A Journey through Chinese Windows and Doors – an Introduction to Chinese Mathematical Art

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Abstract: *We know quite a lot about mathematical art in the West. Hundreds of papers and books were written about Islamic art. Some researchers explored mathematical patterns in tribal art from both Americas as well as from Africa. However, we know very little about mathematical art in Chinese culture.*

There are numerous examples of artistic creations in China that have very strong mathematical background. We can find them in geometric patterns on silk fabrics, on Chinese ceramic ware, in furniture as well as in windows and doors of Chinese buildings.

In this paper we are going to look at some of these patterns from a *transformation geometry point of view. Our main target will be geometric patterns in wooden lattices in windows and doors. We will be interested which of the plane symmetry groups are represented in these patterns and how they can be modeled with dynamic geometry software. All examples analyzed in this paper are modeled with Geometer's Sketchpad. The paper will demonstrate numerous examples taken from real buildings as well as their computer models.*

1. Introduction

In any well established culture we can find intriguing patterns that have a strong mathematical background. Quite often their authors didn't even have the slightest idea of what a mathematician could see in their artworks. This is true with geometric patterns on Iban kilims, Indonesian batics, African pottery, New Guinean tapa rugs and many others. In some other cultures artistic decorations were, and still are, created with the use of geometry principles. This is the case of art in most of the Arab countries, and in the West.

In China, the well known Chinese lattices are often works of high artistic value and in many of them we can find principles of Euclidean and transformation geometry. These interesting objects will be the theme of this paper. We will look at them from a transformation geometry point of view, a discipline that makes an intensive use of transformations like rotations, reflections, glide reflections, translations and dilations. Although the modern theory of symmetry groups and crystallography was not known in China we may expect that the creators of Chinese lattices used some principles that are similar to those that we can find in the papers of Weyl, Coxeter, Conway (see [1],[7]) and other geometers of the twentieth century.

This paper has three major aspects. (1) We will look at Chinese lattices from a historical point of view. We present a very brief overview of their history and we examine briefly Daniel Sheets Dye classification. (2) We look at Chinese lattices from a symmetry groups point of view. In this paper we will concentrate on planar symmetry groups only and we will show how to classify 2D patterns into seventeen symmetry groups. (3) We will show how patterns from selected Chinese lattices can be modeled with Dynamic Geometry software. All our models will be created using GSP 5.0. However, we will mention also some other DG programs and their usefulness in creating such models.

The literature for this project can be divided into two major groups – publications related directly to Chinese lattices and publications related to the mathematical theory of symmetry groups.

There is very little literature describing Chinese lattices. Two major publications describe a kind of visual classification of Chinese lattices (see [2]), and artistic features of a big collection of Chinese lattices, mainly doors and windows from old buildings (see [6]).

Daniel Sheets Dye, the author of the first publication, was for some time Professor in the West China Union University in Chengdu. He made drawings of about 3000 Chinese lattices from various buildings in Chengdu and in the neighboring area. He developed a classification of Chinese lattices based on their visual features. This classification is not at all related to the classification of symmetry groups that is used in mathematics. However, it happens from time to time that a majority of examples from a Sheet's class falls into one or more specific symmetry groups.

Weidu Ma, the author of the second publication, collected a large number of real wooden lattices, mostly windows and doors, restored them and wrote a monograph of a great artistic value. He is a well known collector of Chinese antiques and owner of the Guanfu Classic Art Museum, the first private museum in China.

Between publications on the mathematical foundations of symmetry two publications should be mentioned. The first one by Conway and others (see [1]) presents a very enlightening and modern approach to symmetries of things, not only planar patterns. It is probably the most beautiful mathematical book ever written. This book forms mathematical foundations for this paper.

Slightly older is the book by Claire Horne (see [3]), a student of The Textile Institute at the University of Leeds. This book presents a classical approach to the symmetry of planar patterns based on crystallographic approach. Finally in two papers Majewski and Wang ([4], [5]) present an algorithmic approach to creation of Chinese lattice patterns.

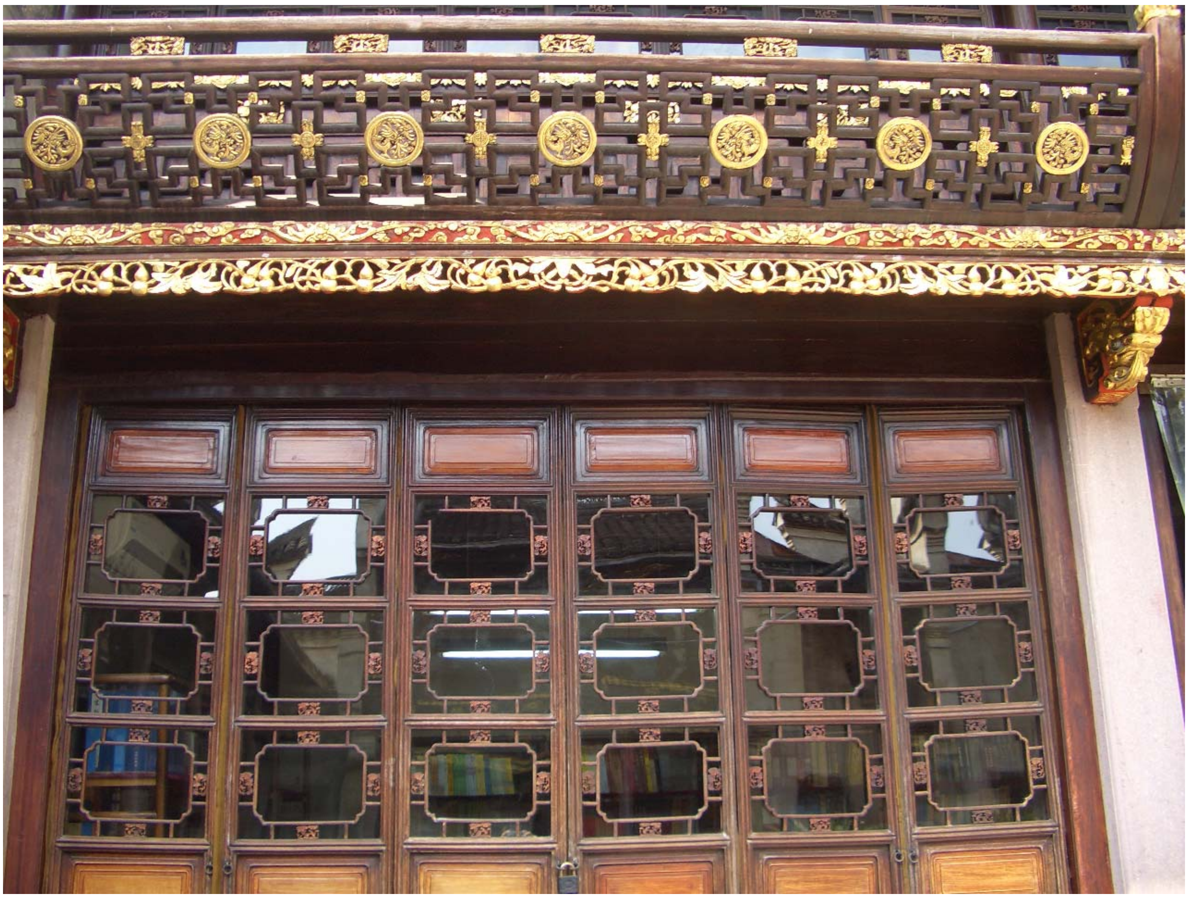


Fig. 1 *Modern examples of Chinese lattices (bottom: rectangles; top: loops)*

2. A Glimpse at the History of Chinese Lattices

Chinese lattices are made out of wood – a material that is not long-lasting. Therefore, the only evidence of the existence of such lattices in the deep past are pictures or reliefs on old bronzes, ancient ceramics and other things that were found in graves. One of the first such findings comes from a grave from the Chou dynasty, that is about 1000 years B.C.

There is more evidence of Chinese lattices during the Han dynasty. They can be found in pictures on grave bricks, clay house models, patterns on tomb walls. There are more of images of such lattices from the Tang dynasty – pottery, clay houses, paintings, printings, and fabrics. From later dynasties there are left examples of real wooden lattices with remarkable and sometimes extremely complex patterns.

It is interesting what kind of mathematical figures were used through centuries and how they were arranged. Let us look briefly into these matters.

- Prior to the Han dynasty: vertical and equally spaced bars in lattice were used.
- Han dynasty (206 B.C. to 220 A.D.): window lattices were not much different from those in the past. However, sometimes they were covered by paper. It is also important to notice that during this period mica was first used to cover spaces in the lattice. During this period squares, circles, ovals, waves, ice-ray patterns, and thunder-scroll in lattice design were used. These patterns were often interlocked and superimposed. From this period we have also the famous Han line pattern.
- Tang dynasty (618 A.D. to 907 A.D.): bars equally spaced, vertical, and crossed by parallel horizontal bars on top, middle and at the bottom. The Han line pattern gave beginning to the crossed frames pattern.
- Song dynasty (960 A.D. to 1279 A.D.): the circle and square in sophisticated combinations were used. The gold coin motif was used in rectangular and hexagonal interlock. During this period we have further development of patterns with octagon-square framework superimposed with honeycomb and sometimes with multiple twisting. At the end of this period multiple borders and flower patterns were used. Over-ornamenting becomes a fashion.
- Ming dynasty (1368 A.D. to 1644 A.D.): the lattice design continued to use octagon-square pattern. Octagonal patterns became very sophisticated. In fact, some of these patterns today are referred to as Ming patterns. Doors got carvings with one-footed dragons. Some lattices, esp. those with octagon-square patterns got a lot of carvings on the front side while the back side was flat for paper covering.
- Qing dynasty (1644 A.D. to 1911 A.D.): lattices reached its culmination in variety and quantity of designs. The design was very elaborate, with a lot of tricks and ornaments. At this time there were several country centers for lattice design. Here are the five most important ones: Chengtu, sometimes considered as the foremost lattice center in China; Canton was considered as a second place; Wu-Han, the so called Chicago of China; Sochow and Nanjing; and Peking or Peiping. Each of these centers developed its own characteristic style. Designs from Peking had a significant influence on designs of lattices in Korea, Mongolia and Manchuria.

