# Learning about Functions with a Geometrical and Symbolic Software Environment: a Study of Students' Instrumental Genesis along Two Years


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**Abstract**: *Software learning environments, especially those offering extended multi representational capabilities, are more and more complex. That is why researchers are now sensitive to the process of instrumental genesis that transforms this kind of artefact into an instrument for students' mathematical work. The study reported here deals with Casyopée, a geometrical and symbolic learning environment dedicated to functions at upper secondary level. The same students have been observed along their 11th and 12th grade. Learning situations have been designed with the teacher, an experienced user of Casyopée. Consistent with the curriculum, these situations aimed at approaching functions by modelling geometrical dependencies, a task for which Casyopée offers special capabilities. The observation tried to capture how students developed together their use of the software and their mathematical knowledge. Although the situations had been prepared with students' instrumental genesis in mind, the observation in the first year was relatively deceiving: students had little initiative and did not identify clearly key functionalities of the software in relationship with aspects of functions. In the second year, there was a clear improvement: while students used Casyopée more freely, they developed a flexible understanding of functions, associating the objects in the software to mathematical representations. The study suggests that such an instrumental genesis can be a real attainment, but needs to be achieved as a long term process.*


## 1. The Instrumental Genesis of Geometrical and Algebraic Environment

Many studies on the use of technology in mathematics education refer to an instrumental approach (see [1] and [3]). This approach derives from the analysis by psychologists of new uses of a tool by an individual, and his/her associated cognitive changes: in a 'study of thought in relation to instrumented activity', Vérillon and Rabardel (see [7]) stress that a human creation, an 'artefact', is not immediately an instrument. A human being who wants to use an artefact builds up his/her relation with the artefact in two directions: externally s/he develops uses of the artefact and internally, s/he builds cognitive structures to control these uses. After Piaget, Vérillon and Rabardel describe these structures in terms of schemes, which are mental means that a person creates to assimilate a situation. When a person acts on settings through an instrument his/her behaviour has a specific organisation. For that reason, the authors introduce the notion of 'instrument utilisation schemes'. These utilisation schemes have the properties of adaptation and assimilation of the schemes and direct the uses of the instrument by the person. Being mental structures of a person, utilisation schemes are not given with the artefact. They are built in an 'instrumental genesis' which combines the development of uses and the adaptation of schemes: when developing the first uses, a person pilots the artefact through existing schemes, then this primitive experience is the occasion of an adaptation of the schemes, and the better adapted schemes are a basis for developing new uses, and so on. This genesis is both individual and social: a person builds his/her own mental structures, but, generally, an instrument is not used by only one person and therefore the process of adaptation takes place in a social context.

Vérillon and Rabardel 's approach helps to see that instruments are not neutral, because they have an effect on the cognitive functioning of the user. More precisely, in the case of instruments used for the mathematical activity the cognitive structure (utilisation schemes) is made of knowledge about the artefact itself and mathematical knowledge related to the domain of use. For instance, Lagrange (see [4]) described various schemes, calculator oriented or not, algebraic, graphic or symbolic that a user of a CAS calculator (TI-92) can use to search for the properties of a rational function. Each of them mixes the awareness of possibilities and constraints offered by the calculator for a given task and knowledge about the function itself. For example, trying to conjecture the properties of the function by graphical exploration implies to know about the various capabilities offered by the calculator to frame adequately the function (zooming or defining ranges for $x$-axis and $y$-axis) and to anticipate the function's behaviour to adequately use these capabilities. Furthermore, the various schemes for a given task have to be coordinated, for instance conjectures have to be checked against algebraic treatments that another module of the calculator can perform.

