How to Double the Rate of Discovery with Cabri 3D?

Jen-Chung Chuan

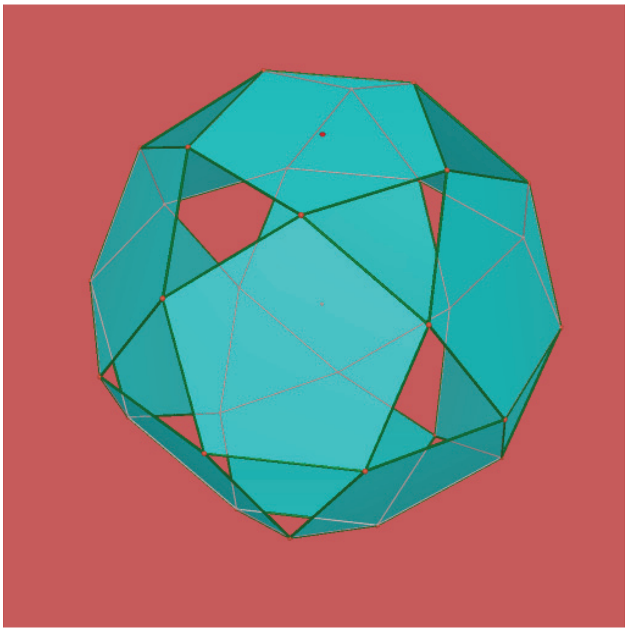
jcchuan@gmail.com

Abstract *Joseph Gergonne and Jean-Victor Poncelet independently noted that given any theorem or definition in projective geometry, substituting point for line, lie on for pass through, collinear for concurrent, intersection for join, or vice versa, results in another theorem or valid axiom, the "dual" of the first. The principle of duality has thus doubled the rate of discovery. In this paper we are to examine what other duality principles are doing the same when using Cabri 3D as a tool of discovery.*

Introduction

There are two ways to increase the efficiency of a mathematical experiment with the computer. One way is to choose a better computing hardware and software. (This is often stated in the research proposal.) The other goes by way of the clever thinking. (This is often found in a mathematics conference.) We now describe seven methods of doubling the rate of discovery with Cabri 3D by making uses of the duality principles in various contexts.

The Seven Methods



12 Pentagons of an Icosidodecahedron

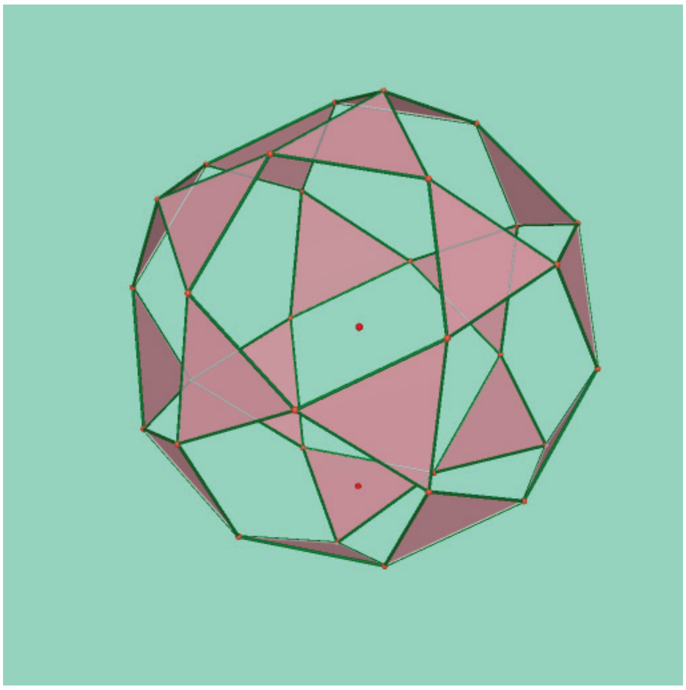
Method 1

Consider the following problem: Constr uct all 12 pentagons of an icosidodecahedron.

Here is one solution involving line reflections only (see the attached file: Pentagonal Faces of an Icosidodecahedron.cg3): Fix one face of a dodecahedron. There are exactly five edges each having exactly one vertex adjacent to the pentagon. By taking the reflections of the pentagon with respect to the edges we obtain the second generation of five pentagons.

After taking the reflections of the second generation pentagons with respect to the same edges, we obtain five “new” third generation pentagons. The process will terminate when the 4^{th} generation pentagon is constructed. The result is a display of 12 identical pentagons in space forming the shape of an icosidodecahedron.