Review of the Chinese lattice would not be complete if we didn't look at some of the above mentioned designs and examine briefly the classification of lattices by Daniel Sheets Dye in his book (see [2]). He classified lattices into 27 groups. Each group represents a particular visual feature that does not always stand for a particular set of mathematical properties. The classification is not precise, and the groups overlap frequently. Here we will mention only the most important of these groups.

- Groups related to the mathematical properties of the base pattern: parallelogram, octagon and hexagon.
- Groups related to the point of focus: lattices with one focus point, two foci, three foci, five foci, no foci at all, wedge lock, presentation, and out-lock.
- Groups with lines and waves: Han line, parallel wave line, opposed wave, wave with loops, and loops.
- Groups with swastika pattern: swastika like, and unlike swastika.
- Groups with specific element placed in a kind of chessboard grid.
- Group with scroll patterns: s-scroll (the so called thunder-scroll) and u-scroll.
- Groups with regular and random, so called rustic, ice-ray patterns.

Some of the mentioned here patterns will be shown later in this paper. We will show also how these patterns can be developed using GSP.

3. A brief Introduction to the Plane Symmetry Groups

As we said earlier the concept of the plane symmetry groups, or wallpaper groups, described by Conway (see [1]) will form the mathematical foundations for this paper. Therefore it is important to examine terminology used in his works and the main concepts. There are a few approaches to plane symmetry groups. One based on crystallography examines repeating plane patterns from crystallography point of view. A very detailed description of this approach can be found in the book *Geometric Symmetry in Patterns and Tilings* by Claire Horne (see [4]). There is also a specific notation to describe symmetry groups for this approach. Recently a different approach was introduced by John H. Conway and William Thurston. This approach is based on MacBeath's mathematical language for discussing symmetries of patterns. A very precise description of this approach can be found in the already mentioned book by Conway and others (see [1]). The authors examine planar patterns for their specific properties: *mirror lines*, *kaleidoscopes*, *gyrations*, *miracles* and *wanderings*. This, a bit playful, terminology will be explained in a moment.

In this paper by planar patterns we understand repeating, regular patterns, covering the whole plane. On many occasions, due to limited space, we show a picture of a finite pattern. In such a case we mean that the pattern has no edges and it extends right, left, up and down up to infinity. Each planar pattern may have assigned a signature, i.e. a string of characters describing specific properties of the pattern.

Two points *A* and *B* in a planar pattern will be considered as of the same kind if we can move the point *A* to *B* and the whole pattern will not change. In such case we will use one of these points to represent all points of the same kind. The same we can say about lines in a pattern (e.g. mirror lines). Two lines will be considered as of the same kind if we can move one line onto another without changing the whole pattern. Figure 2 shows two groups of points of the same kind and two groups of lines of the same kind. We will frequently say that the pattern has one point A (or line *b*) with a specific property meaning that in the reality there exist many points (or lines) of the same kind like A (or *b*).

Mirror lines on a planar pattern are lines of reflection of the pattern. This means the pattern on one side of the line is symmetric with the pattern on the other side. While describing a planar pattern mirror lines are denoted by the symbol ∗. Points, so called kaleidoscopes, where mirror lines intersect are denoted as ∗2, ∗3, etc. This simply means that the pattern has a point where two or three mirror lines cross. Notation like this ∗632 means that the pattern has three kaleidoscopes: one with 6 mirror lines crossing, another one with 3 mirror lines crossing, and another one with 2 mirror lines crossing. A single star means that there is a mirror line but it does not cross with another mirror line. Figure 2 shows two kaleidoscopes with 4 crossing mirror lines, and a kaleidoscope with 2 mirror lines crossing (find them). Therefore, the lattice in the picture is ∗442.

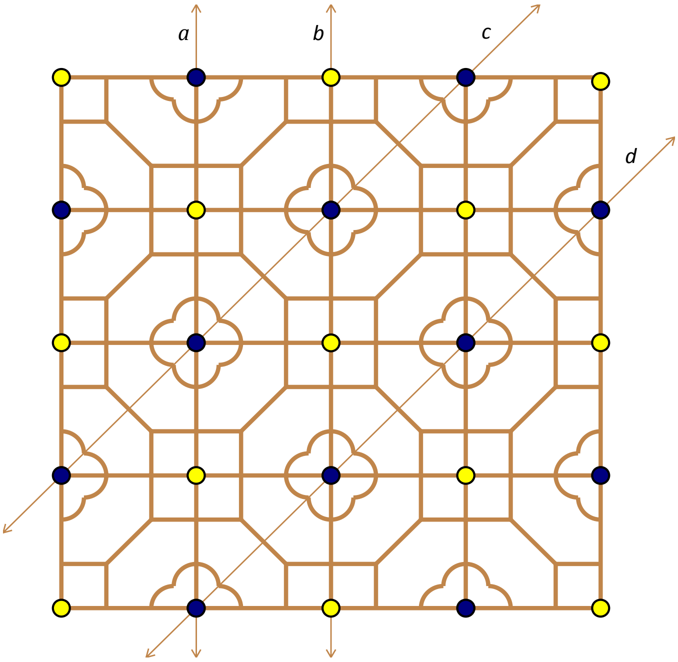


Fig. 2

All points represented by large dots with yellow filling are of the same kind. Points with dark filling are the same kind, but they are not the same kind as the first group.

*Lines **a** and **b** are of the same kind and lines **c** and **d** are of the same kind but they are not the same kind as the first group.*

In order to be consistent with our goals we suppose that the pattern extends left, right, up and down up to infinity covering completely the whole plane.

Lattice origin: modern design, Shanghai area, author's collection

Gyrations are points on a pattern where the whole pattern can be rotated a given angle about the point and the pattern will not change. Gyrations are marked using integer numbers 2, 3, 4, 6. Each number means the number of rotations necessary to obtain full rotation 360 degrees. Therefore, for example, number 4 means that the pattern will not change if we rotate it 90 degrees about the gyration point. Notation like this 333 means that we have three points of different kind and each of them is a 3-fold gyration point. Figure 3 shows a lattice with three gyration points.

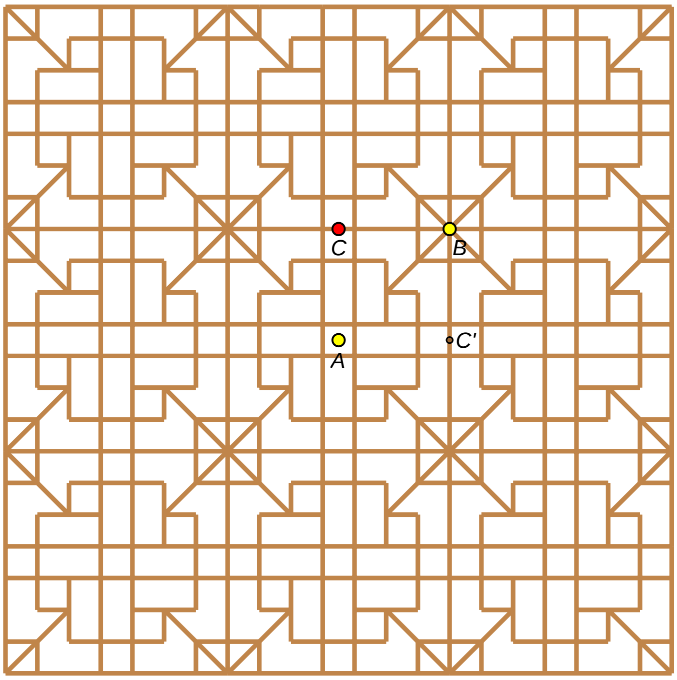


Figure 3

In this lattice there are three gyration points marked as A, B, and C. Point C is a 2-fold gyration point. Therefore, rotation by 180 degrees about C does not change the pattern. The two remaining points A and B are 4-fold gyration points, which means that rotation about 90 degrees does not change the whole pattern. Point C' is the same kind as point C. The signature of this lattice will have 442 and no star as there are no mirror lines.

Lattice origin: Buddhist temple, Mount Omei, Szechwan, 1875AD

"It is rather unusual window. The use of double bar occurs only occasionally in China, but it is frequent in Japan" (Dye, [2]*, pg. 152).*

Lattice signature: 442

Miracles are another type of feature that we can find in planar patterns. This is the situation when we can pass from a point on the pattern to its copy without crossing a mirror line. In mathematical terms the miracle means glide reflection of a pattern, i.e. superposition of reflection and translation along the reflection line. Miracles are marked as ×. It is possible that a planar pattern has 2 different miracles, different – in the sense of not being of the same kind. Figure 4 shows Chinese lattice with mirror line and a miracle.

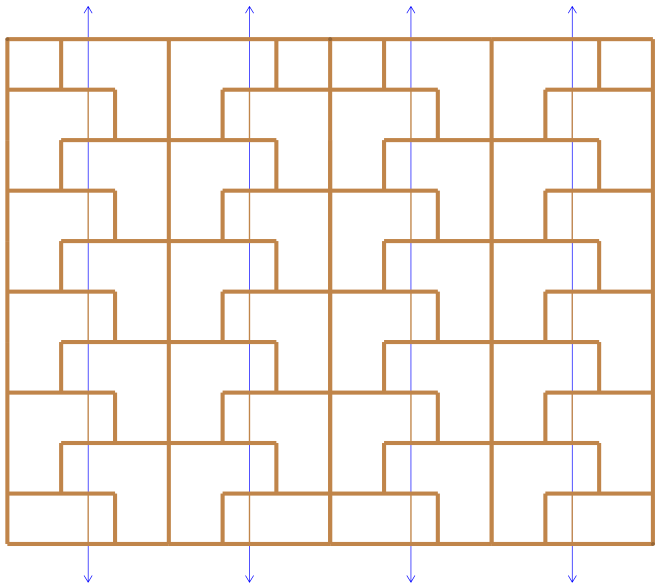


Fig. 4

Chinese lattice with miracle, i.e. glide reflections. The lattice has also a mirror line.

