Polyhedral Tensegrity Structures

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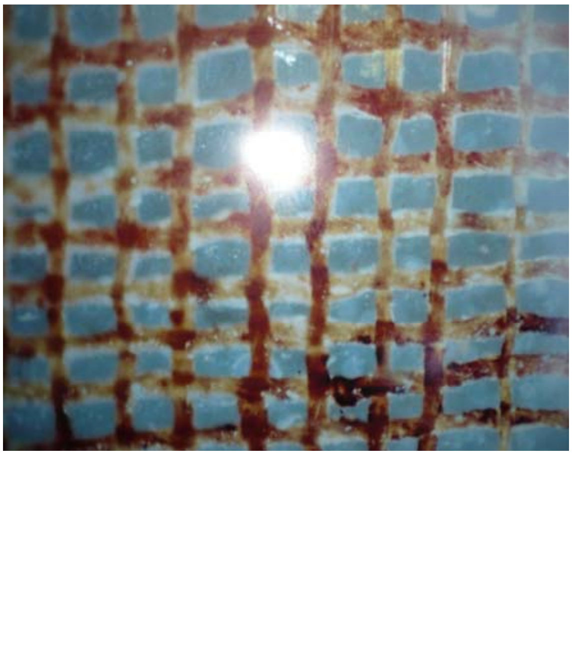
1. Introduction by personal experience



For years I have been admiring the cleverness and mastery of ancient local village carpenters who had constructed a roof covering a cattle shed which is nowadays used after refurbishment as a holiday cottage. Just before falling asleep I stared at a post and four wooden elements forming a strange X. Half of the X is in one plane and half of it is in a plane perpendicular to the initial one. These elements support the roof construction. The upper ones called swords prevent the roof from shifting in the direction of the ridge and the lower ones called struts prevent it from shifting in the perpendicular direction. The roof is stable in spite of changing weather conditions. Sometimes one can hear squeaking noises coming from the wooden elements which indicates continuous action of forces.

Another astonishing fact is how it is possible that filaments of byssus keep together in a woven cloth over two thousand years and they do not separate.

My husband runs a chiropractic practice. Sometimes I have to explain to some prospective patients over the phone how the therapy works. I compare the action of the practitioner on patient’s fascia to an actor performing at a puppet show. The actor pulls one string and the marionette moves its hands, bows, etc. Similarly the practitioner gives an impulse to fascia, then muscles and tendons move and cause bones, in particular those in joints, to move. They move to their natural equilibrium position and a patient feels healthy.

At the time when these particular experiences attracted my attention I had no idea that they all had a common denominator.

2. The idea of tensegrity

The above examples belong to a wide range of constructions, structures and actions which became formalized in the twentieth century, hence not so long ago. Their common denominator is a concept of structure with a property called *TENSEGRITY*. The term was coined from two words: tensional integrity. This concept is based on a balance between tension and compression forces acting within the structure. The realization of this concept appears in Nature since the creation of living organisms or even earlier. It may be recognized, according to recent investigations, in every cell, every tissue, every organ, in most architectural constructions, in mechanical devices like for example bicycles.

However, we only recognize something when it gets its name. The term *tensegrity* was coined by Buckminster Fuller in the middle of the twentieth century. And it was Buckminster Fuller, an architect, who developed an icosahedron based on the tensegrity technology. It was not the first mathematical model of a tensegrity structure, since already in the twenties Karl Ioganson contributed his work to the main exhibition of Russian constructivism in 1921. It was a tensegrity prism, a mathematical model consciously constructed to illustrate a mathematical notion.

Fuller is known by constructing geodesic domes. He had experimented before with incorporating tensile components in his architectural works.

However, it was Kenneth Snelson, a student of arts, attending Fuller's lectures, who created a sculpture which was an answer to a longstanding Fuller's problem. Fuller was rejecting Snelson's priorities, claiming that

Snelson's discoveries would never receive proper attention without Fuller noticing that it was the solution to his problem. The history of discovering and applying more tensegrity structures is rich and might require more time.

The notion was used, but what it actually described. It is relatively easy to recognize designata of the notion, but what about a definition?

Let us quote some of them: [VGJ]

1. "a plurality of discontinuous compression columns arranged in groups of three non-conjunctive columns connected by tension elements forming tension triangles" (Fuller, 1962),
2. "structural framework, a novel and improved structure of elongate members which are separately placed either in tension or in compression to form a lattice, the compression members being separated from each other and the tension members being interconnected to form a continuous tension network"(Snelson, 1965),
3. "A tensegrity system is established when a set of discontinuous compressive components interacts with a set of continuous tensile components to define a stable volume in space".(Pugh, 1976),
4. "Tensegrity system is a system in a stable self-equilbrated state comprising a discontinuous set of compressed components inside a continuum of tensioned components."(Motro, 2003).

The above do not resemble definitions of mathematical objects.

Donald Ingber, professor of pathology from Yale University gives a clear explanation what tensegrity systems are. "The term (tensegrity) refers to a system that stabilizes itself mechanically because of the way in which tensional and compressive forces are distributed….Tensegrity structures are mechanically stable not because of the strength of individual members but because of the way the entire structure distributes and balances mechanical stresses...An increase in tension in members throughout the structure - even ones on the opposite side. The global increase in tension is balanced by an increase in compression within certain members spaced throughout the structure…The structure stabilizes itself through a mechanism that Fuller described as continuous tension and local compression… Tensegrity structures offer a maximum amount of strength for a given amount of building material.” [DI]

3. Mathematical definitions

Let $\mathbf{R}^d$ be d-dimensional Euclidean space. For simplicity let d = 2,3.

