The Nonabelian Tensor Square of a Centerless Bieberbach Group With the Dihedral Point Group of Order Eight: Theory and Calculation

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Abstract

A centerless Bieberbach group is a torsion free crystallographic group with trivial center. In this paper our focus is on the centerless Bieberbach groups with the dihedral point group of order eight. With the method developed for polycyclic groups, we compute the nonabelian tensor square of one of the groups of dimension four. Our approach of the computation of the nonabelian tensor square of the group is using both hand calculations and computer calculation by GAP. We explore how to use GAP to assist our hand calculation and gain insight into this group construction and look what are missing in just doing GAP calculation. We also illustrate the problems and solutions for mapping GAP calculation to hand calculation in computing the nonabelian tensor square of the group.

1 Introduction

A Bieberbach group is a torsion free crystallographic group G which satisfies the short exact sequence

$$1 \longrightarrow L \longrightarrow G \longrightarrow P \longrightarrow 1$$

where P is a point group that is a finite group acting faithfully on a maximal normal free abelian subgroup L of G which is of finite rank and is called a lattice group. It follows that L is a Fitting subgroup of G and its rank or Hirsch length is referred to as the dimension of G.

The nonabelian tensor product $G \otimes H$ of groups G and H has its origins in homotopy theory and was introduced by Brown and Loday [8], extending the ideas of Whitehead [18]. The nonabelian tensor square $G \otimes G$ of a group G is a special case of the nonabelian tensor product where the product is defined if the two groups act on each other in a compatible way and their actions are taken to be conjugation. $G \otimes G$ is generated by the symbols $g \otimes h$, for all $g, h \in G$, subject to relations

$$gh \otimes k = (g^h \otimes k^h)(h \otimes k) \quad \text{and} \quad g \otimes hk = (g \otimes k)(g^k \otimes h^k) \qquad (1)$$

for all $g, h, k \in G$ where $g^h = h^{-1}gh$. The study of $G \otimes G$ was started by Brown, Johnson and Robertson [7]. Following to that, the group $G \otimes G$ has become an increasing interest to many other researchers. Many researchers have investigated general properties and even explicit descriptions of the nonabelian tensor squares of some particular groups such as 2-generator p-group of class 2 [2], 2-generator Burnside group of exponent 4 [11], metacyclic groups [4], 2-Engel groups [3, 6] and infinite two-generator groups of class two [15]. Computing the nonabelian tensor square of a Bieberbach group with a point group P is not in the literature. Only recently in November 2009, Rohaidah [14] successfully got her doctoral degree on her work on computing the nonabelian tensor squares of Bieberbach groups with cyclic point groups. The structure of $G \otimes G$ has been investigated by Brown and Loday [8]. Results in [8], give a commutative diagram with exact rows and central extensions as columns as in diagram (2). In (2), $\nabla(G) := \langle g \otimes g \mid g \in G \rangle$ is a central subgroup of $G \otimes G$. The nonabelian exterior square $G \wedge G$ is a quotient group $G \otimes G/\nabla(G)$. $J_2(G)$ denotes the kernel of $G \otimes G \to G : g \otimes h \mapsto [g, h]$ and $\Gamma(G/G')$ be Whitehead's quadratic functor [18]. Hence by [8], the kernel of $G \wedge G \to G : g \wedge h \mapsto [g, h]$ is isomorphic to Schur Multiplicator $H_2(G)$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & \Gamma(G/G') \arrow[r] \arrow[d] & J_2(G) \arrow[r] \arrow[d] & H_2(G) \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \nabla(G) \arrow[r] \arrow[d] & G \otimes G \arrow[r] \arrow[d] & G \wedge G \arrow[r] \arrow[d] & 1 \\
 & 1 & G' \arrow[r, equal] \arrow[d] & G' \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd} \tag{2}$$

There are several methods used in computing $G \otimes G$, including the definition given in (1). However, the definition has become impractical for large finite groups. Brown, Johnson and Robertson [7] computed $G \otimes G$ for all groups G of order up to 30 using (1) and also with the ToddCoxeter algorithm. Ellis and Leonard [11] gave a more effective computer algorithm to determine $G \otimes H$ for finite G and H and they computed the nonabelian tensor square of the Burnside group of exponent 4 of order 4096 as an example of application. Bacon [1] used crossed pairing method to compute the nonabelian tensor squares of free nilpotent groups of class 2 of finite rank and Beuerle and Kappe [4] used the method to compute the nonabelian tensor squares of the infinite metacyclic groups. This method was also been used in computing the nonabelian tensor squares for nonabelian cases such as for the two-generator two-group of class two [12, 15] and the free 2-Engel groups of finite rank [6]. However, the step in the method can sometimes be difficult and lead to the incorrect identification of the nonabelian tensor square of a group [12].