The necessity of considering students' and teachers' instrumental genesis when introducing new tools in mathematics teaching learning is now widely recognized (see [1], [2], [3]). It is also recognised that when a tool offers a wealth of capabilities deeply connected to mathematical knowledge, the instrumental genesis is likely to be complex and cannot be achieved in the short term. It is especially the case of tools offering means to work both on geometrical and algebraic situations, and articulate these (see [8]). Few studies actually provide specific examples of a genesis of one of this tool and there is little data about how this type of genesis can develop in the long term. In particular, these tools allow activities of modelling that recent curricula encourage for learning functions, a process that can also be achieved only in the long term. The aim of this paper is then to report and analyse an example of instrumental genesis of a geometrical and symbolic environment devoted to functions, Casyopée, in the process of learning about functions. It is especially expected to know the period of time that students need in order to consider Casyopée really as an instrument of their mathematical activity about functions, looking at key capabilities of this environment as constituent of their mathematical knowledge about functions.

## 2. Learning about Functions with Casyopée

As mentioned above, in many countries, the choice generally made by curricula is to privilege functions at upper secondary level, in order that students consolidate their algebraic proficiencies in order to prepare for calculus. For instance according to the French curriculum students should learn:

> *... to identify the independent variable and its set of values for a function defined by a curve, a table of data or a formula, to establish the value of the function for a given value of the variable in each register, to describe the behaviour of a function given by a curve, using a relevant vocabulary or a sketch*[1].

The curriculum insists on the algebraic notation, and on the various equivalent expressions of a function:

[1] Extracts of the French curriculum are the authors' translation. The curriculum can be found at http://www.cndp.fr/secondaire/mathematiques/

*The notation f(x), already introduced before, and f will be systematically used… (Students should learn) to recognize various forms of an expression and to choose the most relevant form for a given work.*

The idea of function has to derive from activities in varied mathematical and non-mathematical fields:

*Learning situations will come for instance from geometry, physics, actual life or historical problems. Students will have to reflect on language expressions like a depends on b in the common language and in mathematics.*

More specifically, the curriculum points out problems related to geometrical dependencies as a basis for learning situations:

*It is possible to study geometrical situations, the independent variable being a length and the dependant variable an area. The problem is then often to look for a maximum, a minimum or simply a value.*

The curriculum also encourages the use of technology:

*Computer tools can help a quasi-experimental approach to the fields of numbers and of geometrical objects. It favors students' more active attitude and commitment to the task. Possibilities for observing and manipulating are much wider. The opportunity of doing a great number of computations and to study as many cases as wanted helps to observe and verify properties.*

The rationales and history of the Casyopée project at an earlier stage have been exposed in [5]. The Casyopée team brings together teachers and researchers to take up the challenge of teaching about functions at upper secondary level, consistent with recent curricula. The team is concerned that while technology is able to offer multi-representational and symbolic manipulative capabilities very effective for solving problems and learning about functions, no tool presently exists that is really adapted for students' use. Dynamic Geometry software offers a means for constructing operational figures and exploring co-variations and dependencies in these figures, but exploration is limited to numerical values. Students are neither encouraged nor helped to use algebraic notation and to work on algebraic models of geometrical dependencies. Computer Algebra Systems (CAS) exist to ease symbolic manipulation, but they are designed for more advanced users and it is difficult for secondary students to recognize functions and other objects as introduced by the curriculum. For instance, in most CAS, functions are considered over the whole set of real numbers without consideration of an adequate set of definition.

The ReMath European project[2] provided the opportunity for the Casyopée team to develop and experiment with a software environment more adapted for students' activities about functions.
Casyopée has two main windows. The first one, (called the symbolic window) provides students with symbolic computing and representation capabilities as well as facilities for proving. The second one consists of a Dynamic Geometry (DG) window. Casyopée's two windows are closely linked, that is to say that objects in one window can be fully used in the other, and that the software provides the student with specific aid to pass objects from one window to the other. Functions of one variable are the central objects of Casyopée. A function is defined by a formula involving a function variable, and an adequate domain. As most other symbolic systems related to functions and numerical graphers define functions over sets of real numbers, without regard to the existence of formulas on this set, this definition is a distinctive feature in Casyopée. It allows consistency with the mathematical definition as well as providing realistic modelling: when designing a function as a model of a situation, often the function is not defined on the whole set of real numbers and often not on the whole set of existence of the formula. Casyopée provides means for creating sets of ordered

