York, Inc.
[3] Maeda, Y. (2010). *Construction of common perpendicular in hyperbolic space.* Proceedings of the Fifteenth Asian Technology Conference in Mathematics, pp.210-218.