Variables and Functions: Using Geometry to Explore Important Concepts in Algebra

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Abstract: *Students find the concepts of* variables *and* functions *in algebra very challenging. Part of the challenge arises from the use of static media, and part arises from insufficient emphasis on how variables* change *and functions* behave*. A geometric approach provides students with experiences in which they manipulate variables directly and continuously, and in which they observe, directly and immediately, the behavior of functions as they vary the independent variables. Sketchpad® dynamic mathematics software makes it easy for students to create geometric variables and functions and to use them to study domain, range, composition, and inverses. Through these experiences, students can more easily progress from an* atomic *view of function (taking a single input variable to a single output variable) to a* collective *view (mapping an entire domain to an entire range).*

Introduction

Mathematics educators have engaged in serious discussion over many years about how we ought to teach algebra, and what elements we ought to emphasize. [1, 2, 4]

Static approach to algebra

In the US, the traditional approach has emphasized variables as particular numbers whose value is unknown, and an emphasis on evaluating expressions and solving equations. This emphasis still predominates in many US high school textbooks.

In this approach, variables are initially viewed as static placeholders, as in the problems shown in figure 1. (In the second problem, x represents two numbers, not just one, but there's still no sense of variation.) As they progress, students learn that these placeholders can take on different values, and develop their concept of variability by extending the static view to encompass many values—at first many discrete values, and eventually many continuous values.

Evaluate $2x^2 - 3x + 1$ for $x = 2$

Solve $2x^2 - 3x + 1 = 10$

Figure 1

Similarly, functions are initially presented as static rules that take as input one particular number, and produce another particular number as output. At least at first, there's no sense of the *behavior* of functions; they just take one number and give you another.

Many of the equations we write are static, with no real sense of variation or behavior. Zalman Usiskin [4] gives the examples shown in Figure 2. He observes that the first is a formula or rule, the second is an equation to solve, the third is an identity, and the fourth is a property. Only the fifth gives a sense of x as a variable, a dynamic quantity that actually changes.

1. $A = LW$
2. $40 = 5x$
3. $\sin x = \cos x \cdot \tan x$
4. $1 = n \cdot 1/n$
5. $y = kx$

Figure 2

Dynamic approach to algebra
Chazan [1] and others argue for a competing approach: to emphasize the role of variables as changing quantities, and the concept of functions and their behavior as the unifying theme of algebra. The dynamic elements are critical here: variables are changing quantities, and functions behave in certain ways.

This approach emphasizes the behavior of functions as a way to characterize the relationships between quantities that vary. Students think of a function as a relationship between changing quantities, and explore its behavior by changing the value of the independent variable.

The difference is emphasis
These are not absolute categories. Students in traditional classes study functions and vary variables, and students in classes using the dynamic approach learn to evaluate expressions and to solve equations. The difference is emphasis: what do we communicate to students as the fundamental object of study in algebra? To some extent, our opportunities are limited by the available media.

Static media
One of the difficulties with moving to a more dynamic approach to teaching algebra has that it's hard to give students the experience of varying a variable, and hard to give them the opportunity to explore and observe the behavior of a function as they change the independent variable. Until recently almost all of our instructional media (textbooks, chalkboards, overhead projectors) have been static media. It's difficult, and unpersuasive, to portray variation with static media, and it's even more difficult to give students direct control of variation. If students could control the independent variable, if they could easily and fluidly change its value at will, they'd gain a dynamic ability to explore the behavior of functions.

Dynamic media
The last 20 years have seen the development of dynamic mathematics software such as The Geometer's Sketchpad®[3], software that provides users with dynamic control of the mathematics they create. Sketchpad® began as a dynamic geometry program, allowing students the powerful ability to manipulate geometry by dragging points and other geometric objects on the screen. More recently, Sketchpad has added important algebra features, allowing the creation of numeric values that can be easily changed and animated, and supporting the creation and graphing of functions.

Still, students' ability to vary geometric objects is much more direct than their ability to vary numeric objects. It's easy for a student to drag a point around on the screen and see how a construction changes. The student has direct control of the point by dragging it with the mouse. She can drag it this way or that, she can drag it quickly or slowly, she can even drag it carefully, while observing the behavior of the construction, to achieve a particular objective.