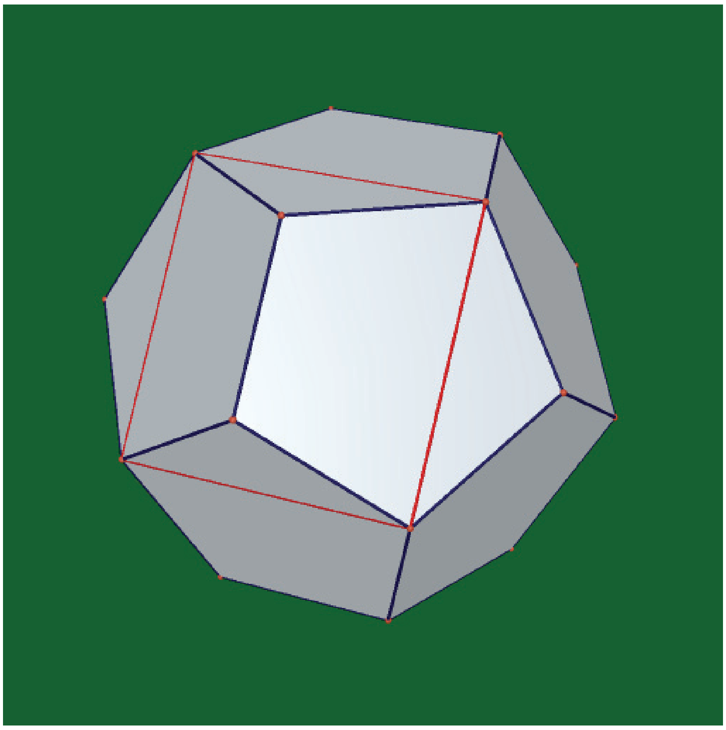
The game, however, is NOT over!



20 Triangles of an Icosidodecahedron

The Principle of Duality now leads us to the bonus part of the discovery: Starting with a regular icosahedron, let us explore the “same” process repeatedly to a fixed face. The result (see the attached file: Triangle Faces of an Icosidodecahedron.cg3) is an aesthetic display of 20 regular triangles of the same size in space forming the shape of an icosidodecahedron.

DUAL PRINCIPLE #1: ALWAYS EXPLORE THE DUAL OBJECT.

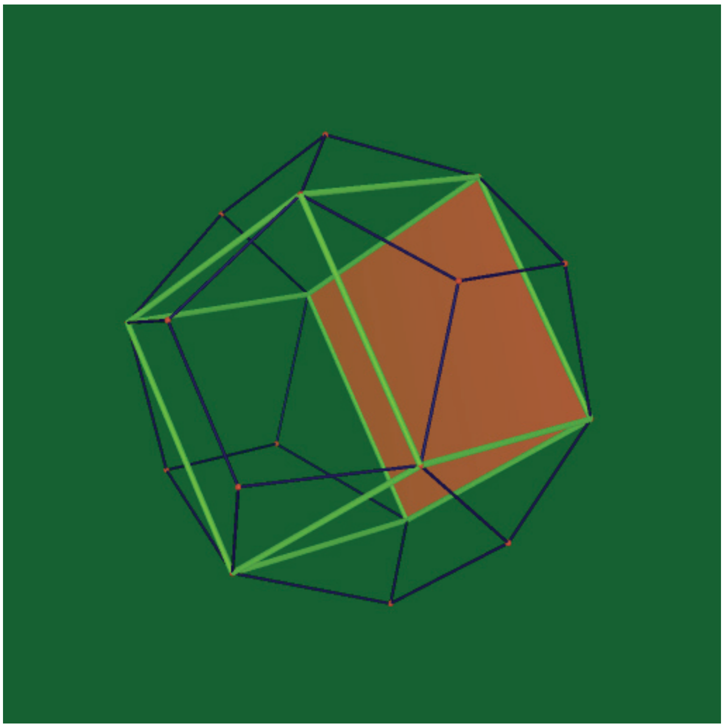


Square on Dodecahedron

Method 2.