More recent method used to compute the nonabelian tensor squares of groups is initiated by Rocco [13] follows by Ellis and Leonard [11] where Rocco introduces the group $\nu(G)$ that is a subgroup of $[G, G^{\varphi}]$ defined below:

Definition 1
Let G be a group with presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. The group $\nu(G)$ is defined to be

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^{\varphi} \mid \mathcal{R}, \mathcal{R}^{\varphi}, [g, h^{\varphi}]^{x} = [g^{x}, (h^{x})^{\varphi}] = [g, h^{\varphi}]^{x^{\varphi}}, \forall x, g, h \in G \rangle.$$

The important fact about $\nu(G)$ is that Rocco [13] and Ellis and Leonard [11] have shown that it is isomorphic to the nonabelian tensor square of group G as given in the following theorem.

Theorem 2 *Let G be a group. The map $\sigma : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$ defined by $\sigma(g \otimes h) = [g, h^{\varphi}]$ for all g, h in G is an isomorphism.*

Blyth and Morse [5] have provide an analysis of the group $\nu(G)$ for arbitrary finite and infinite group G as a tool to compute $G \otimes G$ and other homological functors as in diagram (2). Their results which specialized on polycyclic groups are shown in the following theorem.

Theorem 3 *Let G be a polycyclic group with a finite presentation $\langle \mathcal{G} | \mathcal{R} \rangle$ and polycyclic generating set $\mathcal{B}$. Then*

(i) The group $G \otimes G$ and $\nu(G)$ are polycyclic.

(ii) The group $\nu(G)$ has a presentation that depends only on $\mathcal{G}, \mathcal{R}$, and $\mathcal{B}$.

These results actually give us two methods to compute the nonabelian tensor squares of polycyclic groups that are hand computation within the subgroup $[G, G^{\varphi}]$ of $\nu(G)$, aided with the commutator calculus, commutator identities of $\nu(G)$ (referred to Rocco [13], Blyth and Morse [5]) and a computer method. The computer method is done by computing a consistent polycyclic presentation of $\nu(G)$ using polycyclic quotient algorithm provides by Eick and Nickel [9] and computing the subgroup $[G, G^{\varphi}]$ of $\nu(G)$ using the GAP Polycyclic package [10].

The goal of this paper is to compute the nonabelian tensor square of a centerless Bieberbach group of dimension four with the dihedral point group of order eight. The group will be shown to be polycyclic, hence the method to compute the nonabelian tensor squares introduced by Blyth and Morse [5] mentioned above is chosen. The research has been aided by the used of computational method in group theory via the library of Bieberbach group, CARAT [17] and a computational group theory system, Groups, Algorithms, and Programming (GAP)[16]. GAP provides varieties of algorithmic methods to compute with groups of various types including crystallographic groups and polycyclic groups. Eick and Nickel [9] have shown that the Polycyclic package [10] is significantly more effective than the previously known method for many finite polycyclic groups and it also extends to infinite polycyclic groups. In this paper, we demonstrate how we can use GAP to aid us in the computation.

2 Preliminaries

In this section we list some related definitions and structural results used in computing the nonabelian tensor square of a Bieberbach group of dimension four with the dihedral point group. We start with definitions of a polycyclic presentation and a consistent polycyclic presentation that are taken from Eick and Nickel [9].

Definition 4 ***Polycyclic Presentation***
Let F_n be a free group with generators $g_1, \ldots, g_n$ and R be a set of relations of group G. The relations of a polycyclic presentation F_n/R have the form:

$$\begin{aligned}
g_i^{e_i} &= g_{i+1}^{x_{i,i+1}} \ldots g_n^{x_{i,n}} && \text{for } i \leq I, \\
g_j^{-1} g_i g_j &= g_{j+1}^{y_{i,j,j+1}} \ldots g_n^{y_{i,j,n}} && \text{for } j < i, \\
g_j g_i g_j^{-1} &= g_{j+1}^{z_{i,j,j+1}} \ldots g_n^{z_{i,j,n}} && \text{for } j < i \text{ and } j \notin I
\end{aligned}$$

for some $I \subseteq \{1, \ldots, n\}$, $e_i \in \mathbb{N}$ for $i \in I$ and $x_{i,j}$, $y_{i,j,k}$, $z_{i,j,k} \in \mathbb{Z}$ for all i, j and k.