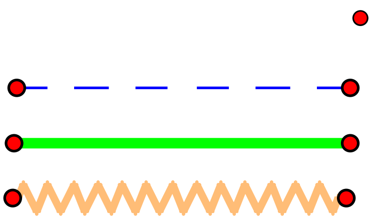
Definition 1. A finite ordered set $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, $\mathbf{p}_i$ € $\mathbf{R}^d$ is called *a configuration*. Points $\mathbf{p}_i$ are called vertices (or nodes) of the configuration.

Definition 2. *A tensegrity graph* **G** is an abstract graph with a set of vertices **V** = {1,2,3,…,n) and a set of edges **E** such that **E** is a disjoint union of three sets $\mathbf{E}_-$, $\mathbf{E}_o$, $\mathbf{E}_+$. Elements of $\mathbf{E}_-$ are called *cables,* elements of $\mathbf{E}_o$ are called *bars* and elements of $\mathbf{E}_+$ are called *struts*. Denote **G** = (**V**; $\mathbf{E}_-$, $\mathbf{E}_o$, $\mathbf{E}_+$.).

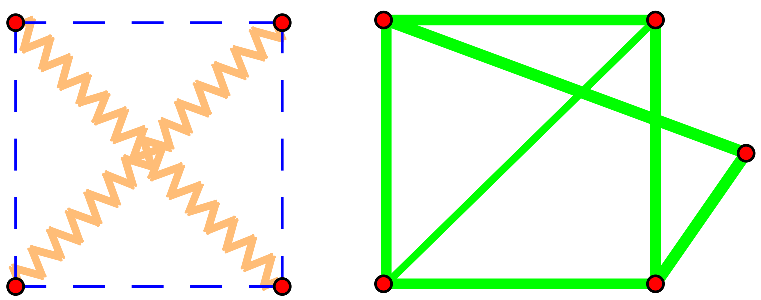
Definition 3. If in the graph **G** the set **V** corresponds to the configuration **p** = $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, $\mathbf{p}_i$ € $\mathbf{R}^d$, then the pair (**G,p**) is called *a tensegrity construction* and denoted by **G(p)**.

For visualization of a construction we use the following convention:

- vertices
- cables
- bars
- struts



Let us consider examples of tensegrity constructions. The first and the third one have even-isometric configurations.



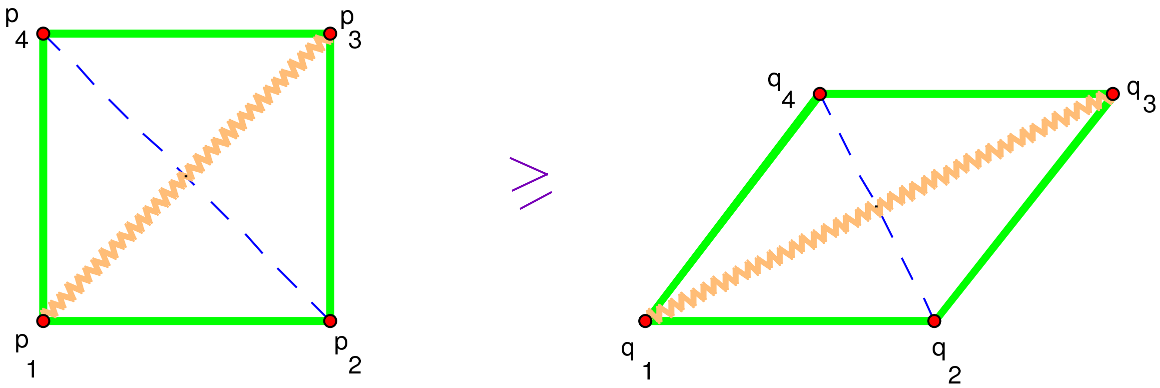
Edges of the second and the third one are bars only. Therefore they are called bar constructions.

The problem of interest is as follows:

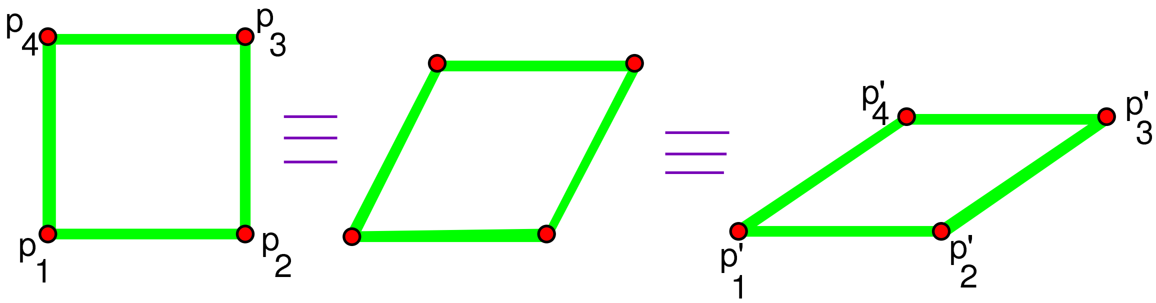
Given a tensegrity construction **G(p)** we would like to describe families **G'(p')** of equivalence classes of configurations **p'** (with respect to even isometries) such that the following constraints are satisfied:

- if $(\mathbf{p_i}, \mathbf{p_j}) \in \mathbf{E_-}$, then $d(\mathbf{p_i}',\mathbf{p_j}') \le d(\mathbf{p_i}, \mathbf{p_j})$ which means that cables can be shortened,
- if $(\mathbf{p_i}, \mathbf{p_j}) \in \mathbf{E_o}$, then $d(\mathbf{p_i}',\mathbf{p_j}') = p(\mathbf{p_i}, \mathbf{p_j})$ which means that bars do not change their lengths,
- if $(\mathbf{p_i}, \mathbf{p_j}) \in \mathbf{E_+}$, then $d(\mathbf{p_i}',\mathbf{p_j}') \ge d(\mathbf{p_i}, \mathbf{p_j})$ which means that struts can extend their lengths.