[2] Specific Targeted Research Project IST4-26751: http://remath.cti.gr

real numbers, possibly including parameters, in order to define domains interactively. These parameters can be treated both formally and numerically by way of animation. Constraints can be set on parameters in order to adapt to all situations: for instance if the parameter is intended to model a measure, it can be defined as positive. Functions can depend on parameters. Expressions (that is to say formulas not involving a function variable but possibly involving parameters) can also be defined and treated. Thus Casyopée treats the algebraic objects generally included in upper secondary curricula on functions in a consistent way.

A wide range of construction capabilities is available within the DG window to build a figure including free points. Measures can be defined as "geometrical calculations" possibly including symbolic objects (e.g. parameters, functions and expressions) created in the symbolic window. Because Casyopée is a DG system based on an underlying symbolic kernel (the free software Maxima), it offers the facility for exporting geometrical functions or expressions that is not provided by DG systems based on numerical calculations: Casyopée can compute a domain and a formula for "geometrical" expressions or functions related to measures, providing the capability to express geometrical dependencies algebraically. This "export" capability that will be illustrated below is intended to provide help for students when modelling geometrical functional dependencies or expressions[3].

## 3. A Long-Term Instrumental Genesis: Questions and Methods

The questions addressed in this paper derive from the aim expressed at the end of the first section: to study a long term genesis of Casyopée.

- It is expected that this genesis will articulate notions about functions and knowledge about Casyopée's functionalities. Then what are Casyopée's key functionalities that students progressively understand along this genesis, in parallel with the development of their mathematical knowledge about functions?
- Since notions related to functions are understood only in a multi-year process, it is expected that a genesis of Casyopée will be more than a year long. Then, what is the state of the process after one year? What can be achieved after two years?

Our method was to study the same class of scientific students at $11^{th}$ and $12^{th}$ grades and to focus particularly on two students Elina and Chloé working as a team in this class. The teacher was a member of the Casyopée team and he used as much as possible Casyopée with his students. He prepared with us classroom activities for the observations, first in the frame of the ReMath project, and then as a part of one author's doctoral thesis in progress. A special observation of the Elina-Chloé team was carried out by way of screen and video recording, and of semi-directed interviews. We consider here three milestones: two observations at key times in each year, and the results of an interview at the end of the second year.

The first observation took place in January of the first year ($11^{th}$ grade) at the occasion of the concluding session of the ReMath experiment. This experiment consisted of six sessions. It was organized in three parts. Consistent with our sensitivity to students' instrumental genesis, each part was designed in order that students learn about mathematical notions while getting acquainted with Casyopée's associated capabilities.

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[3] The existence of two windows, one symbolic and the other geometrical distinguishes it from software such as GeoGebra that provides some algebra *inside* a dynamic geometry window. The export capability and other features linking the two windows distinguish Casyopée from other software that also has two such windows. As a difference with Computer Algebra Systems, the symbolic facilities are available through menus and buttons and not through a command language.