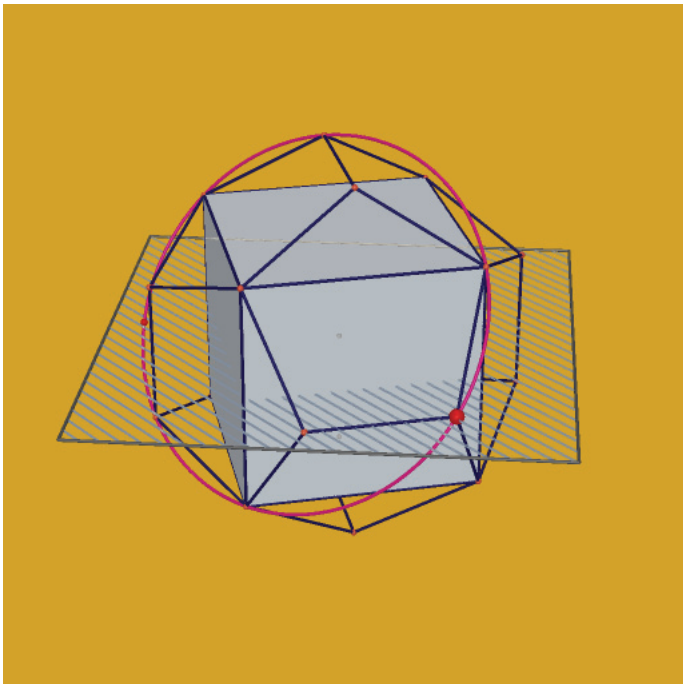
Problem: Find the largest cube contained in a regular dodecahedron.

Solution: Fix an edge. There are four vertices adjacent to the edge. These four vertices lie on the corners of a square. (See the attached file: Square on Dodecahedron.cg3)



Largest Cube in Dodecahedron

The cube erected on the “correct” side with this square as the base will be the desired cube. (See the attached file: Largest Cube in Dodecahedron.cg3)

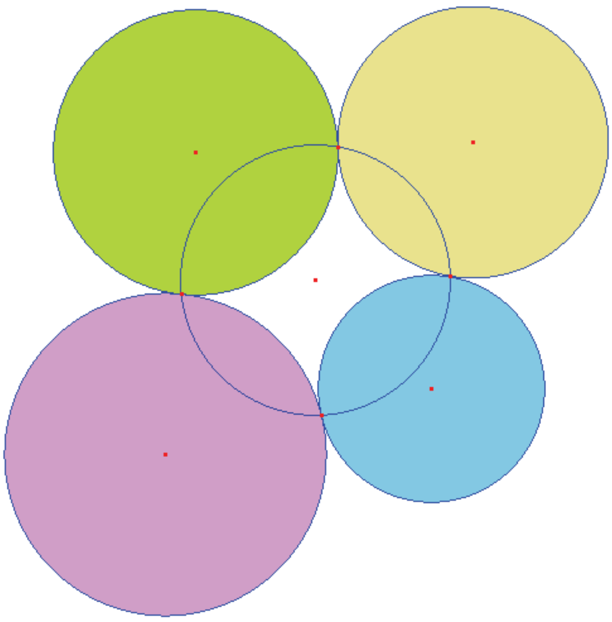


Dual Construction:

Find the smallest dodecahedron containing a given a cube. Here is the procedure: find any point of intersection of any perpendicular bisector of any side of the cube with the circumcircle of any equilateral triangle whose vertices are taken from those of the cube. The desired dodecahedron is formed by taking the convex hull of the cube with this and 11 other similarly constructed points (see the attached file: Smallest Dodecahedron Containing Cube.cg3)