Numeric variables in Sketchpad
Numbers are easy to create, and they're easy to change and to animate. A student can create a number, edit it easily, change it discretely using the + and – keys, or vary it continuously by animating it. And by using it in a calculation such as $x^2 - 5$, she can create a dependent variable that depends upon it, she can plot the two values in Cartesian coordinates, and she can see the graph develop as the independent variable changes (figure 3). Still, the

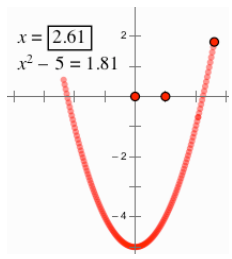


Figure 3

student doesn’t have the same sense of control as she does when dragging a point that determines a geometric construction.

The student can do a bit better by putting a point on the *x*–axis, measuring its abscissa, and using this as the independent variable. Now, in figure 4, she can drag the independent variable and watch the behavior of the function. She can see the value of the dependent variable change numerically, and see its value change on the graph as the plotted point moves. Though achieving dynamic control of a variable is a bit more complicated than achieving dynamic control of a point, she has direct control of the variable by means of the mouse. She can drag to change a number, making it a true variable. In the process she observes how this variation defines the function and creates a graphical representation of the function’s behavior.

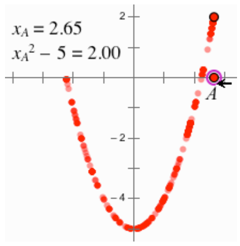
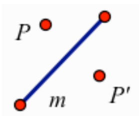
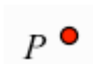


Figure 4

Shortcomings of numeric variables and functions
Even with these powerful ways to vary numbers, students don’t get the same sense of complete control and continuous variation in the numeric realm as they do with points in the geometric realm. This remains an important concern; concrete experience with the dynamic, continuous nature of variables can make a significant difference in students’ ability to investigate the behavior of functions. Current ways of manipulating numbers don’t provide a student with the same tactile experience, don’t give the same sense of control, don’t create the same satisfaction and sense of power, as does the dragging of points.

Geometric variables and functions

Most definitions of function don’t specify that we have to use numbers as the input and output, as the independent and dependent variables. And in fact there’s plenty of precedent for the use of geometric variables and geometric functions. In the numeric realm the variable, the unitary changeable object, is a number; in the geometric realm it’s a point. Students already study functions of points, though they don’t usually call them functions. Most of the time they call them transformations. But in fact these words have the same mathematical meaning. We can say that a function transforms an input variable into an output variable, and we can say that a transformation takes a pre-image and produces an image. The input (pre-image) and output (image) may be numbers, or they may be points. It doesn’t matter what terms we use; the operation of converting an input to a unique output is what characterizes a function.

Variable	Example Function	Input (Pre-image)	Output (Image)	Notation
Number	$f(x) = 2x - 3$	5	7	$f(5) = 7$
Point	Reflection Across *m* 			$r_m(P) = P'$

Advantages of geometric functions
Geometric functions present some powerful advantages as a way to explore important concepts. Students can easily do many important activities and investigations:

- Create a function starting from a new page.

• Manipulate the independent variable by dragging it.
• Observe the behavior of the function as they drag.
• Record the behavior of the function.
• Confine the independent variable to a restricted domain.
• Observe the resulting range of the function.
• Compose two functions.
• Investigate how a restricted domain affects a composite function.
• Experiment to find an inverse function.
• Investigate functions defined by locus constructions.
• Use custom transformations to create a wide variety of other functions.

Observing and recording a function's behavior

With numeric functions, students observe and record a function's behavior by creating a table of values. Looking at a table of numbers doesn't give a good sense of how a function behaves, so students often graph the values in the table. With a geometric function, students can trace both the input and output points, drag the input, and get an immediate image of the function's behavior. The traces recorded on the screen for a geometric function, as in figure 5, are the equivalent of the table of values for a numeric function, but they give a much better sense of the function's behavior.

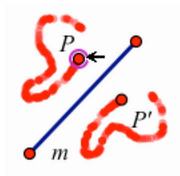


Figure 5

Using a restricted domain

Students often have trouble understanding what it means to restrict the domain of a numeric function, and why one would want to do so. With a geometric function, it's easy, and logical, to want to restrict the domain of a function; doing so gives us a nice picture of the relationship between the variables, by allowing us to see the range that corresponds to any domain. In figure 6, the student has created a function that rotates the input by 90° around center point *C*, has restricted the domain by merging the independent variable *P* to a triangle, and has dragged the independent variable to observe the resulting range.