Definition 5 ***Consistent Polycyclic Presentation***
Let G be a group generated by $g_1, \ldots, g_n$. The consistency of the relations in G can be determined using the following consistency relations.

$$\begin{aligned}
g_k(g_j g_i) &= (g_k g_j) g_i && \text{for } k > j > i, \\
(g_j^{e_j}) g_i &= g_j^{e_j - 1}(g_j g_i) && \text{for } j > i,\ j \in I, \\
g_j(g_i^{e_i}) &= (g_j g_i) g_i^{e_{i-1}} && \text{for } j > i,\ i \in I, \\
(g_i^{e_i}) g_i &= g_i (g_i^{e_i}) && \text{for } i \in I, \\
g_j &= (g_j g_i^{-1}) g_i && \text{for } j > i,\ i \notin I.
\end{aligned}$$

The Bieberbach group of dimension four with the dihedral point group is a polycyclic group. Our method of computing the nonabelian tensor square of the group is based on method developed by Blyth and Morse [5] which is aided by the following Proposition.

Proposition 6 (Proposition 20 [5]) *Let G be a polycyclic group with a polycyclic generating sequence $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Then $[G, G^\varphi]$ a subgroup of $\nu(G)$, is generated by*

$$[G, G^\varphi] = \langle [\mathfrak{g}_i, \mathfrak{g}_i], [\mathfrak{g}_i{}^\epsilon, (\mathfrak{g}_j{}^\varphi)^\delta], [\mathfrak{g}_i{}^\epsilon, (\mathfrak{g}_j{}^\varphi)^\delta][\mathfrak{g}_j{}^\delta, (\mathfrak{g}_i{}^\varphi)^\epsilon] \rangle$$

for $1 \leq i < j \leq k$, where

$$\epsilon = \begin{cases} 1 & \text{if } |\mathfrak{g}_i| < \infty; \\ \pm 1 & \text{if } |\mathfrak{g}_i| = \infty \end{cases} \text{ and } \delta = \begin{cases} 1 & \text{if } |\mathfrak{g}_i| < \infty; \\ \pm 1 & \text{if } |\mathfrak{g}_i| = \infty. \end{cases}$$

3 Computing The Nonabelian Tensor Squares

A centerless Bieberbach group with dihedral point group of dimension four is given in the form of matrix representation by CARAT. Based on the matrix representations and Definition 4, the polycyclic presentation of this group is obtained as follows:

$$\begin{aligned}
G = &\langle a,\ b,\ c,\ l_1,\ l_2,\ l_3,\ l_4 \mid a^2 = l_3^{-1},\ b^2 = l_2,\ c^2 = l_1^{-1}, \\
&b^a = c l_3^{-1},\ c^a = b l_3^{-1},\ c^b = c l_1 l_2 l_4^{-1},\ l_1^a = l_2^{-1},\ l_2^a = l_1^{-1},\ l_3^a = l_3,\ l_4^a = l_4^{-1}, \\
&l_1^b = l_1^{-1},\ l_2^b = l_2,\ l_3^b = l_3^{-1},\ l_4^b = l_4^{-1}, l_1^c = l_1,\ l_2^c = l_2^{-1},\ l_3^c = l_3^{-1},\ l_4^a = l_4^{-1}, \\
&l_j^{l_i} = lj,\ l_j^{l_i^{-1}} = l_j \quad \text{for} \quad j > i, \quad 1 \leqslant i, j \leqslant 4 \rangle
\end{aligned} \tag{3}$$