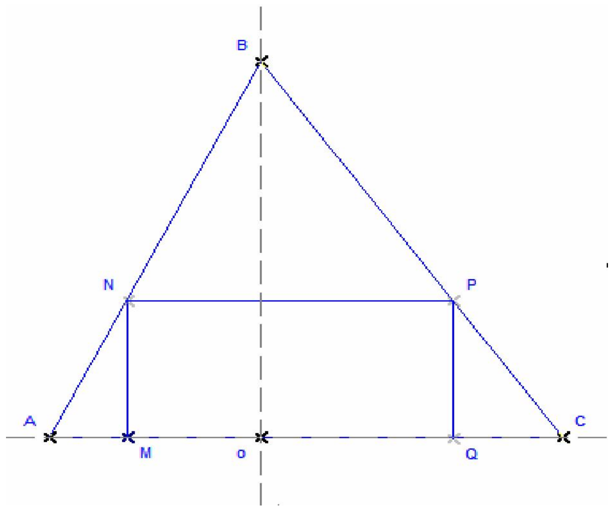
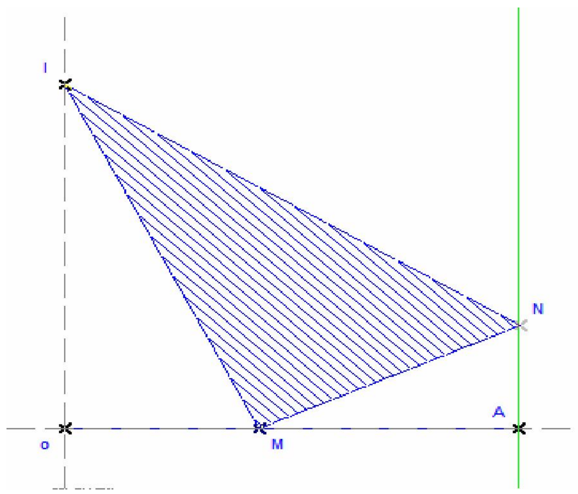
The first part (three sessions) focused on capabilities of Casyopée's symbolic window and on quadratic functions. The aim was that students become familiar with parameter manipulation to investigate algebraic representations of family of functions, while understanding that a quadratic function can have several expressions and the meaning of coefficients in these expressions. The central task was a "target function game": finding the expression of a given form for an unknown function by animating parameters.

In the second part (two sessions) we first aimed to consolidate students' knowledge of geometrical situations and to introduce them to the geometrical window's capabilities. The central task was to build geometric calculations to express areas and to choose relevant independent variables to express dependencies between a free point and the areas. We also aimed to introduce students to coordinating representations in both algebraic and geometrical settings, by way of problems involving areas that could be solved by exporting a function and solving an equation in the symbolic window.

The third part consisted of one concluding session that will be considered here, as a milestone of students' instrumentation. Students had to solve a problem of maximum area taking advantage of all Casyopée features and of all notions they learnt in the previous sessions.

The second observation took place in January of the second year ($12^{th}$ grade), as a second step of a series of activities prepared in one author's doctoral work. The three steps were: (1) a session aiming at the consolidation of Casyopée's use some months after the ReMath experimentation: the goal is to model a variable area in a square; the function at stake is quadratic, (2) a session where students have to use more completely Casyopée's functionalities, especially for the management of parameters and for symbolic calculation, again in a modelling activity, the function at stake being a third degree parametric polynomial (3) a session involving the study of a family of logarithm functions, a more classical task with regard to the curriculum as compared with the geometrical modelling in the two other sessions, the goal being that students become aware of how they can use Casyopée to prepare for the baccalaureate, an exam they have to pass at the end of the second year. The semi-directed interview was conducted at the end of the second year, before the baccalaureate in order to understand the evolution students' relationship with mathematics and with Casyopée.

## 4. Presentation and Comparison of Tasks Proposed in Two Observations

| **Problem 1** ($11^{th}$ grade) | **Problem 2** ($12^{th}$ grade) |
|---|---|
|  |  |