Lattice origin: Chengtu, 1850-1875 A.D.

Lattice signature: ∗×

Wanderings are the last feature of planar patterns. They occur when the pattern does not have mirrors, gyrations, and miracles. This simply means that a base pattern is translated in two different directions filling the whole plane. Wanderings are marked by ○.

Now, when we know how to describe each specific property of a planar pattern we can assign to each pattern its specific signature. The signature of a pattern is a sequence of symbols placed in the following order: wonders, gyrations, kaleidoscopes, and miracles. Not all parts must exist in a signature of a particular pattern. For example signature 2222 represents a pattern that has four points of rotation and has no other features, signature ∗2222 represents a pattern with four kaleidoscopes and no other features, 4∗2 represents a pattern with a 4-fold gyration point and a kaleidoscope with two crossing mirrors, finally ∗× denotes a pattern with one mirror (no crossing mirrors) and a glide reflection.

One of the major facts in plane symmetry groups is the theorem, the so called magic theoremin the Conway's book:

The Magic Theorem[1]

There exist exactly 17 symmetry types in planar patterns. These are types with the following signatures:

∗632	∗442	∗333	∗2222	∗∗
			2∗22	∗×
	4∗2	3∗3	22∗	
			22×	××
632	442	333	2222	○

[1] This theorem is known as Fedorov's Theorem (Fedorov, 1891). In 1897 the theorem was rediscovered by Fricke and Klein, and again in 1924 by Polya and Niggli.

Finally we have to find out how to assign a signature to a planar pattern. There are a few algorithms that allow us to proceed step-by-step and decide to which signature group the pattern belongs. In this paper we will use the algorithm proposed by Conway. We start from mirrors, then we proceed to gyrations, then to miracles, (you remember these are glide reflections), and finally if we didn't find anything then we check for wanders. Here are the steps of the Conway algorithm.

1. Mark any mirror line, find points of intersection of mirror lines, and finally select one point of each type. In the signature write down ∗ and numbers of mirrors passing through each of these points. Numbers can be written in any order, but we organize them from the largest to the smallest.
2. Mark in your pattern gyration points, select one point of each type and write down numbers representing each gyration point. The numbers representing gyration points should be placed before kaleidoscopes, again in the decreasing order.
3. Check if there are miracles and mark a glide reflection line of each type by ×. Put this symbol at the end of a signature. If the pattern has two miracles of a different type, then we will have to use ×× to mark them.
4. Finally, if we didn't find anything up to this moment then we should check if the pattern has wanders, i.e. translations in two directions only. If this is the case use the O as a signature for the pattern.

If the above algorithm did not produce any signature for a given pattern, then the pattern has no symmetries. Such patterns also exist, for example so called rustic ice-ray (fig. 5) quite often used in Chinese lattice design.

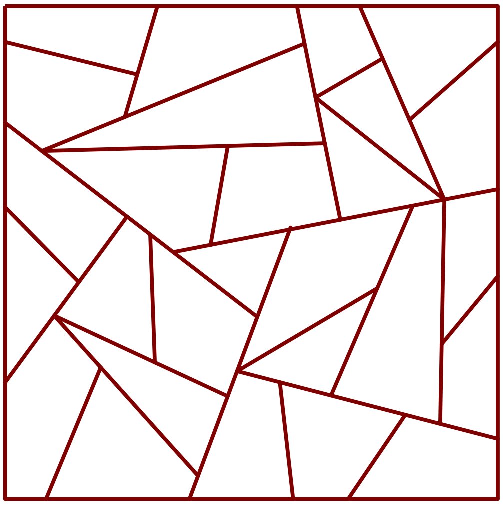


Fig. 5

Rustic ice-ray lattice is completely irregular and it does not have any symmetries. Therefore, it has no symmetry group signature and we are not able to determine how the pattern extends to cover the whole plane.

Lattice origin: a shop window in Zhouzhuang near Shanghai, modern design.

Before proceeding further we have ask the final question – how we know that while developing a signature for a planar pattern we didn't miss any of its features? There is an easy way to find answer to this question. We can evaluate the cost of the signature:

1. Any digit *n* for gyrations costs (*n*-1)/*n*
2. Any digit *n* for kaleidoscopes costs (*n*-1)/(2*n)*
3. Symbols ∗ and × cost 1 each
4. Symbol O costs 2

It was proved that for any plane symmetry group the total cost should be equal 2. For example, the total cost of a signature 632 is 5/6+2/3+1/2=2. Therefore, if we got a signature 2∗2, then the cost of such a signature is 1/2+1+1/4=1.75 and this means that we missed something. A good option would be to check the types listed in the Magic Theorem and see to which type the signature we got is similar. In the case of 2∗2 we can easily find that the only similar signature is 2∗22. Therefore, we missed one kaleidoscope.

Now, we can proceed to a more technical part of this paper and investigate how we can model Chinese lattices.

4. Software Selection

Most of the objects created while working on this project are geometric constructions built out of hundreds of points, lines, sometimes arcs and polygons. Such objects can be created with majority computer graphics drawing tools or dynamic geometry software. Computer graphics programs like Adobe Photoshop or Adobe Illustrator, with Artlandia plug-in, can be used to create all symmetry groups with one click. A similar, but much simpler tool is Inkscape, a shareware tool. It can be easily used to create patterns representing all symmetry groups. Unfortunately, all these utilities are like a black box, and we are not able to see how the construction was created. For a mathematician it is important to see step-by-step all transformations that were used to create a pattern. Therefore, for this project we decided to use Dynamic Geometry software where we can develop all patterns as geometric constructions.

Before starting this project it was necessary to evaluate a number of Dynamic Geometry programs and select the most convenient one for our expreiments. Major candidates for this project were Cabri, Geometer's Sketchpad, GeoGebra, Geometry Expressions, Cinderella and some shareware geometry programs like CaR or CaRMetal. On the beginning programs with unusual toolboxes, strange menu systems, a non-typical way of performing geometric operations were eliminated. This left us with four major players in Dynamic Geometry software.

Cabri – while experimenting with numerous examples we found that it is rather difficult to perform transformations of complex patterns in Cabri. For example, in order to make a reflection of a complex lattice in Cabri we have to make reflection of each component of the pattern separately. However, Cabri has a nice feature known as macro that can be used to create the whole basic pattern and then make its transformations. Unfortunately a different macro should be developed for each model.

GeoGebra – an interesting candidate for this project was GeoGebra. Patterns developed with this program were very accurate and the graphics on the screen, as well as in print, were beautifully rendered. However, there were a few minor things that prevented us from using GeoGebra. One of them was that GeoGebra does not preserve some properties of objects while doing transformations on them. For example, reflection of a yellow polygon will be a polygon with default color, reflection of a big point or thick line will be a point or a line with default colors and sizes. This way, if we wish to create a pattern with some specific colors or sizes we have to assign these parameters every time when we perform reflection, rotation, etc. There is, of course, another way to overcome this problem. In GeoGebra we can select separately groups of points, groups of segments, and groups of polygons and apply to each group some specific properties. However, this method does not work well with complex and multicolored patterns.

Geometry Expressions is software with a very unique approach to geometry. Here we can deal with geometry objects using their coordinates and analytic properties. Therefore, we can declare lengths of segments, their slope, etc. This way we can construct patterns in a completely different way. This is especially convenient while dealing with complex patterns with large number of elements.

Geometer's Sketchpad – authors experimented with GSP 5, beta version at this time. The most important advantage of GSP is the concept of custom transformations, a new addition to GSP. With custom transformations authors were able to define a set of specific transformations for doing more complex patterns. An example of such transformation is the glide reflection. Glide reflection is a superposition of a reflection and translation, or vice versa. Such transformation can be done in any Dynamic Geometry software, but we always have to deal with the intermediate object, effect of the first transformation, and we have to hide it. Custom transformations in GSP 5 eliminate this need completely. From the quality point of view, after all we are talking here about art; GSP was producing the best possible output. Therefore all examples presented in this paper are developed with GSP 5.

5. Constructions of Chinese Lattices with GSP

It would be interesting to take a few examples of real Chinese lattices and see how we can identify their signatures and model them with Dynamic Geometry software. Such process may not be always straightforward. We have to remember that any Chinese lattice is a finite object. Therefore we have to guess how the pattern can be extended to cover the whole plane. In many cases the lattice has additional border or a decoration that replaces part of the pattern. The border can be an object for further investigations of its symmetries. In many examples the base pattern around a decoration was modified to integrate the lattice with decoration. In each such case we have to remove all distractions of the allover pattern, find the base motif of the lattice, i.e. the smallest part that will be mirrored, rotated, translated, and glide reflected. Then we can use the base motif to create something that can be considered as a base tile. Usually this is a square, rectangle, or hexagon. Such tile can be replicated with translations or reflections to cover the whole plane.

Example 1 Lattice with three kaleidoscopes

We will start with a reasonably simple example of a lattice with mirrors only. The lattice has octagonal shape and a flower decoration in the middle. However, after replacing the central decoration by a typical element of the lattice we can easily see how the lattice can be extended to cover the whole plane (fig. 6).