Smallest Dodecahedron Enclosing a Cube

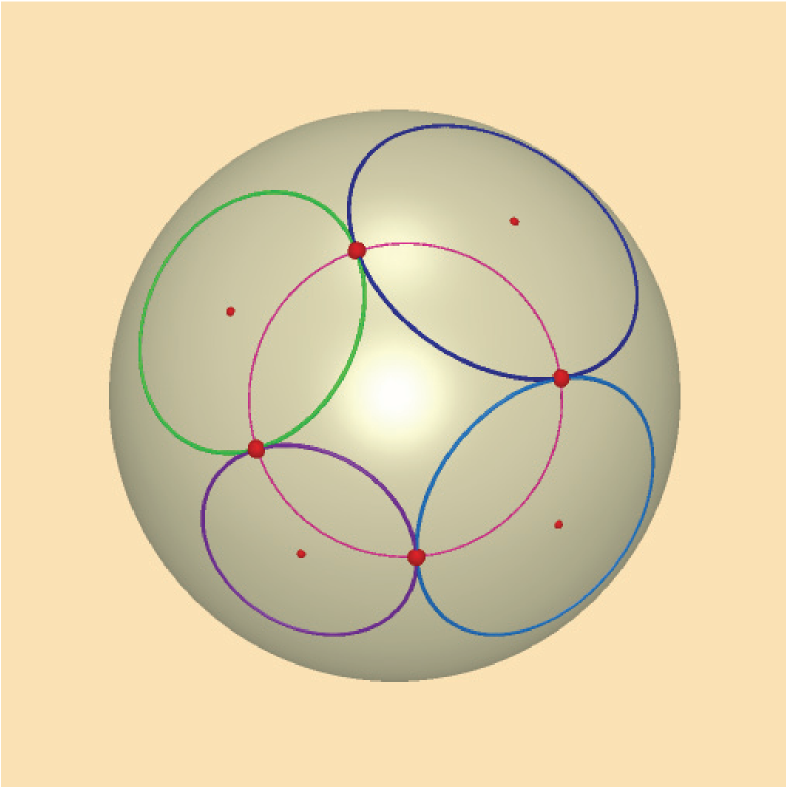
DUAL PRINCIPLE #2: ALWAYS CONSIDER THE CONVERSE CONSTRUCTION BY SWITCHING THE "GIVEN" PART WITH THE "TO CONSTRUCT" PART.



Method 3. Take any chain of four circles in the plane so each touches two others.

The configuration displays the surprising fact that the four points of tangencies lie on a circle (see the attached file: Concyclic Property.fig).

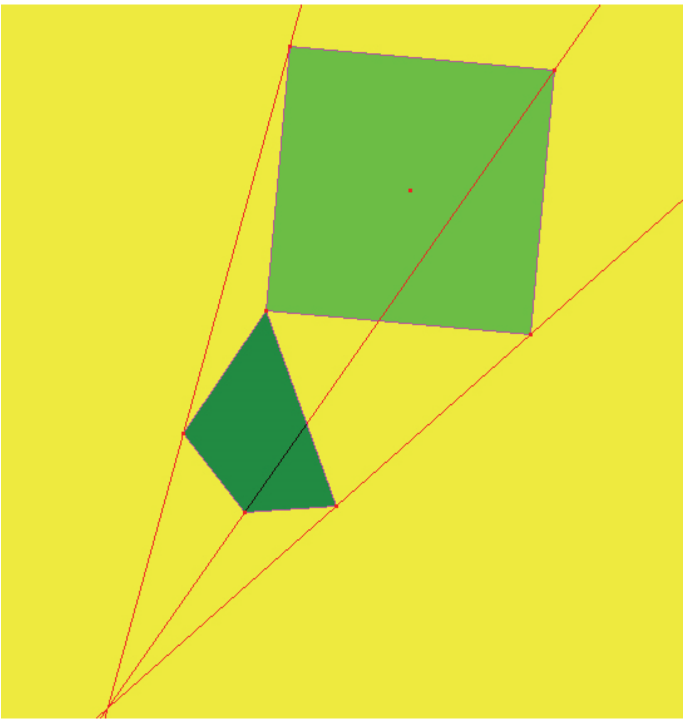
Concyclic Property of Points of Contact in the Plane



Now fix any sphere tangent to the plane containing the circles. After taking the inversion of the circles with respect to the sphere, a theorem in spherical geometry is revealed (see the attached file: Concyclic Property.cg3): for any chain of four circles on the sphere so each touches two other, the four points of tangencies also lie on a circle!