| | |
|---|---|
| Text: Consider a triangle *ABC*. *A(-a;0), B(0;b) and C(c;0), a, b* and *c* being three parameters. Find a rectangle *MNPQ* with *M* on [*oA*], *N* on [*AB*], *P* on [*BC*], *Q* on [*oC*] and with the maximum area.<br><br>Hint of a solution: For all values of the parameters, the maximum area is for *M* at the middle of segment *[oA]*<br><br>See [6] for a more in depth presentation of this problem. | Text: Consider the point *I(0; a), a* being a parameter and the point *A* of coordinates *(10; 0)*. *M* belongs to the segment *[oA], N* is on the parallel to the *y*-axis passing by *A* and the triangle *IMN* is rectangle in *M*. When *M* is in *A*, then *N* is also in *A*. The problem is to find the position of *M* to maximize the rectangle's area.<br><br>Hint of a solution: for values of *a* greater than $\frac{10}{\sqrt{3}}$ the function is decreasing and then the maximum is for $M \equiv o$. For other values, there is a local maximum for a position of *M* inside the segment. This maximum is the absolute maximum for *a* lower than *5* (figure 2), otherwise the maximum is for $M \equiv o$. For $a = 5$ there are two maximums, one for $M \equiv o$ and the other at the middle of *[oA]*. |

**Figure 1.** Tasks proposed in two observations

As said above, both observations were done during sessions where students had to solve optimisation problems involving modelling geometrical dependencies. Figure 1 shows how the problems were proposed to students and gives a hint of the mathematical solutions. Below, we propose a comparative a priori analysis of the two tasks.

In both problems, the figure is defined parametrically. This is consistent with the curriculum in scientific sections, where students have to tackle generic problems and not just numerical cases. In addition, the first part of the ReMath experiment aimed to familiarize students with the notion of parameter and to the associated functionalities in Casyopée. Working on a parametrical figure is expected to give them new insight about this notion.

A solution with Casyopée involves mathematical subtasks in relationship with the corresponding functionalities of the software:

**Table 1.** Mathematical subtasks and Casyopée's functionalities

| **Mathematical subtasks** | **Casyopée's functionalities** |
|---|---|
| • Building a geometrical figure | • Creating objects in dynamic geometry |
| • Exploring and conjecturing | • Creating a geometric calculation, dragging free points, observing numeric values |
| • Modelling a dependency | • Choosing an independent variable, exporting a function |
| • Using an algebraic procedure | • Using Casyopée's algebraic transformations, and justifications |
| • Generalising | • Animating parameters |

The solution of both problems involves these subtasks and the use of the corresponding functionalities, but they differ relatively to the complexity of subtasks. In the first problem there is always one maximum at the middle of the domain, while in the second problem, depending on the values of the parameter, the maximum can be for $M$ at an extremity or at a variable position in the domain (figure 2). Exploring and conjecturing is more complex and has to be coordinated with generalising. The algebraic procedure involves the use of a derivative and a discussion to find algebraically the critical values of the parameter.

## 5. Observations

### The first year

We report on the work of Chloé and Elina for each subtask in the session that we presented above as a milestone in the first year.

Building a geometrical figure:

Students took much time constructing the variable rectangle. Modelling the variable rectangle implied to build a proper rectangle based on a free point on a segment, but students first built a "soft" rectangle, that is to say that the quadrilateral they built was perceptively a rectangle, but did not resist to a variation of the figure by animation of a free point or a parameter. They had difficulty to distinguish between a free point in the plane and a free point on a segment. Thanks to the feedback of the software and to the help of the observer, they recognised that they were wrong, but were slow to correct.

Exploring and conjecturing:

The students confused the creation of a dependant variable representing the area with the choice of an independent variable, two actions accessible in the same toolbar of Casyopée. They did few explorations.

Modelling a dependency:

Students did not understand by themselves the need to choose an independent variable and to export the dependency as an algebraic function. They hesitated on the choice of an adequate variable. They understood the exportation as a way to have a graph of the dependency, rather than the creation of an algebraic model.

Using an algebraic procedure:

After recognizing a parabola, students did not know how to use their previous knowledge about quadratic functions. With the help of the observer they remembered a formula for the abscissa of the vertex. The resulting expression is complex, but can be easily simplified by Casyopée. In spite of this, students could not apply this formula to the generic parametric quadratic expression and they treated only a numeric case, considering the current values of the parameters.