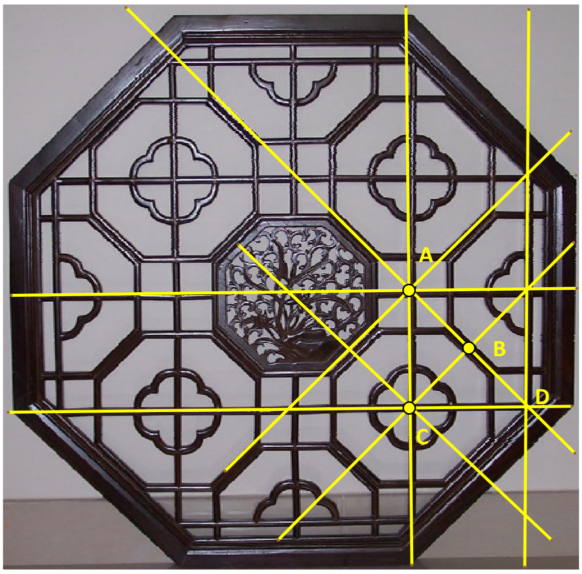


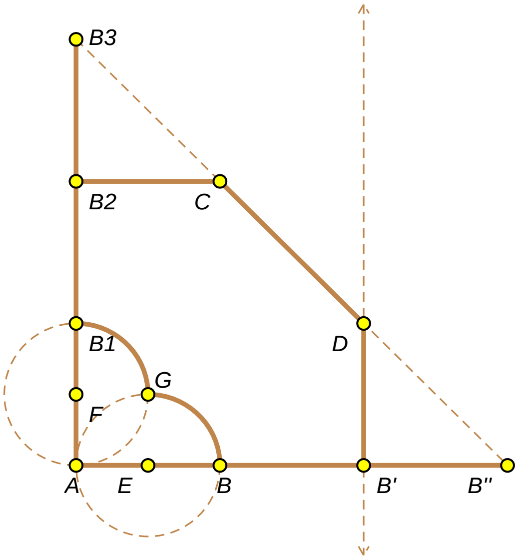
Fig. 6

In this lattice we have three mirrors; there are three kaleidoscopes. Point A is a 4-fold kaleidoscope, point C is another 4-fold kaleidoscope, and finally point B is a 2-fold kaleidoscope.

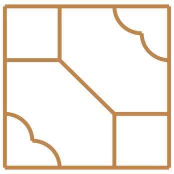
As the base motif can be used the triangle CBD. However, it is more convenient to use the triangle ACD. After reflecting the triangle about the axis AD we will obtain the base tile. Finally, such tile can be replicated all over the plane using reflections about its sides.

Lattice origin: author's property, modern design, lattice was developed in Shanghai.

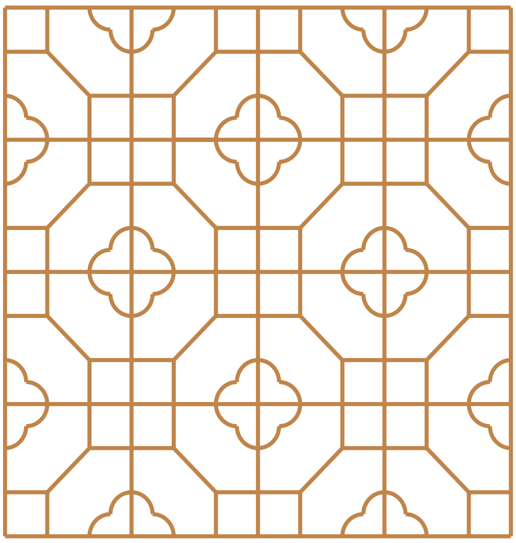
Lattice signature: ∗442



Construction of the base motif



The base tile



Final form of the lattice

Example 1 – Construction of the lattice using GSP

STEP1. Start design by creating a horizontal segment AB, mark it as vector and obtain two translations of the segment by the vector. This will create the bottom part of the base motif – three segments AB, BB', and B'B".

STEP 2. Mark point A as a rotation center, select the three segments and rotate them 90 degrees about point A. This way we obtain the left side of the base motif – three segments AB1, B1B2, and B2B3.

STEP 3. Select points B3 and B" and construct a new segment connecting these points. This will create the diagonal mirror.

STEP 4. Select point B2 and segment B2B3 and construct a line that is passing through the point B2 and is perpendicular to the segment B2B3.

STEP 5. In exactly the same way create the line that is perpendicular to B'B" and is passing through point B'.

STEP 6. Construct the points of intersection of the mirror B3B" with each of the lines created in the two previous steps. These will be points C and D.

STEP 7. Finally construct three segments joining points B2 and C, C and D, D and B'.

The part of the base motif built out of segments is ready. Now we have to create the round shapes connecting point B and B1.

STEP 8. Select the segment AB, and construct a midpoint on it (point E). In the same way we create a midpoint on the segment AB1. This will be the point F.

STEP 9. Create a circle with the center in the point E and radius AE. Create another circle with the center in the point F and radius AF.

STEP 10. Construct the point of intersection of both circles. This will be the point G.

STEP 11. Select points B and G (note, the order is important), then select the circle passing through these two points and construct the arc. Select points G and B1 and the circle passing through these points, and again construct the arc. In order to see better what we got select each circle, the two construction lines, and the mirror, and use the Display>Line Style menu to make them thin and dashed.

Final construction of the base motif is shown on the left (top).

In the next steps we will develop the base tile of the lattice. First hide all dashed lines and all points. We do not need them anymore.

STEP 12. Double click on the slant segment (former CD), select the whole base motif and reflect it about the segment CD. Now we got the base tile.

STEP 13. Replicate the tile by reflecting it by its right and bottom sides. The final pattern is shown on the left (bottom).

Before starting the next example it would be useful to summarize our observations from the last construction. Here are some major points that are worth to take under consideration.

1. **Draw or construct.** It is a very tempting to start designing a Chinese lattice by drawing the base motif by hand. Most of these patterns are built out of segments and any Dynamic Geometry software has a coordinate system with grid. Therefore drawing a base pattern seems to be the most straightforward approach. The consequence of such decision can be vital for further developments. In a figure that was created by drawing its angles and lengths are not fixed. If we were very accurate in our drawing then probably some points are fixed with some segments. Now, if we try to move one of our points or segments, even slightly, then the whole design changes and we get a completely new design or a big mess of segments and points. If we construct the base motif by selecting a base unit first, rotating the base segment by a given angle, translating it given vector, making segments perpendicular, parallel, etc., then our construction becomes firm. We will be not able to destroy it by any modifications of the base pattern. Simply, such modifications may not be even possible. Finally, we can think about a third, mixed approach. Construct major parts of the base pattern but leave free some of its components. This way, later by moving the free element we will be able to change the pattern in a controlled way.
2. **Leave as few points as possible**. The final outcome of our creation is a lattice; this is the shape where we have segments, arcs and sometimes polygons. Points are necessary only when we design the base motif and then when we perform rotations. We do not need them for any other operations. Therefore, as soon as we do not need a point we should hide it. This way we will not be copying it while doing further transformations, e.g. translations, or reflections. In fact, all rotations are done about points located on the edge or a corner of a pattern. Therefore, all points that are inside of a pattern should disappear as soon as they are surrounded.
3. **Do not multiply unnecessary things.** While performing various transformations we may produce objects that duplicate some of the existing objects. For example, if we reflect a pattern about a segment AB and while performing this operation we select the segment with its ending points A and B, then after transformation we will get two new points A' and B' that duplicate points A and B, i.e. they have exactly the same location as A and B. We do not need them, but they are there. The consequence of such careless construction is that we get a final pattern with bulky content that slows quite significantly operations on large patterns.
4. **Setup all parameters as soon as you can.** It can be rather difficult to hide all labels in a design with a few hundreds of segments, or change a color of some segments to blue and some other to red, fill some polygons with a color or remove their edge. In some programs, for example, hiding all labels may not even be possible. Therefore, we should setup majority of parameters before or during construction of the base motif. For example, in GSP we can turn off displaying labels for objects; declare colors for all objects, setup opacity of polygons and how their borders will be displayed. After developing our first segment or point we can declare its width and this parameter will be repeated for all new segments or points until we will change it. All decisions regarding sizes, colors and other things should be done as soon as possible and before finishing the base motif.

Now we can proceed to the next example. In the previous example we dealt with a lattice where we had mirrors only. In this example we will see how a lattice with gyrations only can be created.

Example 2 Lattice with four gyration points

In this example we will show how to construct a lattice with four gyration points. The example we chose is a very traditional and very Chinese pattern. Figure 7 shows clearly how the base motif should be selected. All gyration points are 2-fold that suggests that gyrations points should be located in the middle of sides rather than on vertices of the base motif. However, you may agree that in practice a more natural would be to produce the square with a swastika and the square with horizontal line in the middle and then replicate this whole pattern. We will create at first the base motif, that is shown in figure 7 and then we will show how the pattern can be replicated.

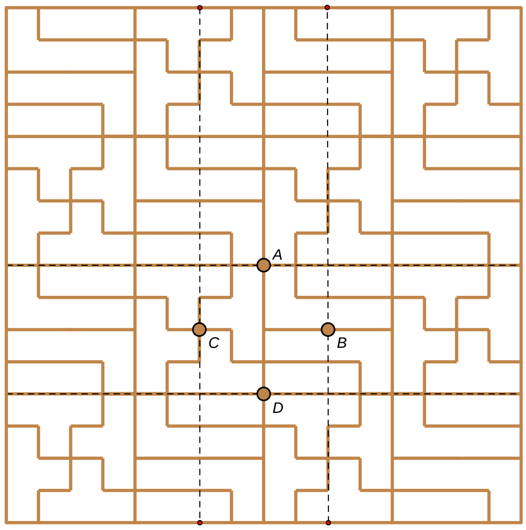


Fig. 7

A lattice with four gyrations points. Each point is 2-fold, i.e. the pattern can be rotated 180 degrees around this point.

Lattice origin: temple in Pao-kuang Buddhist monastery, Szechwan, 1875AD

Lattice signature: 2222

"This is an unusual and clever method of separating swastikas. By stepping one of the crossed waves to the right by one step and down by one step the two ways are made to coincide."

(Dye, [4]*, pg. 252, 254)*

Example 3 Lattice with signature 442

This rather unusual lattice has completely no mirror lines, and it forms a pattern that looks like two overlapping lattices. There are three gyration points: two 4-fold, i.e. with rotations about 90 degrees, and one 2-fold, i.e. with rotation about 180 degrees. In the figure 8 we show the image of a smaller version of original lattices shown in [4]. In the table on next page we will show how the lattice can be constructed using GSP.