The polycyclic presentation of G given above can be shown to be consistent by Definition 5. The proof is omitted here. Since G has a consistent polycyclic presentation, we now can construct this centerless Bieberbach group G with GAP Polycyclic package [10]. GAP commands will be written in teletype font with double semicolon if we want to suppress the output, except the output is of interest. The # symbol is used for comment in GAP for clarity and we fix the GAP object G which is a finitely presented group that is isomorphic to G. Follows are the commands to construct the group G in GAP.

```
gap> F := FreeGroup("a","b", "c", "l1", "l2", "l3", "l4");;
gap> a := F.1;; b := F.2;; c := F.3;; l1 := F.4;;
>    l2 := F.5;; l3:=F.6;; l4:=F.7;;
gap> R:=[a^2/l3^-1, b^2/l2, c^2/l1^-1, b^a/(c*l3^-1),
>        c^a/(b*l3^-1), c^b/(c*l1*l2*l4^-1),
>        l1^a/l2^-1, l2^a/l1^-1, l3^a/l3, l4^a/l4^-1,
>        l1^b/l1^-1, l2^b/l2, l3^b/l3^-1, l4^b/l4^-1,
>        l1^c/l1, l2^c/l2^-1, l3^c/l3^-1, l4^c/l4^-1,
>        l2^l1/l2, l3^l1/l3, l4^l1/l4,  l3^l2/l3, l4^l2/l4, l4^l3/l4,
>        l2^(l1^-1)/l2, l3^(l1^-1)/l3, l4^(l1^-1)/l4,
>        l3^(l2^-1)/l3, l4^(l2^-1)/l4, l4^(l3^-1)/l4];;
gap> # Construct the finitely presented group
gap> Gfp:=F/R;
<fp group on the generators [ a, b, c, l1, l2, l3, l4 ]>
gap> iso := IsomorphismPcpGroup(Gfp);
gap> G:=Image(iso);
Pcp-group with orders [ 2, 2, 2, 0, 0, 0, 0 ]
```

It should be noted here that GAP cannot run in getting the group iso if the presentation of G is not consistent.

Next we compute a finite presentation for the nonabelian tensor square $G \otimes G$ of G. Actually GAP can compute the nonabelian tensor square of the group G easily and fast. The following commands give the output of the non-abelian tensor square of G by GAP. Here GAP object of NonAbelianTensorSquare(G) is isomorphic to $G \otimes G$ of G.

```
gap> ts:=NonAbelianTensorSquare(G);
Pcp-group with orders [ 2, 0, 0, 0, 0, 4, 8, 8 ]
gap>  List(Igs(NonAbelianTensorSquare(G)),Order);
[ infinity, infinity, infinity, infinity, infinity, 4, 8, 8 ]
```

The above GAP output tells us that the nonabelian tensor square of G has eight generators in which five of them are of infinite orders, two of them are of order eight and one of them is of order four. However the result does not give us the insight of the structure of the nonabelian tensor square of the group such as its exact generators and presentation. With the theory of the computation of the nonabelian tensor square of polycyclic group by Blyth and Morse [5], we compute exactly the generators and the presentation of the nonabelian tensor square of the group G. Since our computation is within $\nu(G)$, we use $[G, G^{\varphi}]$ and $G \otimes G$ also $[g, h^{\varphi}]$ and $g \otimes h$ for $g, h \in G$ interchangeably.

By Proposition 6, the subgroup $[G, G^\varphi] \cong G \otimes G$ of $\nu(G)$ is generated by the following set:

$$\begin{aligned}
&\{[a, a^\varphi],\ [b, b^\varphi],\ [c, c^\varphi],\ [l_4, l_4^\varphi], [a, b^\varphi],\ [a, c^\varphi],\ [a, l_1^\varphi],\ [a, l_2^\varphi],\ [a, l_4^\varphi],\\
&[b, c^\varphi],\ [b, l_1^\varphi],\ [b, l_3^\varphi],\ [b, l_4^\varphi], [c, l_2^\varphi],\ [c, l_3^\varphi],\ [c, l_4^\varphi],\\
&[a, b^\varphi][b, a^\varphi],\ [a, c^\varphi][c, a^\varphi],\ [a, l_1^\varphi][l_1, a^\varphi],\ [a, l_2^\varphi][l_2, a^\varphi],\ [a, l_4^\varphi][l_4, a^\varphi],\\
&[b, c^\varphi][c, b^\varphi],\ [b, l_1^\varphi][l_1, b^\varphi],\ [b, l_3^\varphi][l_3, b^\varphi],\ [b, l_4^\varphi][l_4, b^\varphi],\\
&[c, l_2^\varphi][l_2, c^\varphi],\ [c, l_3^\varphi][l_3, c^\varphi],\ [c, l_4^\varphi][l_4, c^\varphi]\}
\end{aligned} \tag{4}$$