Generalising:

As said above, the students conjectured the optimal position for a generic triangle in the geometrical window, but could not prove it in the symbolic window, because they could not take advantage of Casyopée as an algebraic tool.

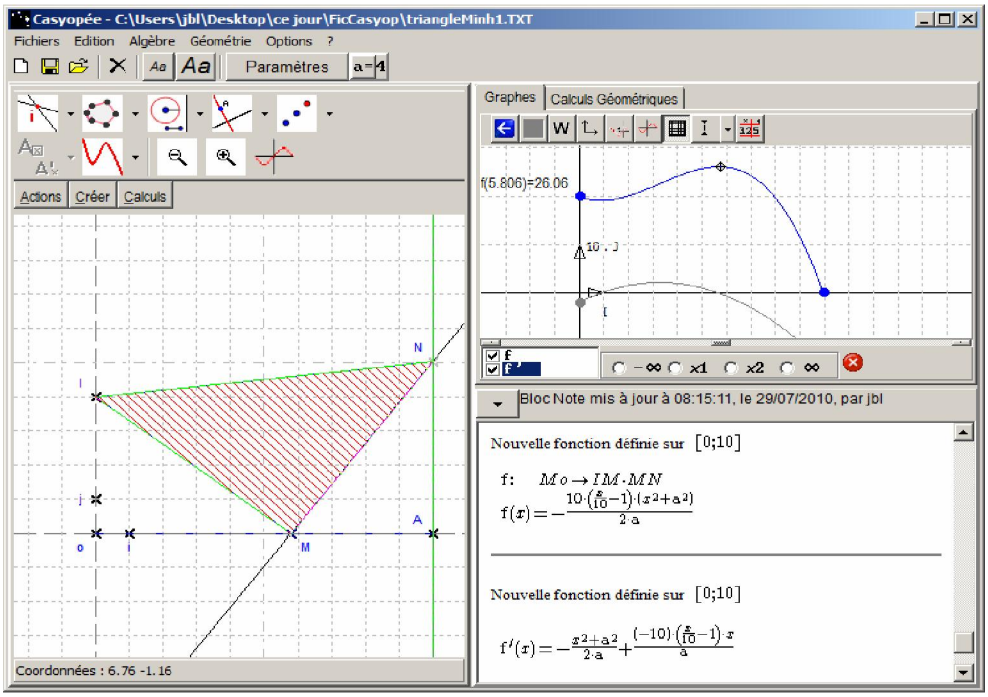



**Figure 2.** The problem 2's figure and graph of the function for a case $\frac{10}{\sqrt{3}} < a < 5$

This observation was certainly deceiving after the careful preparation in six sessions. The outcome was that the instrumental genesis was to be considered on a longer period. Comparing with other students, Chloé and Elina were considered representative of a majority of students, and also particularly positive relative to the use of a new tool in spite of their difficulties. That is why we choose to centre the subsequent observations and this article on this team.

**The second year**

We report now on the work of Chloé and Elina in the session that we presented above as a milestone in the second year, again considering the subtasks described in table 1.

Building a geometrical figure:

Difficulties in using dynamic geometry remained, but students corrected easily their mistakes.

Exploring and conjecturing:

As said above, the exploration is more complex in this problem, and students did a lot of exploration, corresponding to different cases and values of the parameter. They commented, using the relevant functional language: “growing”, “decreasing”…

Modelling a dependency:

There was a much better understanding of the process of modelling and of the associated functionalities of Casyopée, as shown by the following extract:

Chloé: *Choosing the independent variable? Last time we did it with the altitude?*
Elina: *No, the distance OM, I think that it will be a good variable.*
Chloé: *Yes* (She chooses this variable and then exports the function).
Elina: *Its domain*?
Chloé: *It is the set of real numbers, Oh, no it is [0;10]*
Elina: *Look* (points to the screen) *It is here.*

Using an algebraic procedure:

The students proposed a procedure using the derivative. They easily used Casyopée functionalities like "expanding", "factoring" and also the justifications for the sign of the derivative.
Generalising:

The students animated the parameters in order to study the different cases.