In this case we will say that **G(p)** *dominates* **G'(p')**. In other words it means that the construction **G(p)** with longer cables and shorter struts dominates the construction **G'(p')**.We write **G(p)** ≥ **G'(p')**.

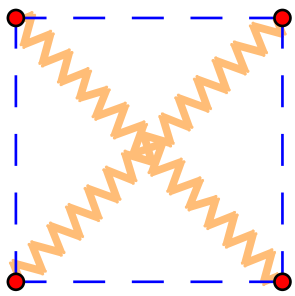


Note that in the family of bar constructions the relation of dominance is an equivalence relation.

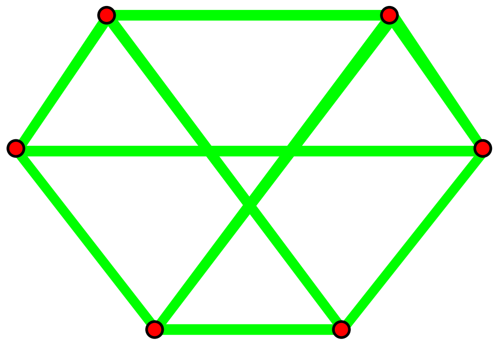


Definition 4. We say that **G(p)** is *globally rigid* in $\mathbf{R}^d$ if **G(p)** ≥ **G'(p')** in $\mathbf{R}^d$ implies that configurations **p** and **p'** are even-isometric in $\mathbf{R}^d$.

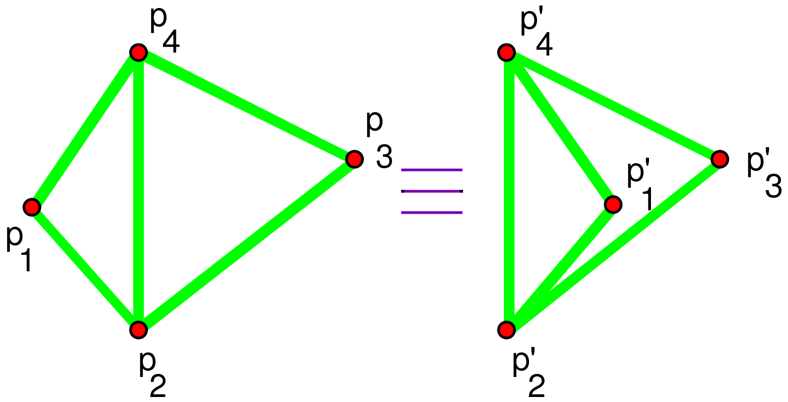
The construction is globally rigid. The bar constructions above are not globally rigid .The whole continuous family of constructions depending on the angle between struts is creating non-isometric constructions such that each one dominates the other one because the bars are all of equal lengths and there are neither struts nor cables.



There are constructions which are globally rigid on a plane, like the one at the picture.



Global rigidity depends on where the construction is considered.



The above construction is globally rigid on a plane and is not globally rigid in the space.

Definition 5. For a given tensegrity construction **G(p)** if there exists a number s, s>0 ,such that for every configuration **p'**, if **G(p)** dominates **G(p')** and configurations **p** and **p'** are closer than s, then **p** and **p'** are congruent, then **G(p)** is said to be *rigid.*

The notion of rigidity also depends on the ambient space, since there are rigid tensegrity constructions on the plane which are not rigid in the space.

Definition 6. If a tensegrity construction is not rigid then we say that it is *elastic*.

Deciding on type of rigidity can be a tedious and serious mathematical job.

How to formalize the notion of stress?

To each edge of the graph underlying the construction we associate *a stress coefficient* . This number is non-negative if the edge is a cable, it is non-positive if the edge is a strut. There are no constraints on these coefficients if the edge is a bar. To obtain an n by n matrix **S(G(p))** we fill in the remaining places with zeroes.

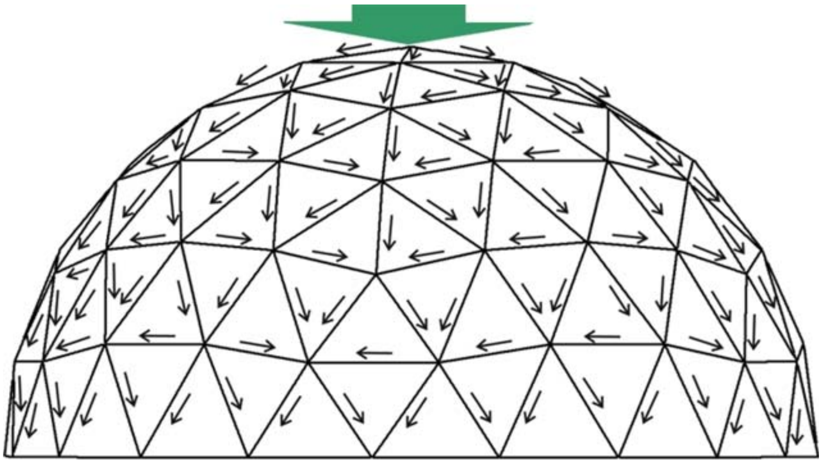
Definition 7.This matrix **S(G(p))** is called *the proper stress* of the tensegrity construction **G(p)**.