Concyclic Property of Points of Contact on Sphere

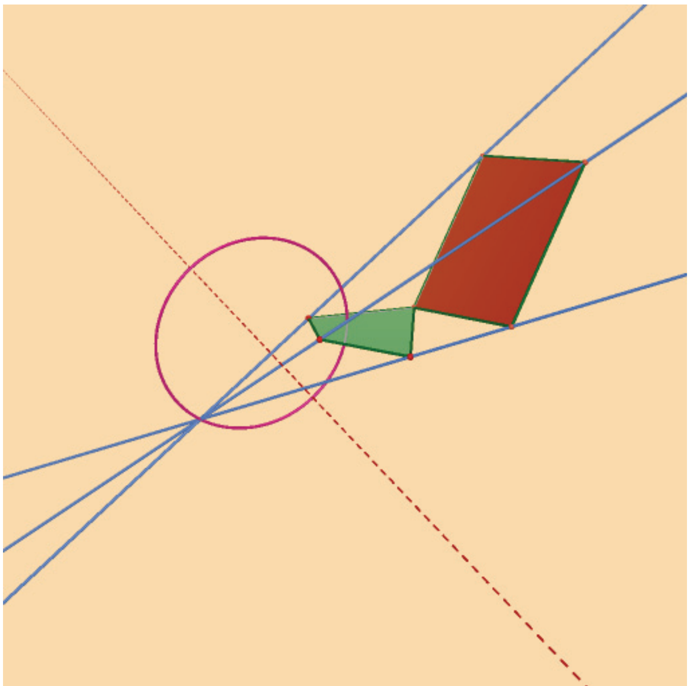
DUAL PRINCIPLE #3: IT IS PROFITABLE TO USE INVERSION AS A DICTIONARY TO TRANSLATE A RESULT IN PLANE GEOMETRY INVOLVING CIRCLES ONLY, INTO A RESULT IN SPHERICAL GEOMETRY.



Every Quadrilateral Can be Projected on a Square, Planar Case

Method 4. Problem: What quadrilateral can be projected onto a square?

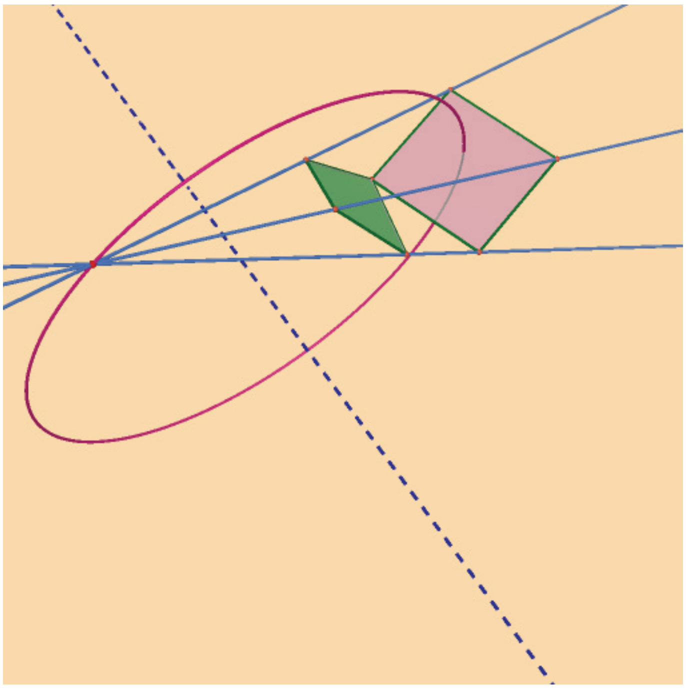
Solution: The power of elementary mathematics is vividly displayed in the classic work on elementary mathematics “Triumph der Mathematik” (English translation: 100 Great Problems of Elementary Mathematics). The construction process is implemented in the attached Cabri II Plus file Every Quadrilateral is Perspective to a Square.fig.



Every Quadrilateral Can be Projected on a Square in Space

Since the natural setting for projective geometry is 3D instead of 2D, it is natural to see if the construction can be carried out in space. To our surprise, we have discovered that the slightly modified construction leads to not one single perspectivity, but an animation displaying a whole family of such correspondences between the quadrilateral and squares with all view points lying on a circle. (See the attached file: Every Quadrilateral is Perspective to a Square.cg3)

DUAL PRINCIPLE #4: EMBED THE OBJECT IN A SPACE ONE DIMENSION HIGHER.



Exchanging the Roles of Stationary and Moving

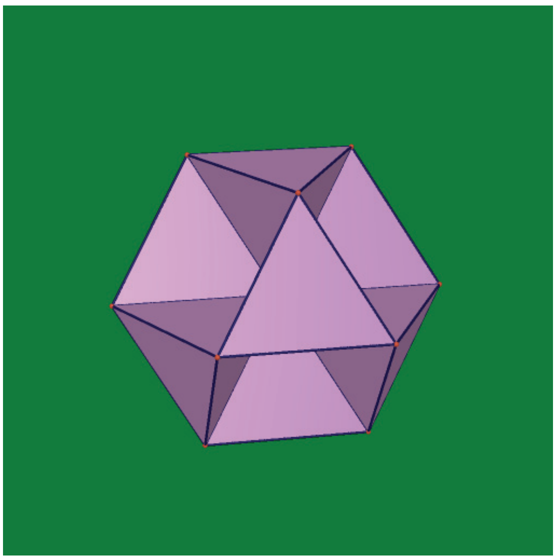
Method 5. As in the example of Method 4, if the square were kept stationary, the center of perspectivity in the relative motion is seen to travel to travel in a different circle in space (See the attached file: Relative Motion.cg3).