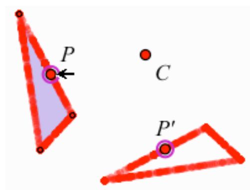


Figure 6

Composing functions

When they compose numeric functions, students often get confused about which function is applied first, in part because the functions are expressed in symbolic form. This symbolic form is somewhat abstract; at least for a beginning algebra student, looking at the symbols doesn't communicate a clear sense of the function's behavior. Geometric functions are much more concrete, with much more obvious behavior. In figure 7 the student has reflected the independent variable across mirror m, rotated around center point C by 90°, and then dragged the input. To get an even better picture of the behavior of the composite function, the student might decide to use a restricted domain, as in figure 8.

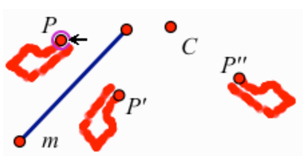


Figure 7

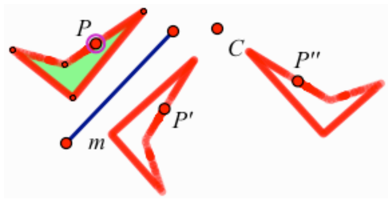


Figure 8

Finding an inverse function

Students often have trouble understanding just what it is that inverse functions do. Concrete examples with geometric functions can help. In figure 9, the student is trying to find the inverse of a rotation around point C by 120°. She has composed this function with a second rotation, about point D by an adjustable angle. The distinguishing feature of an inverse of a geometric function is very concrete: it must put point P" exactly back on top of point P. By experimenting with the position of point D and the adjustable angle, the student maneuvers point P" closer and closer to P, and finds that she can get the two points to coincide by dragging D to the same location as C and by making the second rotation 240°. She checks the result by dragging the independent variable P, and confirms that an inverse of rotation by 120° about C is rotation by 240° about C.

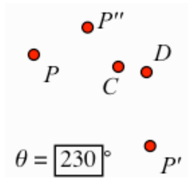


Figure 9

Similarly, a student can start with a dilation by a scale factor of ½, and experiment to determine that the inverse is dilation by a scale factor of 2.

Locus constructions as functions

When a student creates a point locus construction, she defines a point on a path as the driver and another point as the driven object that forms the locus. For instance, in figure 10, the students has constructed segment AB with point A on a circle, and has put point D on the segment. We can consider point A as the independent variable (restricted to the circle as its domain), and point D as the dependent variable. The student can drag A around the circle to see the range, or she can use the Locus command to create the path that corresponds to the range of this function.

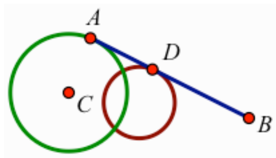


Figure 10

Similarly, in figure 11, the student has attached A to a circle, constructed the perpendicular bisector of AB, and constructed point D, the intersection of this bisector with the extended radius through point A. The independent variable (point A) is restricted to the circle as its domain; by constructing the locus we see the corresponding range in figure 12. (Note that this range consists of all points equally distant from point B and the circle, measuring the distance to the circle along a radius.)

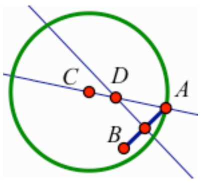


Figure 11

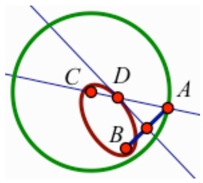


Figure 12

Because the locus construction creates the function's entire range as an object, it's possible to change the definition of the function (for instance, by changing the size of the circle or by dragging point B) to see the effect on the range. Students find it particularly interesting to drag point B outside the circle!

In a locus construction such as this, students are encouraged to make an important conceptual leap, from viewing a function as an operation that that takes a single input point to a single output point to a more collective view, a global view in which the function maps the entire domain to the entire range. Such a conceptual leap is much harder for students to make when they're working with numeric functions, but it's a natural transition in the geometric realm.

Custom transformations

Our early examples involved Sketchpad's built-in transformations, which are isometries and similarities. Sketchpad also allows students to create custom transformations, which can be used to create any transformation whatsoever in which the position of one point affects the position of another. This allows tremendous flexibility and creativity, and in this age of computer graphics students find the results very exciting.