Definition 8. We say that *the construction is in the equilibrium* if for every vertex i of the graph the sum of vectors $\mathbf{p_j} - \mathbf{p_i}$ with the corresponding stress coefficients is the zero vector.

The theory of just defined structures is being developed in specialized papers and is rather difficult. These definitions are due to Robert Connelly. [RC] He is famous by his example of a flexor, that is a non-convex polyhedron that admits a continuous family of bends such that they do not destroy the faces. Only the edges act as hinges.

4. Some properties and examples

All tensegrity structures are systems that stabilize themselves mechanically. Tension forces acting on one of the elements cause increased tension in other members throughout the whole structure – even far away from the stressed element. This can be illustrated by the following example. [LR]

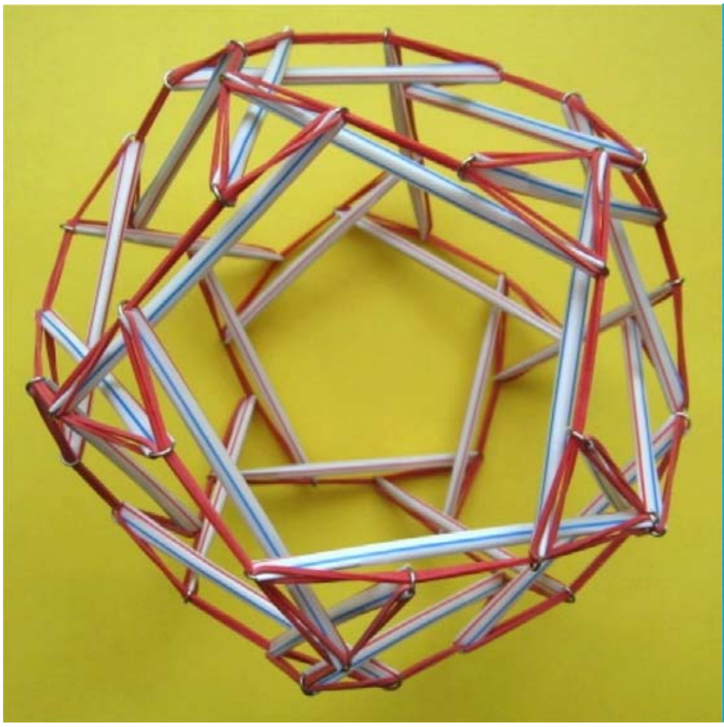


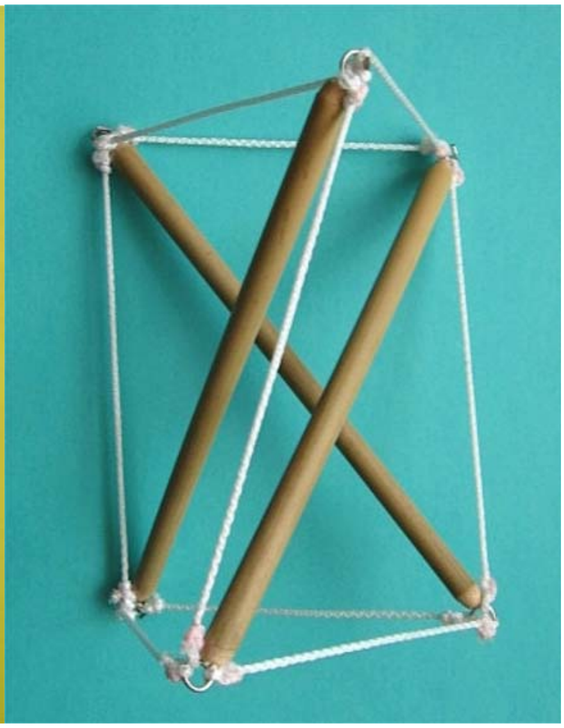
From the point of view of forces acting on tensegrity structures, we may divide them in categories. In one category there are frameworks made up of rigid bars. The bars making up the construction are connected in polygons, mostly triangles, sometimes pentagons or hexagons. Each bar is oriented.

Examples of such constructions are geodesic domes and other roof supporting constructions.

In the other category of tensegrity structures there appears the phenomenon known as prestress. In those structures the elements that bear tension only are distinct from those which bear compression. Independently from external forces like gravity all elements: cables, bars and struts are in tension and compression. Therefore it is said they are prestressed.

Models of such structures are made of soda straws and rubber bands or sticks and strings.