DUAL PRINCIPLE #5: ALWAYS EXPLORE THE RELATIVE MOTION BY MAKING STATIONARY OBJECTS MOVING AND VICE VERSA

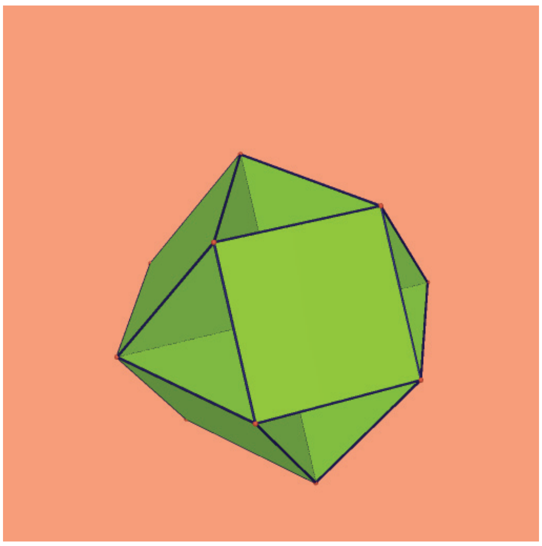
Method 6. Consider this pair of models

(a) Octahemioctahedron (see the attached file: Octahemioctahedron.cg3)

(b) Cubohemioctahedron (see the attached file: Cubohemioctahedron.cg3)