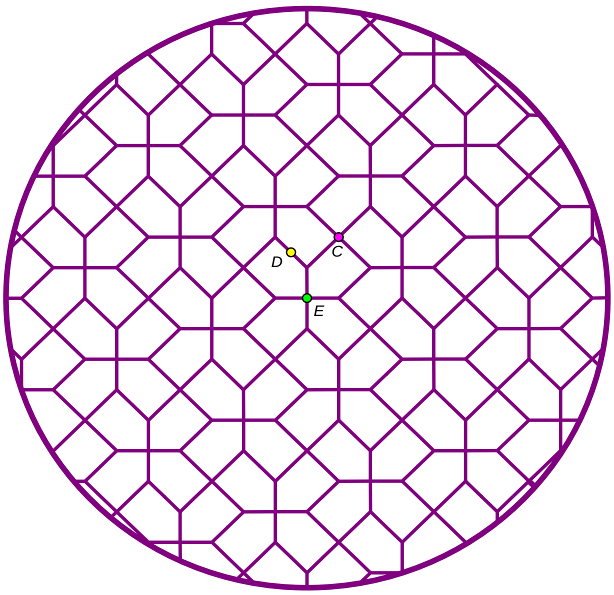


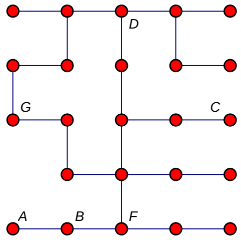
Fig. 8

In this lattice there are two 4-fold gyrations points (C and E). Point D is a 2-fold gyration point. The base motif should be taken as a rectangle between these three gyration points, where points C and E should be corner points to allow 90 degrees rotations. Point D can be located on the side of the rectangle or on one of its corners.

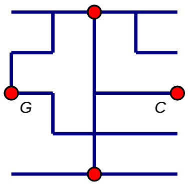
Lattice origin: Chengtu, Szechwan, 1875 A.D.

Lattice signature: 422

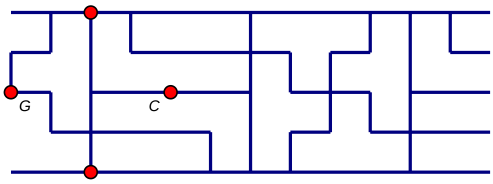
"This very unusual mosaic is most striking. It would be better as a tile than as a wooden lattice. It may be analyzed in several ways, depending upon what component elements are picked out of the pattern", (Dye, [4]*, pg. 341).*



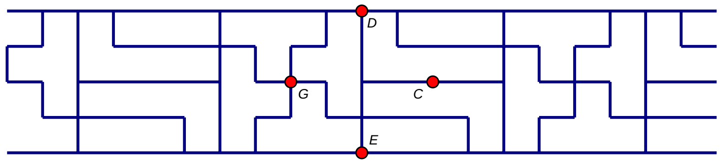
Base motif



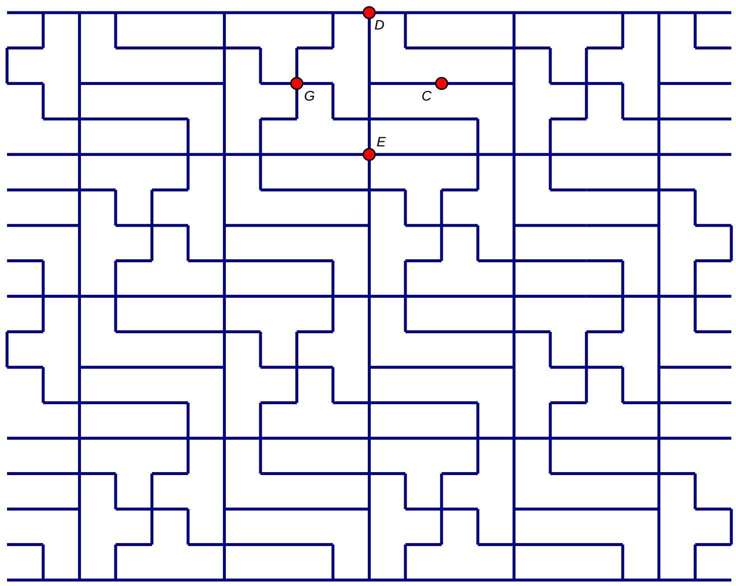
Base motif after cleaning it from unnecessary points



Extended pattern obtained from steps 4 and 5



Pattern extended in both sides after step 6.



The final pattern needs still some cleaning

Example 2 – Construction of the lattice using GSP

STEP 1. Start design by creating a horizontal segment AB, mark it as vector and obtain four translations of the segment by the vector. This will create the bottom part of the base motif.

STEP 2. Create the remaining segments of the base motif as it is shown in the figure on the left. All segments in this construction should be translations or rotations obtained from the base segment AB.

STEP 3. Clean the base motif by hiding most of the points, leave the rotation points only, i.e. points C, D, G, and F. You do not need labels also. Change the line width to medium. Change the size of the base motif to a smaller one.

STEP 4. Double click the point C to make it a rotation point, select the whole motif and rotate it 180 degrees about the point C.

STEP 5. Repeat a few times step 4 by rotating the last obtained copy of the base motif. Hide all unnecessary points.

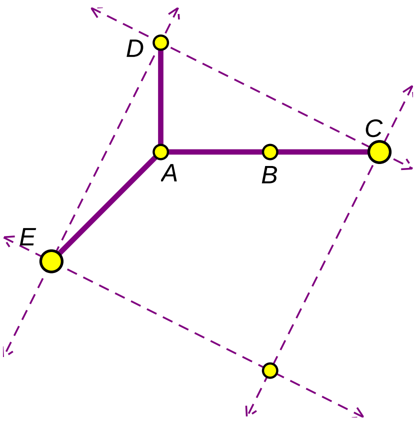
STEP 6. Double click the point G to make it a rotation point and select again the original base motif. Rotate it 180 degrees about the point G. Repeat this process the same number of times as you did step 5.

STEP 7. Double click the point E, select the whole pattern without the horizontal line passing through the point E and reflect it 180 degrees about the point E.

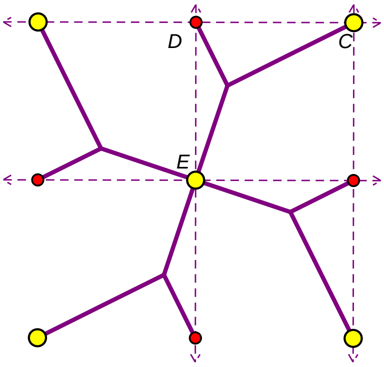
STEP 8. Continue rotating the pattern about any of the points on the top and/or bottom edge until you will get a pattern that satisfies your needs.

STEP 9. If you wish to remove some of the unnecessary fragments (left and right sides) and make your lattice similar to the one shown in the figure 7, select the side strips and delete them. You cannot delete anything that was part of the original motif.

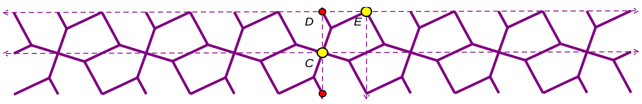
Example 3 – Construction of the lattice using GSP



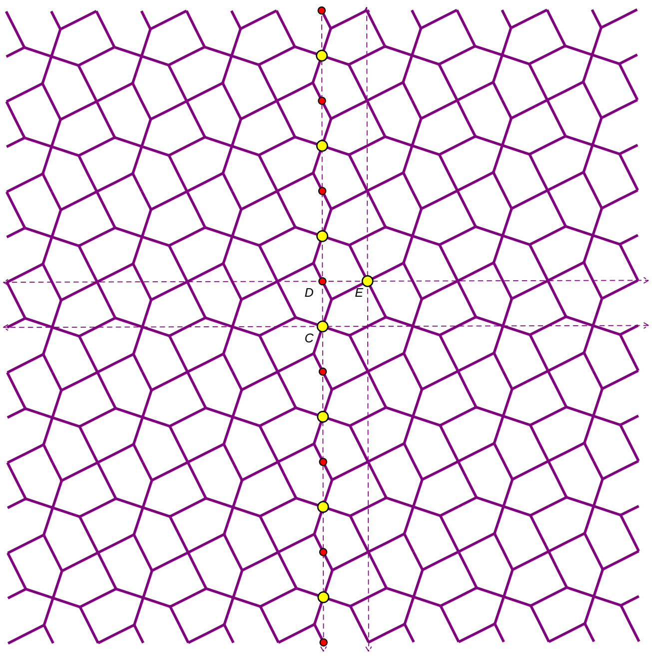
The base motif



The complete tile



Horizontal strip of tiles



The final pattern

STEP 1.*Create segment AB, and translate it right to get its extension BC.*

STEP 2. *Rotate segment AB 90 degrees left about the point A, this will create segment AD.*

STEP 3. *Construct a construction line passing through C and D, and two perpendicular lines to it passing through points C and D.*

STEP 4. *Rotate point C 90 degrees about the point D. This will create point E.*

STEP 5. *Connect points E and A to obtain the last piece of the base motif.*

STEP 6. *Construct a line parallel to line DC and passing through the point E. This will show how the base motif is located on the plane.*

STEP 7. *Rotate the whole construction in order to have construction lines horizontal and vertical. Clean the whole construction by hiding points A and B. You can also hide labels for other points, but leave the points for now.*

STEP 8. *Rotate the base motif 90 degrees, without the construction lines, three times about the point A. You should get a tile shown on the left.*

STEP 9. *Hide all corner points (large) and point E. You will not need them anymore. Leave only the points that are in the middle of each side of the tile. You can also hide all construction lines. I need them to show how the next steps of the construction will be performed.*

STEP 10. *Create multiple copies of the tile by rotating it 180 degrees to the right and left. You should obtain a strip similar to the one shown on the picture left. Hide all points leaving only two points: one on the center of the bottom edge and another one on the center of the top edge. You will use them to extend the pattern vertically.*

STEP 11. *Rotate up and down the whole pattern 180 degrees about each of the two points left.*

STEP 12. *Continue replications of the pattern down and up until you will get a reasonably large rectangle covered by the pattern.*

6. Further Explorations

While exploring various examples of Chinese lattices we can notice that many of them do not show features of any of the seventeen symmetry groups. Sometimes a border, some extra decorations, or a display added in the middle of the lattice changes the whole allover pattern to such extend that the lattice looks like a single title. Of course we can always take the tile and replicate it up, down, left and right to create a repeating pattern. Depending on how we replicated the tile, we can get a pattern that can be classified into one of the symmetry groups. However, sometimes it is useful to remove distracting elements, see what will be left after such cleaning, adjust the remaining irregularities and convert the lattice to a form that can be classified into one of the symmetry groups. Let us see how it can be done in practice.