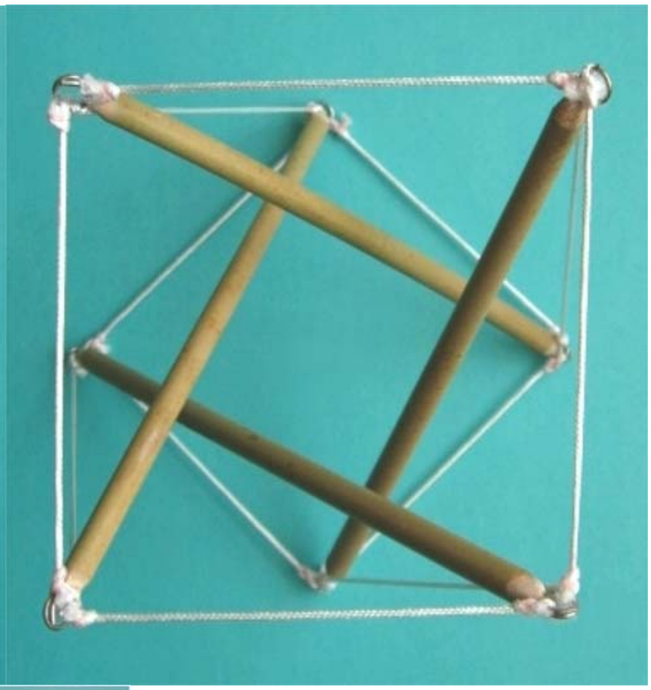
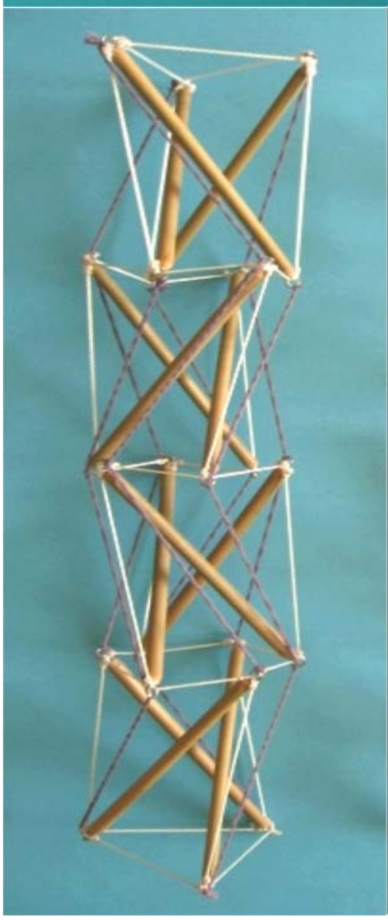
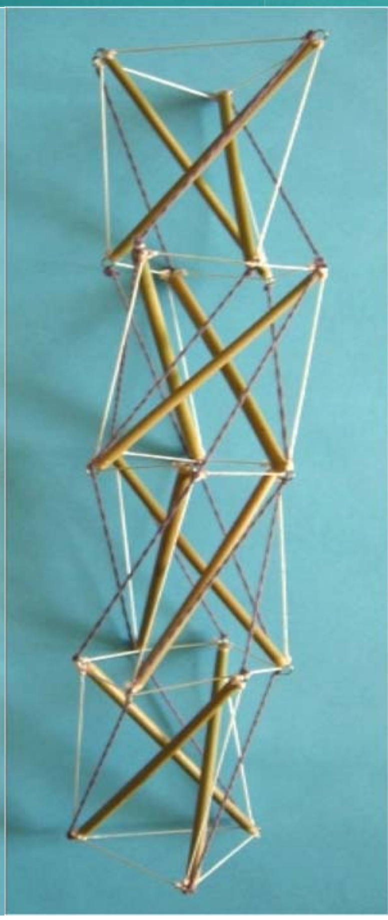


In biotensegrity, which relates to the tensegrity theory within biological sciences, other structures are also considered.

If we look at the fibers of some of our muscles under a microscope, we can see a resemblance to the woven tower, which was pleated out of stripes of paper. The stripes take a form of pieces of a helix. These forms may be extremely compressed without being destroyed.

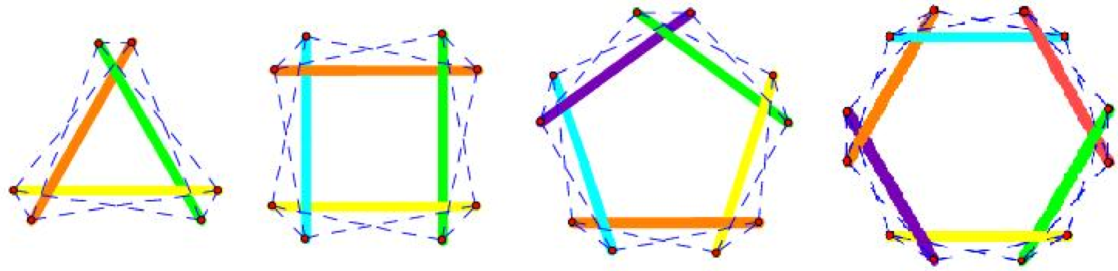
5. Weaving as a genesis of tensegrity structures

Let us look again at the tensegrity prism mentioned above. How is it constructed? Let us take three parallel bars of equal length. Join three ends on one side by three cables and three other ends by three other cables, to receive a regular triangular prism. Twist one of the bases. Now join by cables opposite vertices of parallelograms in such manner that lengths of these cables are of minimal lengths. This structure is called a stable tensegrity triangular prism and it dominates other such structures with the same bars and cables in the bases. Depending on the direction of the angle of twisting we in fact may obtain two types of prisms, an L-prism and its mirror image, an R-prism. One may experiment with copies of this prisms by building towers of prisms. One type of tower may be built of L-prisms only. The other type may be constructed by alternating types L and R. Even simple observation indicates distinct properties of these constructions.