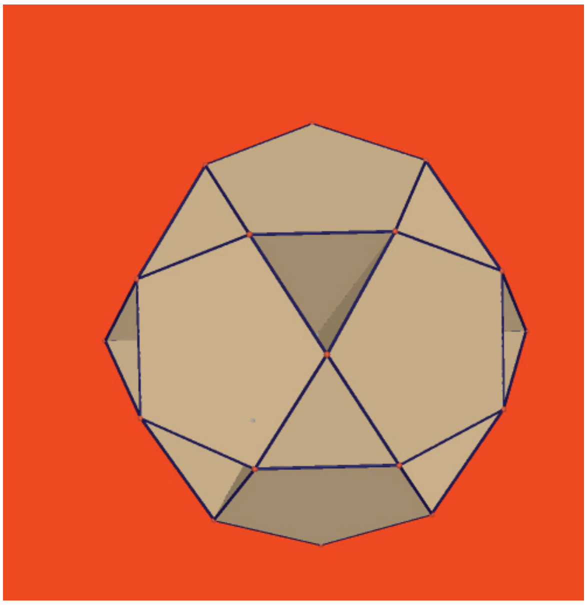


Octahemioctahedron

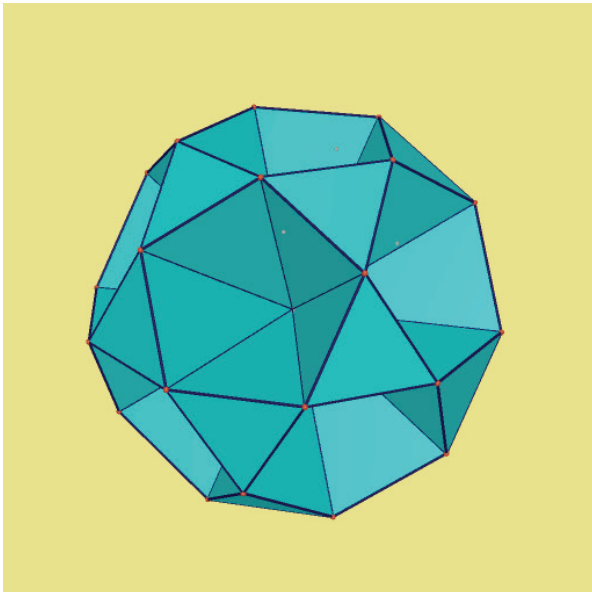


Cubohemioctahedron

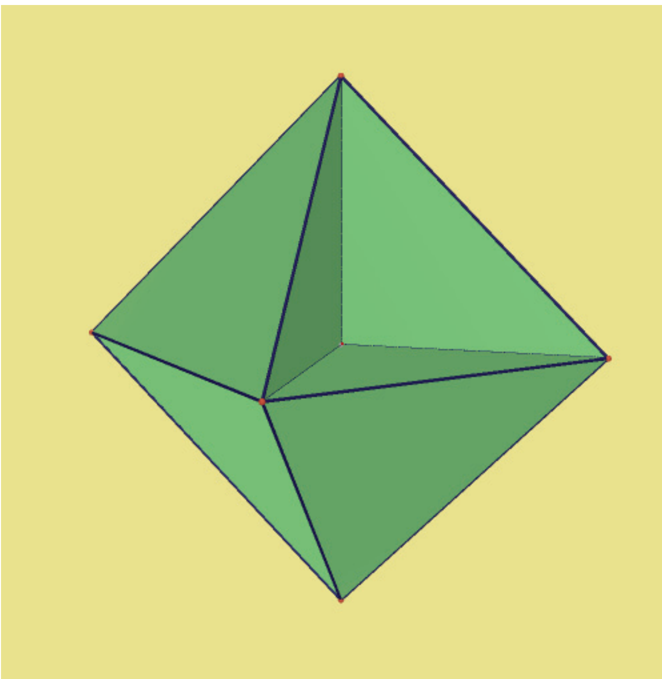
These are examples of Yin-Yang Models, first investigate with Cabri 3D at ATCM 2007 (http://sylvester.math.nthu.edu.tw/d2/yin-yang/) It was shown that both models were constructed with line reflections only, after the fundamental block is built. The construction also leads to the constructions of the dual pair Small Dodecahemidodecahedron-Small Icosihemidodecahedron (see the attached file: Small Icosihemidodecahedron.cg3)



Small Dodecahemidodecahedron



Small Dodecahemidodecahedron



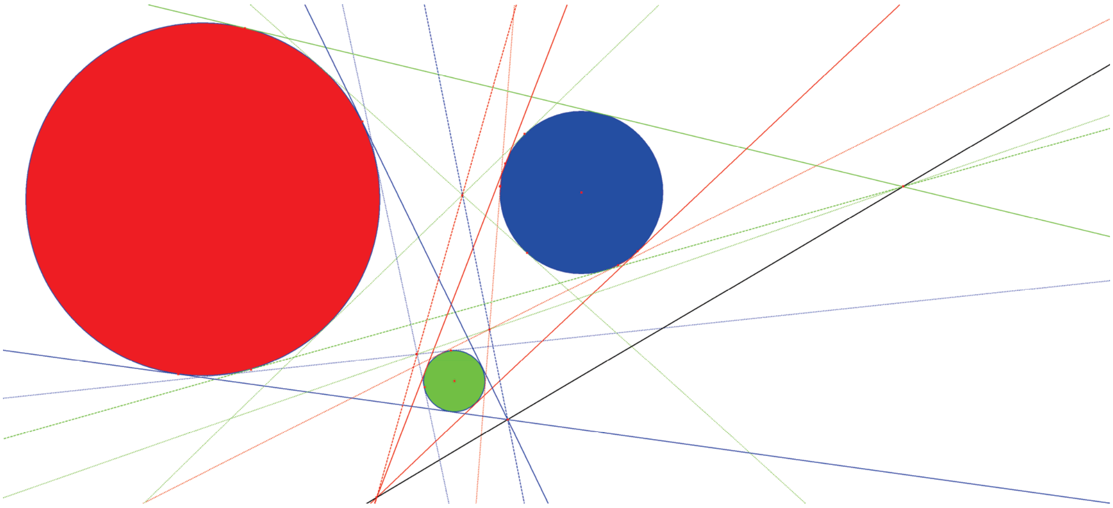
Tetrahemihexahedron

Since Tetrahedron is self-dual, the construction for the “dual of” Tetrahemihexahedron leads to the same model. (See the attached file: Tetrahemihexahedron.cg3)

Dual Principle #6: Yin Yang are interconnected everywhere.