This set is not independent where some of the generators are identities and some of them are products of powers of other generators. By hand computation with commutator calculus and commutator identities of $\nu(G)$ ([13], [5]), we can show that, for example, $[c, l_3^\varphi] = [b, l_3^\varphi]$ as below:

$$\begin{aligned}
[c, l_3^\varphi] &= [a^{-1}bal_3\ ,\ l_3^\varphi] && \text{by relation of } G\\
&= [a^{-1}, l_3^\varphi]\ [[a^{-1}, l_3], (bal_3)^\varphi]\ [b, l_3^\varphi]\\
&\quad [[b, l_3], (al_3)^\varphi]\ [a, l_3^\varphi]\ [[a, l_3], l_3^\varphi]\ [l_3, l_3^\varphi] && \text{by commutator calculus}\\
&= [a^{-1}, l_3^\varphi]\ [b, l_3^\varphi]\ [l_3^2, (al_3)^\varphi]\ [a, l_3^\varphi]\ [a, a^\varphi]^4 && \text{by relation of } G\\
&= [a^{-1}, l_3^\varphi]\ [b, l_3^\varphi]\ [l_3, (al_3)^\varphi]\ [[l_3, al_3]\ ,\ l_3^\varphi]\\
&\quad [l_3, (al_3)^\varphi]\ [a, l_3^\varphi]\ [a, a^\varphi]^4 && \text{by commutator calculus}\\
&= [a^{-1}, l_3^\varphi]\ [b, l_3^\varphi]\ [l_3, l_3^\varphi]\ [l_3, a^\varphi]\ [[l_3, a]\ ,\ l_3^\varphi]\\
&\quad [l_3, l_3^\varphi]\ [l_3, a^\varphi]\ [[l_3, a]\ ,\ l_3^\varphi]\ [a, l_3^\varphi]\ [a, a^\varphi]^4 && \text{by relation and commutator calculus}\\
&= [a, a^\varphi]^2\ [b, l_3^\varphi]\ [a, a^\varphi]^4\ [a, a^\varphi]^{-2}\ [a, a^\varphi]^4\\
&\quad [a, a^\varphi]^{-2}[a, a^\varphi]^{-2}\ [a, a^\varphi]^4 && \text{by relation of } G\\
&= [b, l_3^\varphi].
\end{aligned}$$

We can see that the above hand computation is quite lengthy and tedious. As to that, we make GAP to assist our computation in computing the rest of the independent generators of $G \otimes G$. The computation of the independent generators are shown in the following lemma.

Lemma 7 *Let G be the Bieberbach of dimension four with the dihedral point group which has a polycyclic presentation as in (3). Then the nonabelian tensor square $G \otimes G$ of the group G is generated by the following set*

$$\{a \otimes a,\ c \otimes c,\ a \otimes b,\ a \otimes c,\ a \otimes l_1,\ b \otimes l_3,\ c \otimes l_2,\ (a \otimes c)(c \otimes a)\}.$$

Proof. In GAP, we first construct the group `nu` which is isomorphic to $\nu(G)$ and for more efficient computation, we find a group `Nu` which is isomorphic to `nu`.

```
gap> # Construct the group Nu which is isomorphic to group nu
gap> nu:=NonAbelianTensorSquarePlus(G);
Pcp-group with orders [ 2, 2, 2, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0, 0, 2,
0, 0, 0, 0, 4, 8, 8 ]
gap> nuiso:=IsomorphismPcpGroup(Image(IsomorphismFpGroup(nu)));;
gap> Nu:=Image(nuiso);
Pcp-group with orders [ 2, 2, 2, 0, 0, 0, 0, 2, 2, 2, 0, 0, 0,
0, 2, 0, 0, 0, 0, 4, 8, 8 ]
```

We then construct a group L which is isomorphic to G and and a group R which is isomorphic to G^φ in $\nu(G)$. The GAP object CommutatorSubgroup(L,R) is isomorphic to the subgroup $[G, G^\varphi] \cong G \otimes G$ of $\nu(G)$.

```
gap> # Copy of $G$ in Nu
gap> L:=Subgroup(Nu, Igs(Nu){[1..7]});
Pcp-group with orders [ 2, 2, 2, 0, 0, 0, 0 ]
gap> # Copy of $G^\varphi$ in Nu
gap> R:=Subgroup(Nu, Igs(Nu){[8..14]});
Pcp-group with orders [ 2, 2, 2, 0, 0, 0, 0 ]
gap> CommutatorSubgroup(L,R);
Pcp-group with orders [ 2, 0, 0, 0, 0, 4, 8, 8 ]
```