In figure 13 is the starting point for a simple example: the student wants to rotate point P around center C by an angle that depends on the distance PC. She uses the parameter θ to determine the amount of the rotation: 10° per cm of distance. To define the transformation, she multiplies the distance by θ, and then divides by 1 cm so that her units are still degrees. She rotates point P by the calculated angle to create point P', the dependent variable. By dragging the independent variable P, she determines that the function behaves as expected, so she selects both variables and uses them to define a custom transformation. In figure 14 she's applied this custom transformation to a segment and to a circle to see how it affects both of those objects.

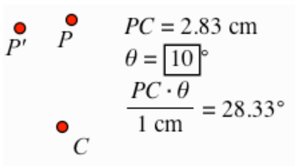


Figure 13

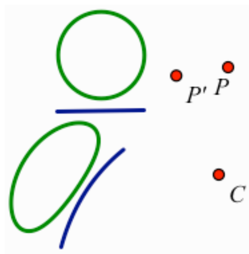


Figure 14

Now the student has a treat in store. She pastes a picture from the web into her sketch and applies the custom transformation to the entire picture, with the result shown in figure 15 (with θ = 5°) and in figure 16 (with θ = 45°).

The use of a picture allows the student to visualize how this function affects the entire plane, by comparing the remembered pre-image with the resulting image. The student can also see the close connection between the "atomic" transformation of the single point P and the "collective" transformation of an entire two-dimensional region. Furthermore, the student can easily create and experiment with a wide variety of interesting functions based on easily-understood definitions.

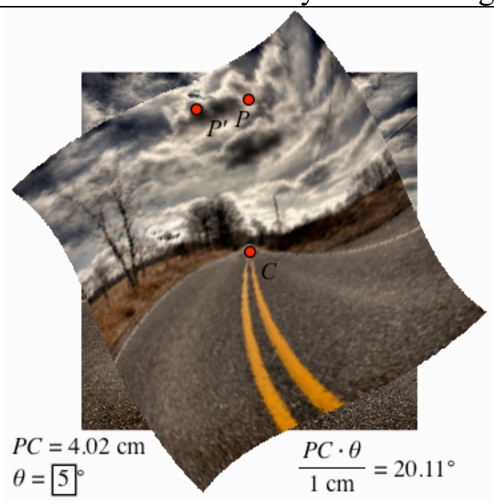


Figure 15

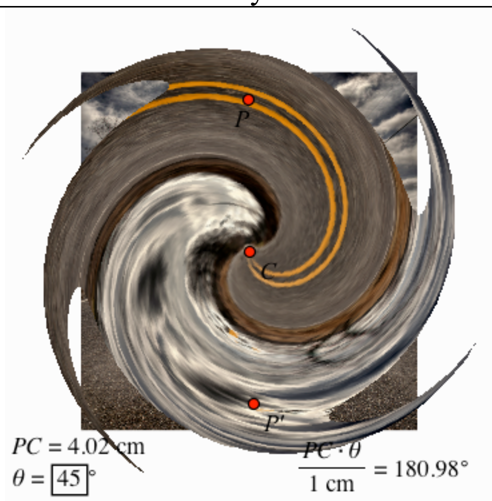


Figure 16

Returning to the Numeric Realm

Students who have mastered these concepts in the geometric realm will be prepared to understand the workings of numeric functions. They will be able to apply their visualizations of domain, range, composition, and inverses to numeric functions. Later, by situating geometric functions on the Cartesian plane, they will be well prepared to relate complex functions, and functions of two real variables, to their work with geometric functions.

Conclusion

The geometric functions shown here have tremendous potential for developing students' understanding of functions, domain, range, composition, and inverses. They broaden students' experiences by erasing the misconception that functions apply only to numbers. They provide concrete, geometric opportunities to vary the independent variable, to create a visual "table of values," to restrict the domain to a particular shape and observe the range, to compose two functions, to understand the inverse of a function in terms of composition, to understand a function as a mapping of an entire set of points (the domain) to another entire set of points (the range), and to create and view a function that maps one entire region of the plane to another.

If we use activities like these to give students a solid grounding working with geometric functions, they will be far better prepared to understand functions in the algebraic realm. Their concrete experiences dragging variables and observing the behavior of functions, and their visual images of domain, range, composition, and inverses will be of great benefit to their continued mathematical study.

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