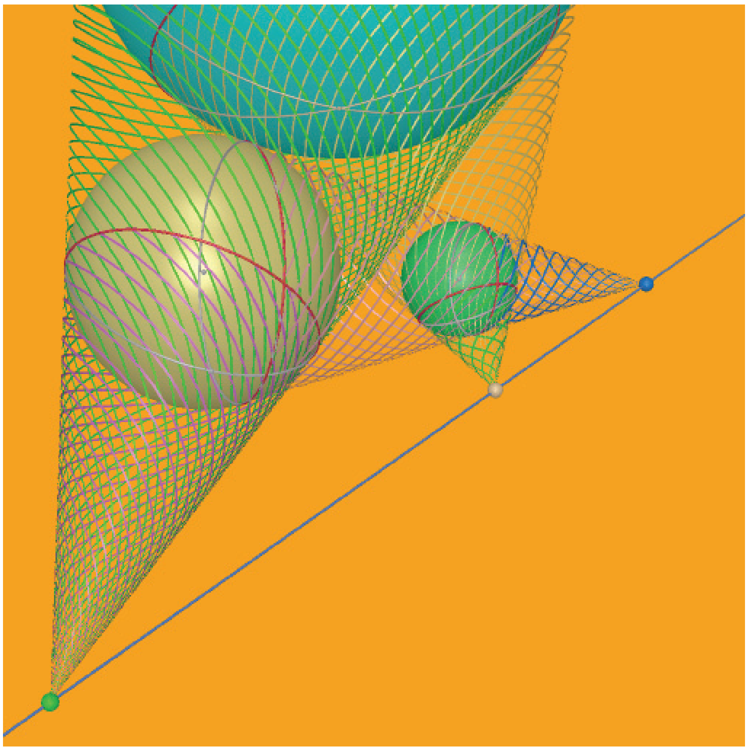
Method 7. Given three circles, what can be deduced?

The answer: Six centers of similitude of three circles lie by three on four straight lines, as given in this illustration:

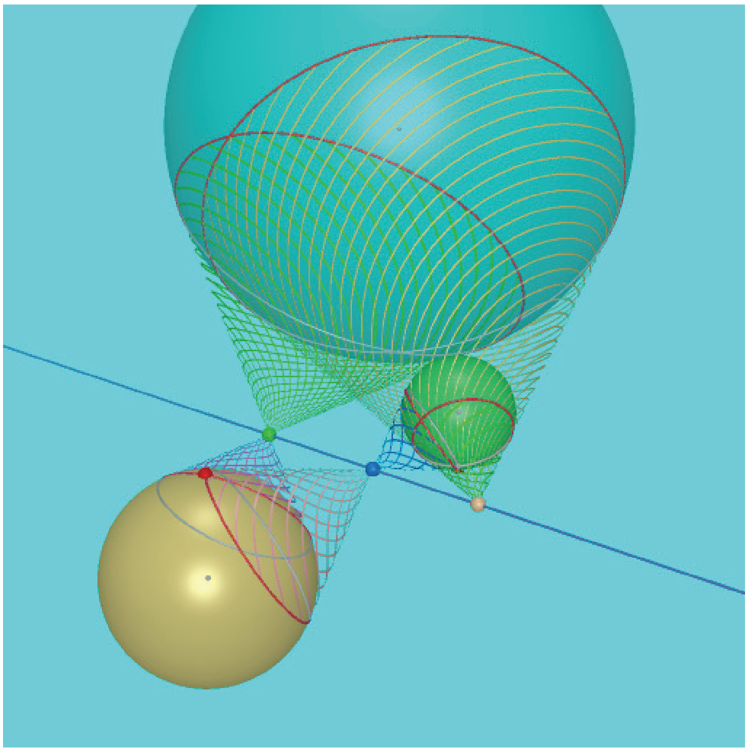


Six Centers of Similitude of three Circles lie by three on Four Straight Lines

Given three spheres, what can be deduced? The answer: Six centers of similitude of three spheress lie by three on four straight lines is given in these illustrations: (See the attached files: Three-Sphere Theorem.cg3 and Internal External Internal.cg3.)



Collinear Property of Vertices of External Common Tangent Cones



Collinear Property of Vertices of Common Tangent Cones, Two Internal One External

DUALITY PRINCIPLE #7: EXPLORE THE POSSIBILITY OF REPLACING THE "CIRCLE" WITH THE "SPHERE".

Conclusion

Mathematics promotes discoveries. Due to the lack of structure, Visual Geometry escapes systematical modern mathematics investigations. Results in Classical Geometric Constructions remain to be isolated from one another. This paper attempts to link a portion of the constructions together under the theme of the Principle of Duality implemented by the Cabri 3D environment.