Example 4

Let us start by presenting a picture of a lattice that looks like a single tile (fig. 9).

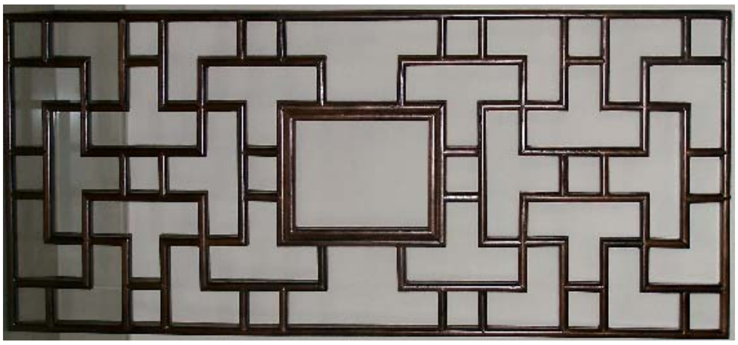


Fig. 9

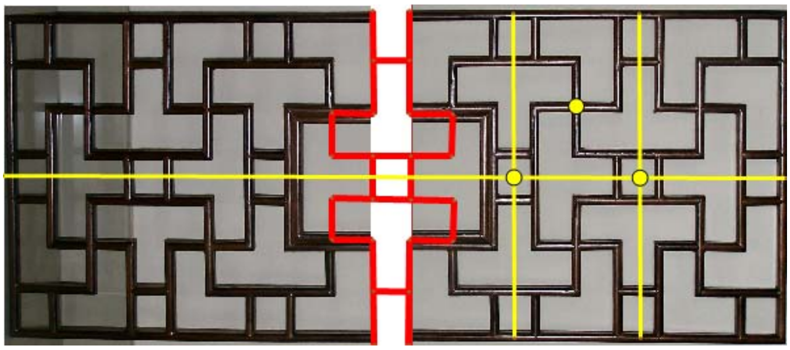


Fig. 9a

Fig. 9

A lattice that looks like a single tile.

If we cut the lattice vertically, exactly in the middle, and move the two parts apart, then we will be able to extend the allover pattern to get a design that can be easily classified in one of the symmetry groups (see fig. 9a).

Here we can notice that there are two kaleidoscopes each with two mirrors and one 4-fold gyration point.

Lattice origin: author's property, modern construction, Shanghai.

*Lattice signature: 4*2*

Construction of such a lattice is similar to those that we developed in our previous examples. Its fundamental steps are shown in the figure 10 and the final lattice is presented in the figure 11.

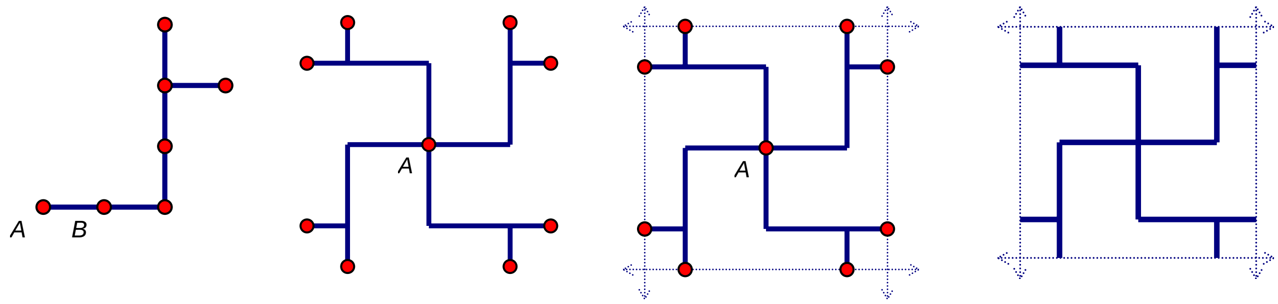


Fig. 10 – *base motif* *Crete the base tile* *Add mirrors* *Hide all points*

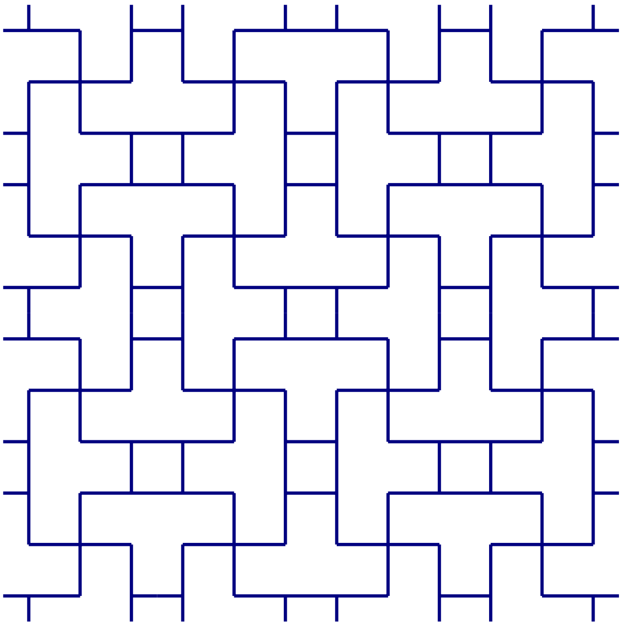


Fig. 11

Final construction of the lattice from example 4. The pattern presented here was described by Daniel Sheets Dye as Han Line and a very similar example to the one shown here we can find in [4], pg. 212.

Lattice origin: Shanghai, modern construction

*Lattice signature: 4*2*

*Presented here lattice contains series of superimposed frames. This can be easily noticed if we use different color for each series of frames, e.g. red for chains of frames going up and down, and green for chains of frames going left and right. However, such lattice will no longer have the 4*2 signature.*

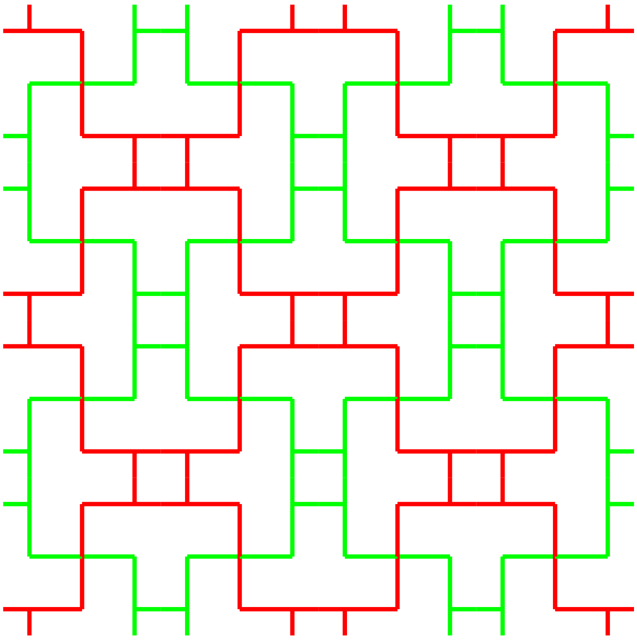


Fig. 12

Lattice from example 4 after further modifications. There are two sets of superimposed frames. Each set has different color.

*Lattice signature: 2*22*

A very similar example of lattice was presented in [4]*, pg. 212. It is a good example of the Han line pattern.*

We could continue experimenting with this lattice for a while. It is possible, for example, to fill spaces between segments using different colors and check what signature will have the obtained lattice. However, it is a time to wrap up the whole discussion.

7. Summary

In this paper we explored selected examples of Chinese lattices, we did show how to assign a plane symmetry group signature to a lattice, and finally we demonstrated how Chinese lattices can be modeled using Geometer's Sketchpad. We concentrated only on selected types of Chinese lattices as well as on selected symmetry groups. It is absolutely impossible to examine all major types, or examples, of Chinese lattices in a conference paper. Before writing this paper about 500 images of Chinese lattices was examined. These were mainly examples presented in [2] and [6], as well as on the photographs that we made during our trips through China, Hong Kong and Macau.

There are many things left that are worth of further research. We didn't even mention a specific type of Chinese lattice that is known as symmetrical ice-ray. This type of lattices can be modeled using algorithmic methods. We didn't mention also rosette patterns and freeze patterns, their classification using symmetry notation, and applications of them to the Chinese lattices. We didn't mention applications of color theory in symmetry groups, and tessellation patterns obtained by filling Chinese lattices by different colors. Finally, we didn't even try to generalize the concept of Chinese lattice into 3D and develop a three dimensional Chinese lattice (if such exist). Applying the 3D concept to Chinese lattices design seems to be very intriguing but we have to wait until a 3d version of GSP will be available.

Therefore, the presented paper can be just a starting point to further explorations of Chinese lattices and mathematical art with Chinese roots. We will leave these tasks for later time and for many other people who are interested in these topics.

8. References

[1] Conway, H., Burgel, H., Goodman-Strauss C. (2008). *The Symmetries of Things*, Wellesley, Massachustes: A K Peters Ltd.

[2] Dye, D. Sheets, (1974). *Chinese Lattice Designs,* New York: Dower Publications Inc.

[3] Horne, C. (2000). *Geometric Symmetry in Patterns and Tilings*, Cambridge, England: Woodhead Publishing Ltd.

[4] Majewski M., Wang J. (2008), *Deconstructing Chinese Lattices with MuPAD*, Proceedings of ATCM 2008, pg. 77-86.

[5] Majewski, M., & Wang, J. (2009). *An Algorithmic Approach to Chinese Lattice Design*, The Electronic Journal of Mathematics and Technology, Volume 3, Number 1, 83-101.

[6] Weidu, Ma (2006). *Classical Chinese Doors & Windows*, Beijing: China Architecture & Building Press, Baiwanzhuang.

[7] Weil, H. (1989). *Symmetry*, Princeton, New Jersey: Princeton University Press.