We see a clear improvement, both in the use of Casyopée's functionalities and in the mathematical abilities.

**The interview at the end of the second year**

We report here some of the two students' answers. The first outcome is that after two years of use, the students saw Casyopée as a tool whose appropriation had not been easy:

*We did not know all functionalities… tools… in Casyopée. We obtained expressions, but we did not know how to manage them. We did not know which functionalities to use.*

They recognized that these difficulties are linked to the understanding of the mathematical content.

*The most difficult is to choose an independent variable. It is important to choose an appropriate variable.*

They also indicated that these difficulties have been overcome thanks to a continuous use of Casyopée and with the help of the teacher.

*I downloaded Casyopée from Google and I use it sometimes (at home) for training… At the beginning it was hard to find functionalities to use it, but now it works… thanks to the help of the teacher. He explains us how to solve problems.*

In spite of the difficulties observed by the first uses, the students also expressed positive feelings relative to specificities of Casyopée, especially the help for modelling and the link between algebraic and geometrical windows.

*Choosing variables is the interesting part… To perform all the process is great: constructing the figure, table of variation, calculation of the derivative… As compared to a calculator, Casyopée is more straightforward and quicker... We have the algebraic and geometrical sides together… We see better how a function "reacts", it is convenient and interesting…*

Students identified clearly different functionalities and how they could help exploring and proving freely.

*We can try different variables, animate the figure, and visualize functions (several at the same time), draw a table of signs, find the derivative…*

Actually, in this interview, the students express in their own words their instrumental genesis.

## 6. Synthesis and Discussion

We recapitulate this genesis in the table 2 by identifying links between students' progress in the use of Casyopée's functionalities and the development of their mathematical knowledge.

**Table 2.** Joint development of mathematical knowledge and knowledge about Casyopée

| Mathematical subtasks | Progress in the use of Casyopée | Development of mathematical knowledge |
|---|---|---|
| Building a geometrical figure | Quicker correction of mistakes after observing unexpected behaviour of the figure. | Better understanding of the functional dependencies in the figure. |

| | | |
|---|---|---|
| Exploring and conjecturing | Correct definition and use of a geometrical calculation for the area | Understanding of the triangle area formula as expressing a co-variation between the mobile point M and the value of the area. |
| Modelling a dependency | Spontaneous and easy use of the buttons for choosing a variable and exporting a function. | Understanding a functional dependency. Distinguishing a functional dependency between co-variations. |
| Using an algebraic procedure and generalising | Easy use of Casyopée's algebraic transformations, and justifications.<br>Animation of parameters. | Understanding the different behaviours of the function depending on the parameter.<br>Understanding a parametric function as a family of functions. |

Table 2 shows how an instrumental genesis deeply links knowledge about an artefact (here Casyopée) and mathematical knowledge (here the idea of function as a model of a functional dependency). In the first year, the use of the artefact was seen as complicated and the help of the teacher or of the observer was needed at crucial steps. In the second year, the students better identified the underlying mathematical notions and the corresponding functionalities of Casyopée. In this year the students had to prepare for the baccalaureate and the activities with Casyopée could have been seen as far from standard exercises proposed in this important exam. Nevertheless, students recognized the contribution of these activities to their learning as well as the help that the artefact could bring even for standard tasks.

The outcome of the study is then that an instrumental genesis of a geometrical and symbolic software environment is a real attainment with regard to mathematical learning, but can be achieved only as a long term process. Further work will be done in order to precise up to what point Chloé and Elina's genesis is representative of other students' genesis, using the data collected during the study, and to study in more depth the mathematical knowledge about functions built during this genesis.

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