We find generators Comm(L,R) that are isomorphic to commutators in the set (4). Below are the GAP commands.

```
gap> a:=Igs(Nu)[1];;    ap:=Igs(Nu)[8];;   b:=Igs(Nu)[2];;
>    bp:=Igs(Nu)[9];;   c:=Igs(Nu)[3];;    cp:=Igs(Nu)[10];;
>    l1:=Igs(Nu)[4];;   l1p:=Igs(Nu)[11];; l2:=Igs(Nu)[5];;
>    l2p:=Igs(Nu)[12];; l3:=Igs(Nu)[6];;   l3p:=Igs(Nu)[13];;
>    l4:=Igs(Nu)[7];;   l4p:=Igs(Nu)[14];;
gap> list:=[a,b,c,l1,l2,l3,l4,ap,bp,cp,l1p,l2p,l3p,l4p];;
gap> # Commutators isomorphic to the commutators in the set (4).
gap> t1:=Comm(a,ap);;     t2:=Comm(b,bp);;     t3:=Comm(c,cp);;
>    t4:=Comm(l4,l4p);;   t5:=Comm(a,bp);;     t6:=Comm(a,cp);;
>    t7:=Comm(a,l1p);;    t8:=Comm(a,l2p);;    t9:=Comm(a,l4p);;
>    t10:=Comm(b,cp);;    t11:=Comm(b,l1p);;   t12:= Comm(b,l3p);;
>    t13:=Comm(b,l4p);;   t14:=Comm(c,l2p);;   t15:= Comm(c,l3p);
>    t16:=Comm(c,l4p);;
>    t17:= Comm(a,bp)*Comm(b,ap);;   t18:= Comm(a,cp)*Comm(c,ap);;
>    t19:= Comm(a,l1p)*Comm(l1,ap);; t20:= Comm(a,l2p)*Comm(l2,ap);;
>    t21:= Comm(a,l4p)*Comm(l4,ap);; t22:= Comm(b,cp)*Comm(c,bp);;
>    t23:= Comm(b,l1p)*Comm(l1,bp);; t24:= Comm(b,l3p)*Comm(l3,bp);;
>    t25:= Comm(b,l4p)*Comm(l4,bp);; t26:= Comm(c,l2p)*Comm(l2,cp);;
>    t27:= Comm(c,l3p)*Comm(l3,cp);; t28:= Comm(c,l4p)*Comm(l4,cp);;
```

We interpret the results of the above commands given by GAP and write the commutators in set (4) that are not independent in terms of dependent ones. We have the following:

```
gap> t2 = t1^-4*t3*t18^2; true    gap> t4; id
gap> t8=t7*t3^4; true             gap> t9=t5^-2*t6^2; true
gap> t10 = t1^2*t3*t5^-1*t6*t7^-1; true
gap> t11=t3^-4*t7^2*t14^-1; true  gap> t13=t1^4*t3^4*t5^-2*t6^2; true
gap> t15=t12; true                gap> t16=t1^4*t3^4*t5^-2*t6^2; true
gap> t17=t1^-4*t18; true          gap> t19=t18^2; true
gap> t20=t18^2; true              gap> t21; id
gap> t22= t3^2*t18^2; true        gap>  t23=t3^4; true
```

```
gap> t24=t18^2; true                gap> t25; id
gap> t26=t3^4; true                 gap> t27=t18^2; true
gap> t28; id
```

Hence we have the generating set (4) is reduced to the following set

$$\{[a,a^{\varphi}],\ [c,c^{\varphi}],\ [a,b^{\varphi}],\ [a,c^{\varphi}],\ [a,l_1^{\varphi}],\ [b,l_3^{\varphi}],\ [c,l_2^{\varphi}],\ [a,c^{\varphi}][c,a^{\varphi}]\}$$

We then check whether the above set generates $G\otimes G$ and we also check the order of each generator by the following commands.

```
gap> list:=[t1, t3, t5, t6, t7, t12, t14, t18];;
gap> CommutatorSubgroup(L,R)= Subgroup(Nu, list);
true
gap> List(list,Order);
[ 8, 8, infinity, infinity, infinity, infinity, infinity, 4 ]
```

Hence this prove the lemma. ■

The main goal of the computation of the nonabelian tensor square $G\otimes G$ of the centerless Biebergroup G with dihedral point group is to determine its presentation. The presentation of $G\otimes G$ is given by the following theorem.