环视中国式门窗－中国数学艺术之探索

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摘要：我们熟悉了解许多关于西方的数学艺术，成百篇论文和许多书籍讲述伊斯兰艺术，有些研究者已经探索了美洲的一些民族艺术中的数学模式，也包括非洲的。然而，我们却对中国文化所隐含的数学艺术了解甚少。

实际上在中国，存在许多艺术创造实例，它们具有极其强烈的数学背景。我们可以发现显现在丝绸织物、陶瓷制品与中国家具中的几何图案的数学艺术，而这些数学艺术也体现在中国建筑的门窗之中。

本文将从变换几何的角度关注这些模式。我们研究的主要对象是中国门窗的木制格子结构的几何模式，着力于探索这些模式中的平面对称群的表现形式，以及如何运用动态几何软件将其模型化。文章中所有探索分析的实例都可通过几何画板加以实现，所涉及的许多实例都出自于实际建筑，也包括它们的计算机模型。

9. Appendix – Examples of Chinese lattices representing all 17 symmetry groups

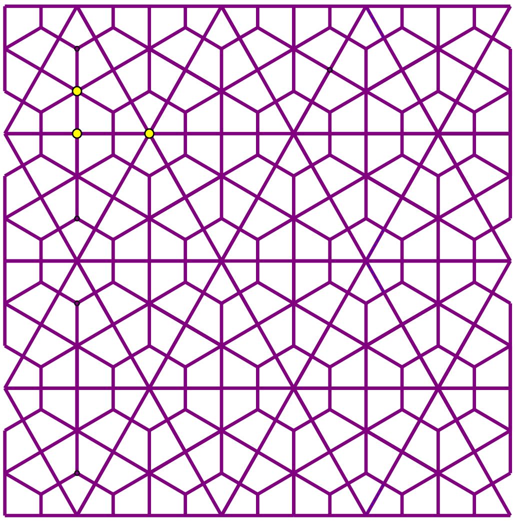
In this appendix we will show examples of Chinese lattices representing every one of 17 symmetry groups. All examples are based on illustrations from [2], [6], as well as our photographs. The order of examples will be similar to the order of signatures in the Magic Theorem. Therefore we will have:

Page 1: ∗632, ∗442, ∗333, ∗2222, ∗∗, 2∗22,
Page 2: ∗×, 4∗2, 3∗3, 22∗, 22×, ××,
Page 3: 632, 442, 333, 2222,○

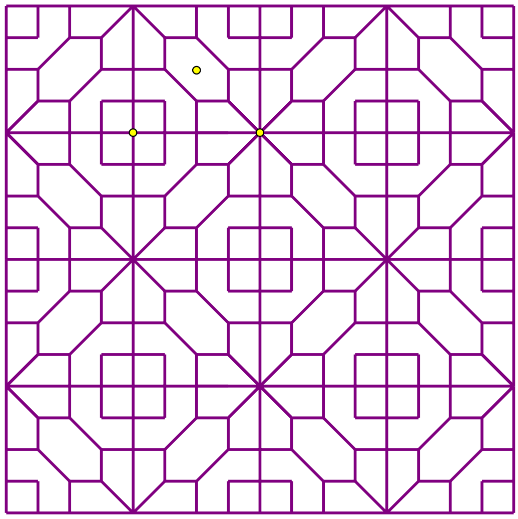
Comments about examples

∗632	In Dye classification this pattern is referred to as *Compound Hexagon* or *Ming pattern* ([4], pg. 76), there are a few interesting modifications of this pattern. Lattice origin: Shaohing, Chekiang, 1850.
∗442	In Dye classification this pattern is referred to as *Separated Octagon-Square* ([4], pg. 62, 63). Lattice origin: Buddhist temple, Mount Omei, Szechwan, 1875.
∗333	*Pepper-Eye Wave* pattern ([4], 78,79). Lattice origin: Chengtu, Szechwan, 1875.
∗2222	This is expanded version of the *Recurving Wave*, ([4], pg. 236, 237). Lattice origin: Coastal China, 1750.
∗∗	The author could not find a real example of Chinese lattice from this symmetry group. The pattern was created to show how such lattice could look.
2∗22	*Oblong Octagon-Square Superimposed* ([4], 62, 63). Lattice origin: Minshan, Szechwan, 1875.
∗×	Similar to *Voided Waves* pattern ([4], pg. 216, 217). Lattice origin: Chengtu, Szechwan, 1800.
4∗2	*Paired Swastikas of Crossed Thunder-Clouds* ([4], 274, 275). Lattice origin: Canton, Kwantung, 1850.
3∗3	*Coat-of-Mail for Door Gods* ([4], 72, 73). Lattice origin: Chengtu, Szechwan, 1800.
22∗	*Separated Waves* ([4], 216, 217). Lattice origin: Chengtu, Szechwan, 1875.
22×	*Man-Character Prop* ([4], 196, 197). Lattice origin: created by Dye in the Kwangyuan style, 1875.
××	The author could not find a real example of Chinese lattice from this symmetry group. The pattern was created to show how such lattice could look.
632	*Superimposed Hexagon Systems* ([4], 76, 77). Lattice origin: Shaohing, Chekiang, 1850.
442	*Squared and Centered Swastikas* ([4], pg. 272, 273). Lattice origin: Chinese silk design, 1920.
333	The author could not find a real example of Chinese lattice from this symmetry group. The pattern was created to show how such lattice could look.
2222	*Swastikas with Divided Ten-Character* ([4], 252, 253). Lattice origin: Chengtu, Szechwan, 1875.
○	The author could not find a real example of Chinese lattice from this symmetry group. The pattern was created to show how such lattice could look.

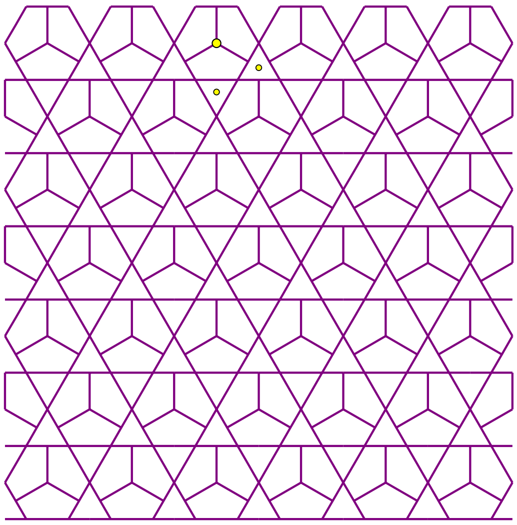
In all examples yellow circles were used to mark kaleidoscopes and red circles to mark gyration points.



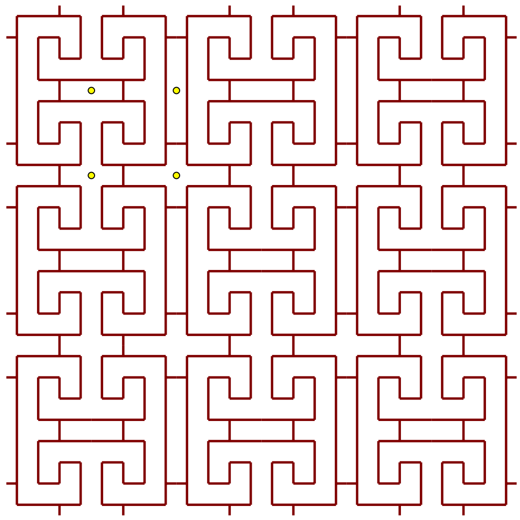
∗632 (Dye, pg. 76)



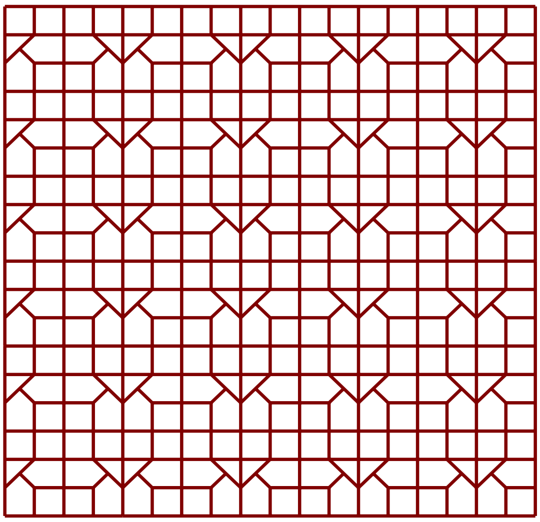
∗442 (Dye, pg. 63)



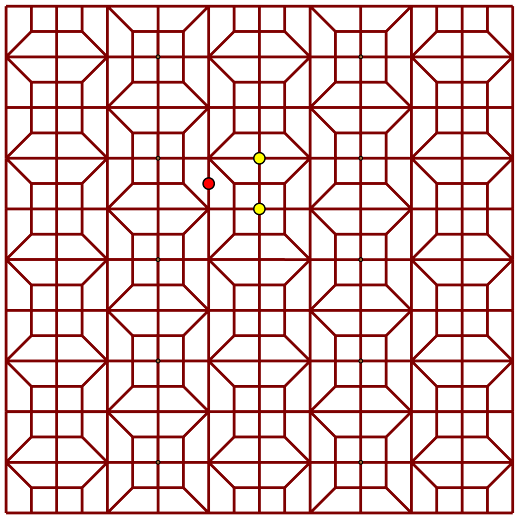
∗333 (Dye pg. 79)



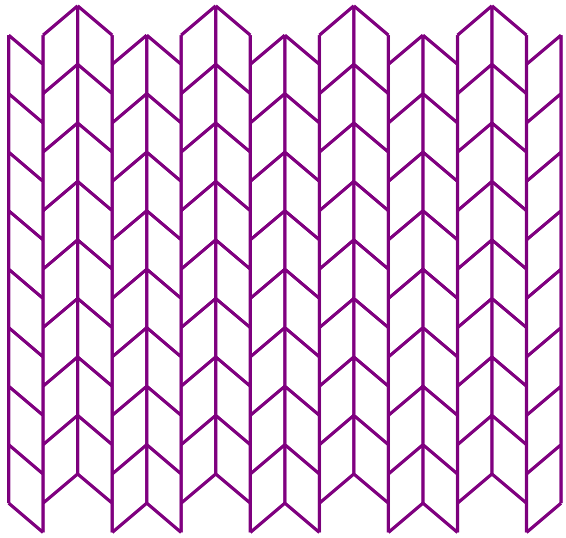
∗2222 (similar to Dye pg. 236)