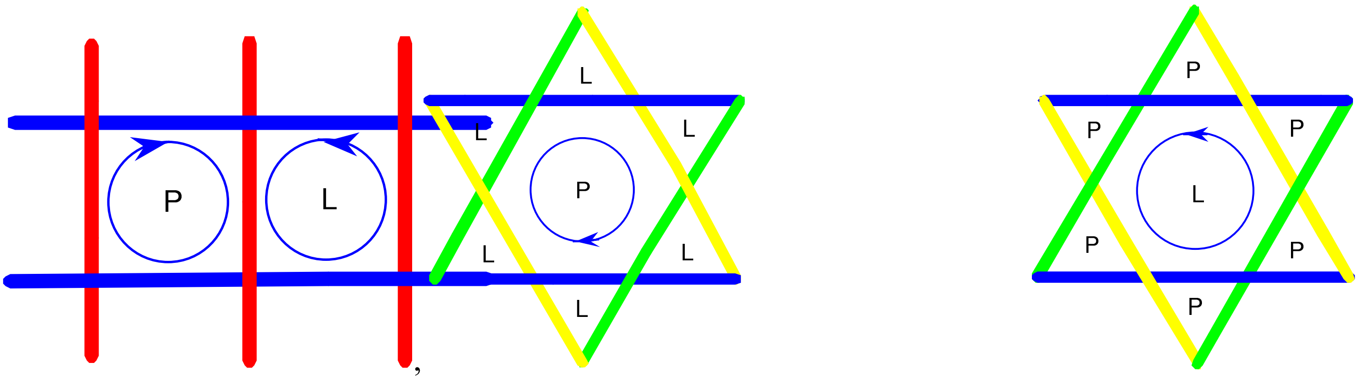



We may similarly construct other regular prisms and consider their projections along a line parallel to the line passing through centers of bases.

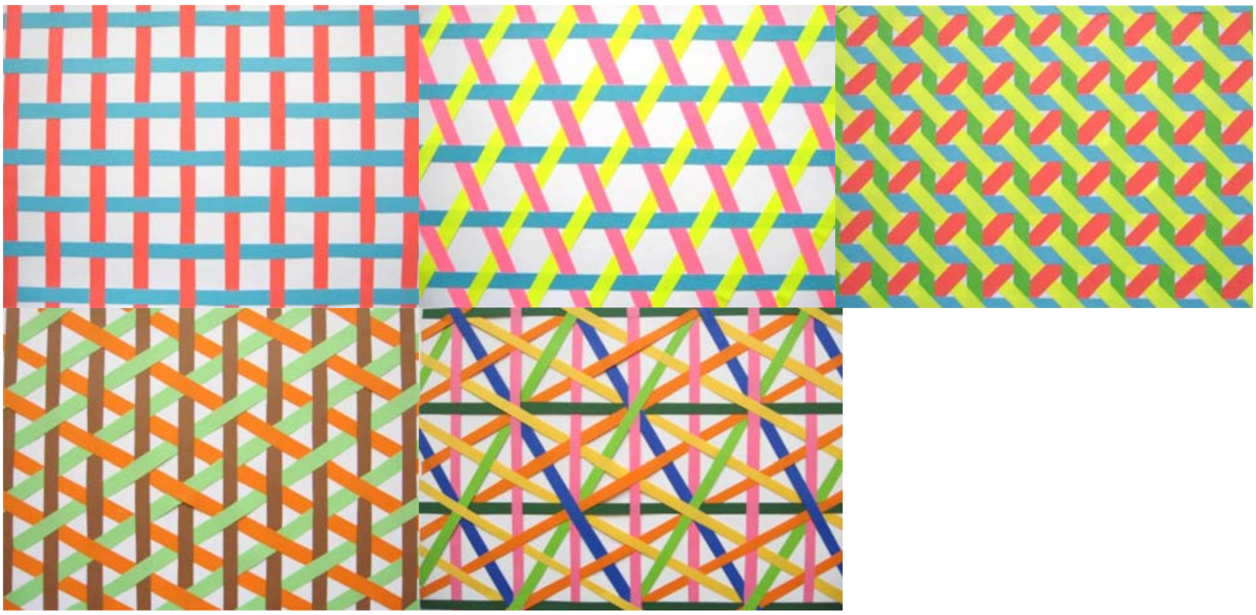
They somehow resemble small fragments of woven cloths joined with additional elements which are cables.



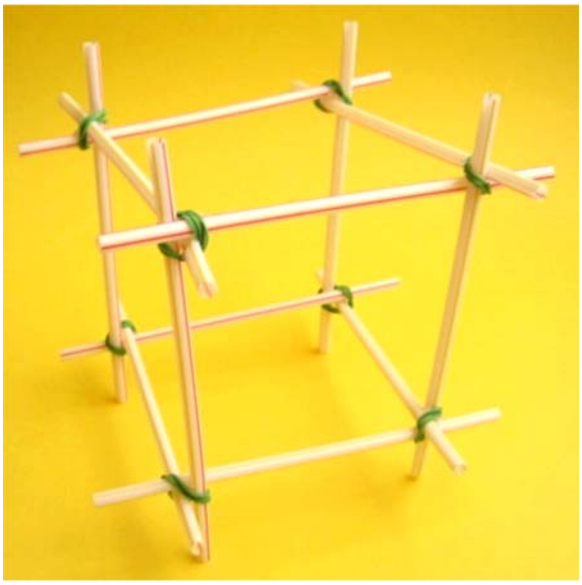
Like in the prisms we could obtain their left or right version, in the same way meshes of the woven fabric may be considered as left or right ones.