Theorem 8 *Let G be the Bieberbach of dimension four with the dihedral point group which has a polycyclic presentation as in 3. Then the nonabelian tensor $G\otimes G$ of the group G is nonabelian and is given as follows*

$$\begin{aligned} G\otimes G = \langle\, g_1,g_2\ldots g_8\ |g_1^8 = g_2^8 = g_3^4 = [g_4,g_5] = [g_4,g_7] \\ = [g_5,g_7] = [g_6,g_7] = [g_6,g_8] = [g_7,g_8] = 1, \\ [g_4,g_6] = [g_5,g_6] = g_2^4g_6^2,\ [g_4,g_8] = [g_5,g_8] = g_2^4g_8^2, \\ [g_i,g_j] = 1 \ \textit{for}\ 1\le i\le 3,\ 1\le j\le 8\,\rangle, \end{aligned} \tag{5}$$

where

$$\begin{aligned} a\otimes a = g_1,\ c\otimes c = g_2,\ (a\otimes c)(c\otimes a) = g_3, a\otimes b = g_4 \\ a\otimes c = g_5,\ a\otimes l_1 = g_6,\ b\otimes l_3 = g_7,\ c\otimes l_2 = g_8. \end{aligned}$$

Proof. By Lemma 7, $G\otimes G$ is generated by the set

$$\{g_1,\ g_2,\ g_3,\ g_4,\ g_5,\ g_6,\ g_7,\ g_8\}. \tag{6}$$

Next, we determine the relations of $G\otimes G$.
Lemma 7 gives us that g_1 , g_2 are of order eight and order of g_3 is four, hence we have

$$g_1^8 = g_2^8 = g_3^4 = 1. \tag{7}$$

By commutator identities of $\nu(G)$ in Rocco [13], we have $g_1,\ g_2,$ and g_3 are central in $\nu(G)$. Hence, we have

$$[g_i,g_j] = 1 \ \text{for}\ 1\le i\le 3,\ 1\le j\le 8. \tag{8}$$

The commutator of the elements of $G\otimes G$ that are not in the central of $\nu(G)$ are as follows: $[g_4,g_5]$, $[g_4,g_6]$,$[g_4,g_7]$, $[g_4,g_8]$, $[g_5,g_6]$, $[g_5,g_7]$, $[g_5,g_8]$, $[g_6,g_7]$, $[g_6,g_8]$ and $[g_7,g_8]$. Now we compute the above commutators. Again hand computations are very lengthy and tedious and are ommited here and we make GAP to assist the computation.

```
gap> Comm(t5,t6); id                gap> Comm(t5,t7)=t3^4*t7^2; true
gap> Comm(t5,t12); id               gap> Comm(t5,t14)=t3^4*t14^2; true
gap> Comm(t6,t7)=t3^4*t7^2; true    gap> Comm(t6,t12); id
gap> Comm(t6,t14)=t3^4*t14^2; true  gap> Comm(t7,t12); id
gap> Comm(t7,t14); id               gap> Comm(t12,t14); id
```

With the above results, we have $[g_4, g_5] = [g_6, g_7] = [g_6, g_8] = [g_7, g_8] = 1,\ [g_4, g_6] = [g_5, g_6] = g_2^4 g_6^2$ and $[g_4, g_8] = [g_5, g_8] = g_2^4 g_8^2$

Hence with these results and by (6), (7) and (8), it is proved that $G \otimes G$ has a presentation as in (8) and by its relations, $G \otimes G$ is not abelian. ■

4 Conclusion and Future Work

This paper gives the computation of the nonabelian tensor square of a centerless Bieberbach group of dimension four with the dihedral point group of order eight. We can see that calculations along with the theories of the nonabelian tensor square of groups, particularly of polycyclic groups can give us the generators and the presentation of the nonabelian tensor squares of the group that computer computational cannot give. However, the computer computational, particularly GAP, is powerful in helping us to make the computation faster and more efficient. GAP is very helpful especially when we are dealing with higher dimension of, not restrict to only Bieberbach groups with dihedral point group but to other general Bieberbach groups and polycyclic groups as well. In future, we will explore GAP in computing other homology functors such as $G \wedge G, \nabla(G), J_2(G)$ and others as in (2).

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