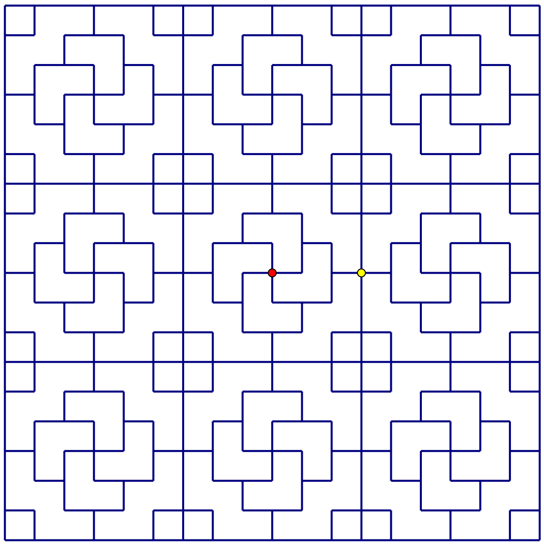
∗∗ (no real example)



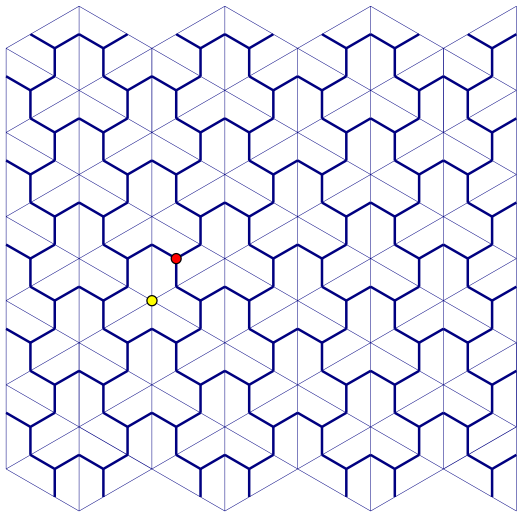
2∗22 (Dye, pg 63)



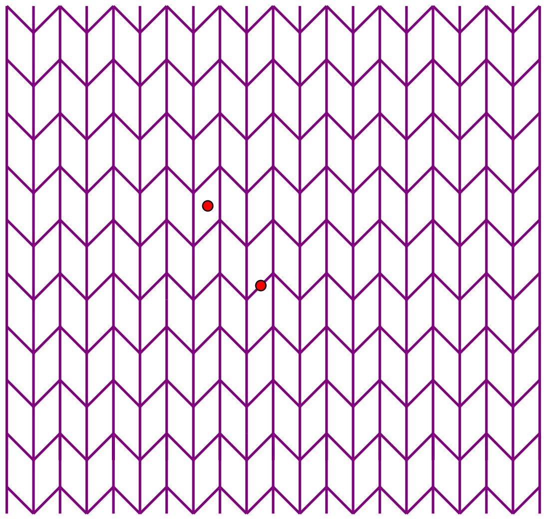
∗× (Dye, pg. 216)



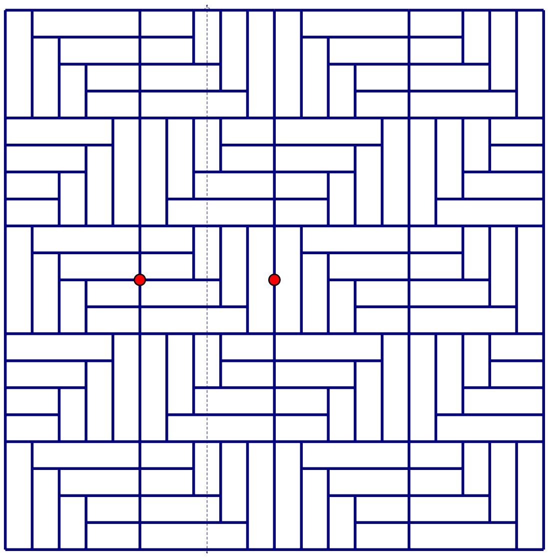
4∗2 (Dye, pg. 275)



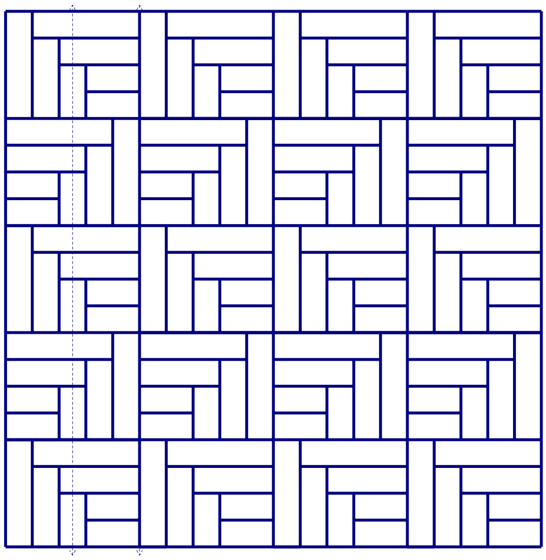
3∗3 (Dye, pg. 72)



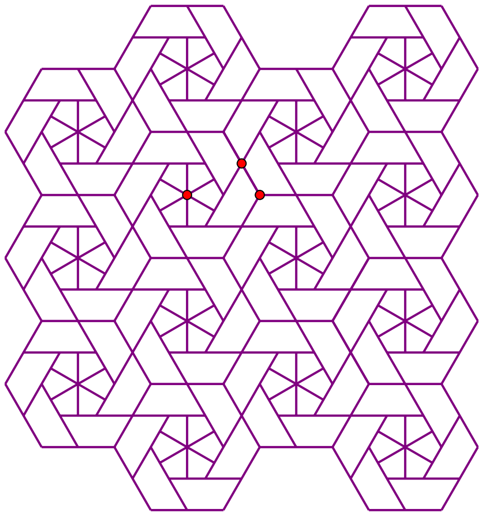
22∗ (Dye, pg. 216)



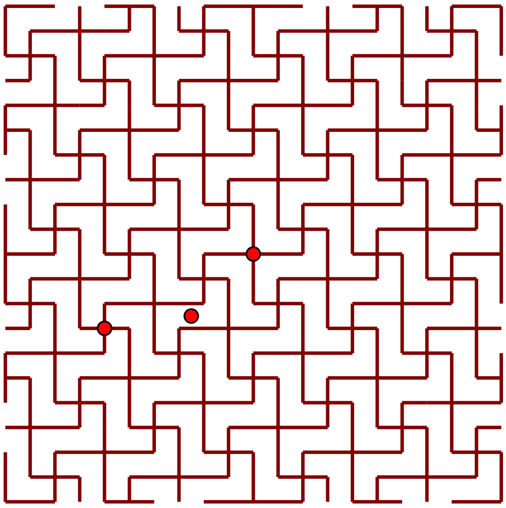
22× (Dye, pg. 197)



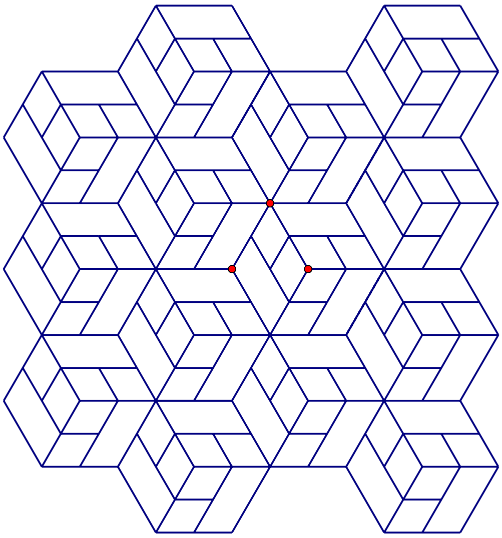
×× (no real example)



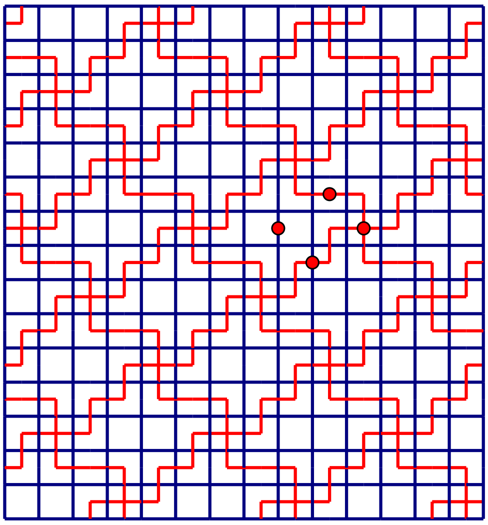
632 (Dye, pg. 76)



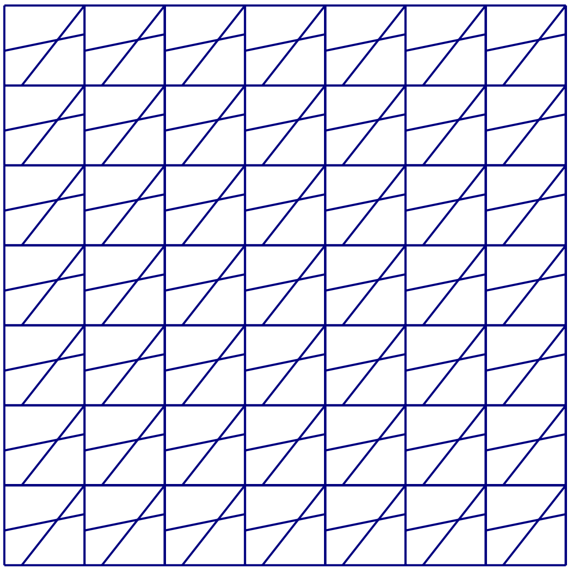
442 (Dye, pg. 272)



333 (no real example)



2222(Dye, pg. 252)



O *(no real example)*