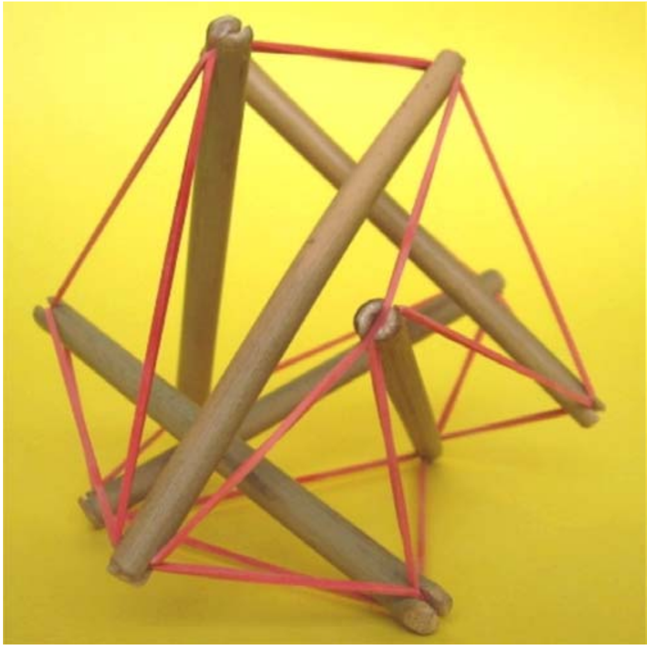
If we use Archimedean mosaics on a plane as underlay for weaving, we get the following patterns.

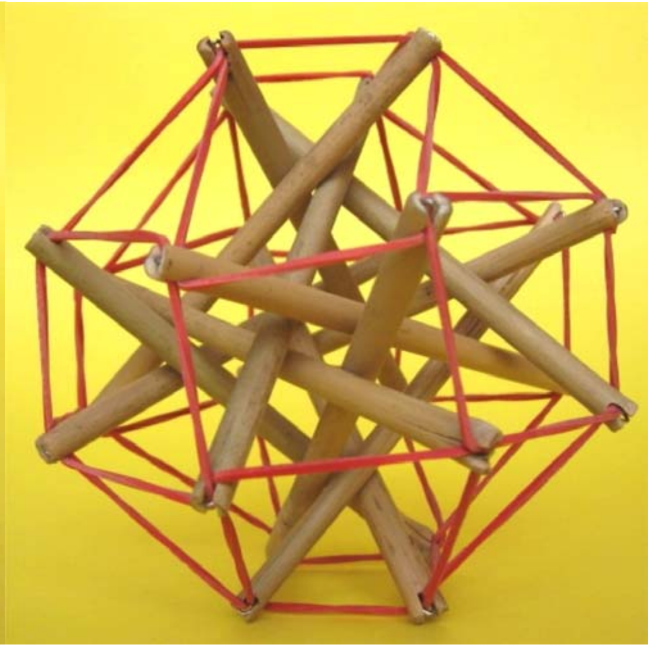


It was Kenneth Snelson who experimented with weaving not only in the plane but also in the space. He also noticed the helical behaviour of bars in the nodes of woven constructions. He modelled his weaving on some Archimedean space fillings. His pleated polyhedra also had two versions L and R.



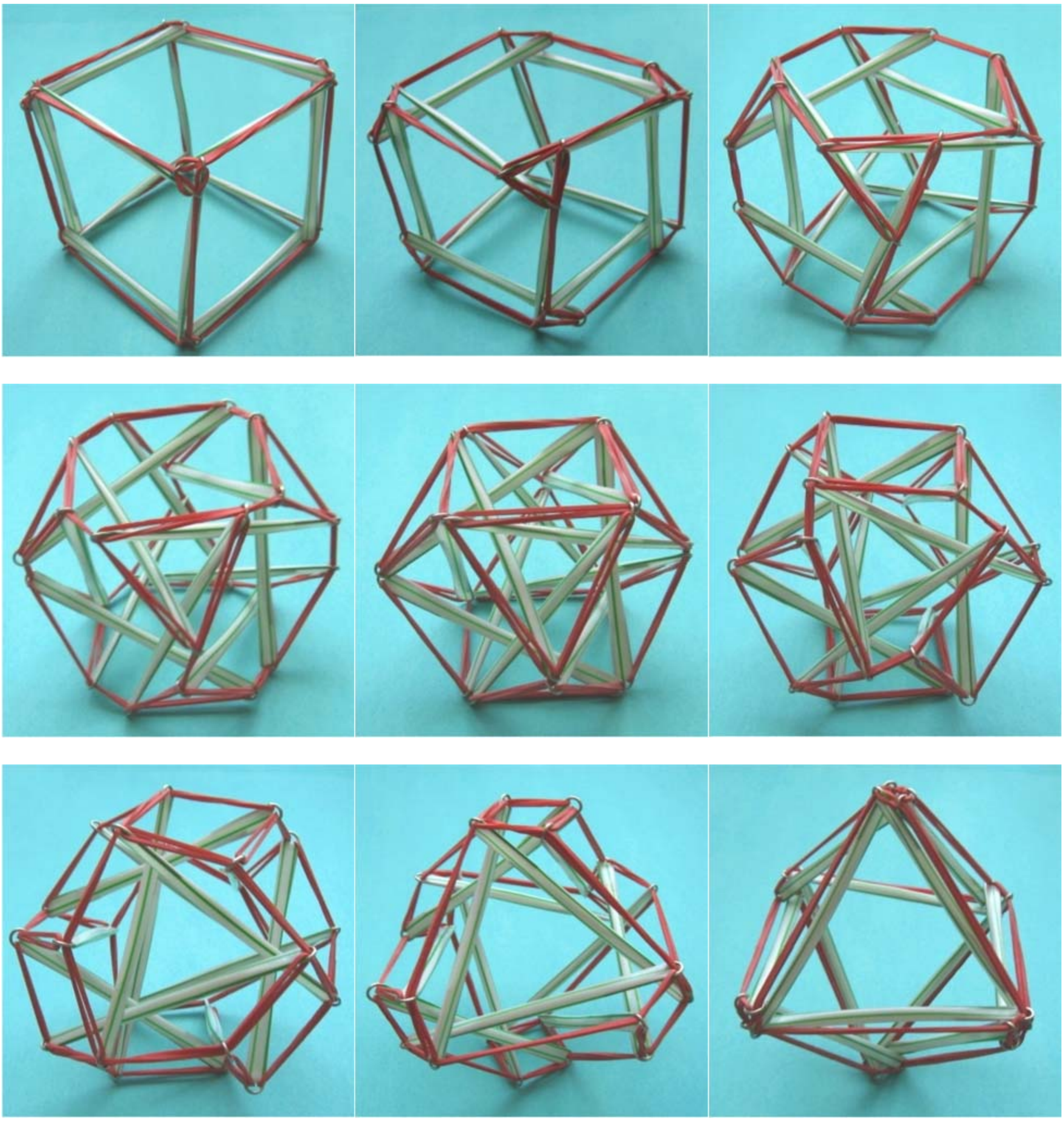
Now, by joining ends of bars by cables in such a spatial woven structure one may obtain examples of polyhedral tensegrity structure. In each case they appear in two versions L and R.





6. Transformations of regular tensegrity polyhedra

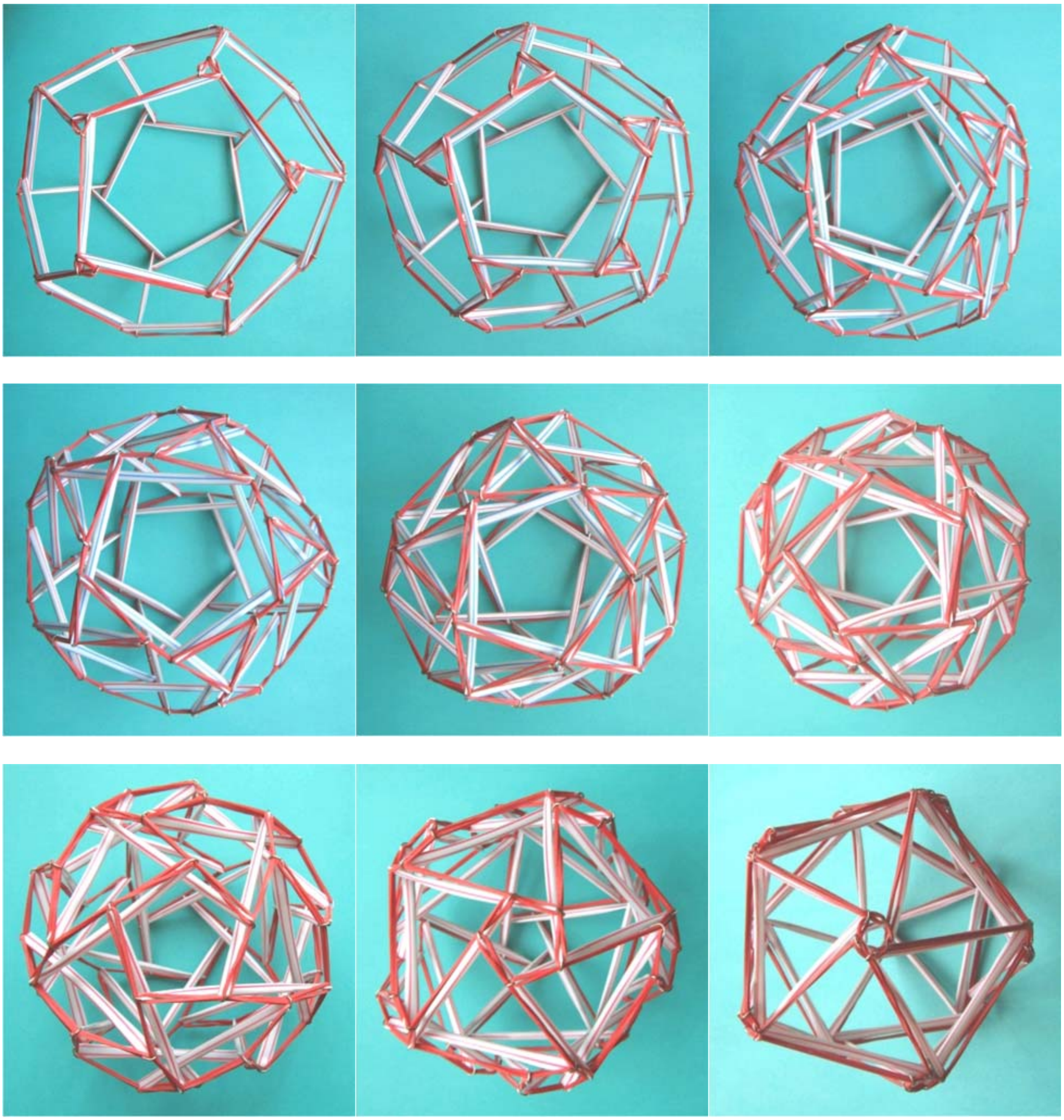
Experimenting with models of tensegrity polyhedra may be a source of nice surprises. We used soda straws and rubber bands models. The steps of transformations are depicted on the sequences of photographs. To begin with we started with an R- model of a cube and ended up with an L-model of a regular octahedron.



Now start with an R-model of a dodecahedron. After performing a sequence of transformations we ended up with an L-model of an icosahedron. (see next page)

These transformations are not unique of that type. One may for example discover relationships between other Archimedean polyhedra. They reveal a small mystery hidden behind most precious gems which are Platonic solids.

In this paper some graphics from [EK] were used.



7. Bibliography

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