From friezes and wallpapers to decorating rods and further to circular and spiral mosaics

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Abstract

Our aim is to describe a relationship between frieze groups, wallpaper groups, rod groups and symmetry groups of circular and spiral mosaics.
All is known about frieze and wallpaper groups and they were approached in many ways. However, rod groups are far less popular and symmetry groups of circular and spiral mosaics of the Gaussian plane were not so extensively investigated and popularised.
The relationship between these groups is given by means of an interpretation dictionary translating isometries of the Euclidean space into isometries of the Euclidean plane restricted to a stripe and further, into the inversive transformations of the Gaussian plane. In this dictionary axes of transformations are important since f.e. distinct translations of the Euclidean plane may correspond to translations, rotations and screws of the Euclidean space.
We thoroughly investigate a few significant examples of creating rod groups and symmetry groups of spiral mosaics from wallpaper groups.The lecture uses little formalism and illustrates every notion by numerous pictures and diagrams.

1 Introduction

In this paper we would like to establish connection between some discrete subgroups of isometries of the Euclidean plane Iz_2 , isometries of the Euclidean space Iz_3 and the inversive transformations of the Gaussian plane $Inv(\mathbb{C})$. In order to visualize the transformations we use the invariant subsets of the plane and the space, respectively, on which the groups are acting. These subsets are called, correspondingly, friezes, wallpapers, decorated rods (or tubes), circular and spiral mosaics.

For the description of groups related to friezes and wallpapers we use the fact that every isometry of the plane can be expressed in terms of a complex function

$$f(z) = az + b\overline{z} + c$$

where $a, b, c \in \mathbb{C}$, $ab = 0$, and $|a|^2 + |b|^2 = 1$.

A convenient way to compose such functions is given by multiplication of matrices of the form:

$$\begin{pmatrix} a & b & c \\ \overline{b} & \overline{a} & \overline{c} \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{C}$, $ab = 0$, and $|a|^2 + |b|^2 = 1$.

2 Friezes and wallpapers

We are mainly interested in the following situations:

Let Tr_k be the subgroup of translations of Iz_k, $k = 2, 3$.

G is of our interest if and only if the following conditions hold:

1. $G \cap Tr_k = \langle \tau \rangle$ or $G \cap Tr_k = \langle \tau_1, \tau_2 \rangle$, τ_1, τ_2 – translations by linearly independent vectors,
2. $J = G \diagup G \cap Tr_k$ is finite.

In case $k = 2$, G is either a frieze group for $G \cap Tr_2 = \langle \tau \rangle$ or a wallpaper group for $G \cap Tr_2 = \langle \tau_1, \tau_2 \rangle$.

Chapters of many books have been devoted to the classification of frieze groups and wallpaper groups and to the description of friezes and wallpapers (see [1, 2, 3, 4, 6]).

Let us briefly present these classifications in a diagrammatic and a pictorial way, in terms of their patterns that are preserved by given symmetries.

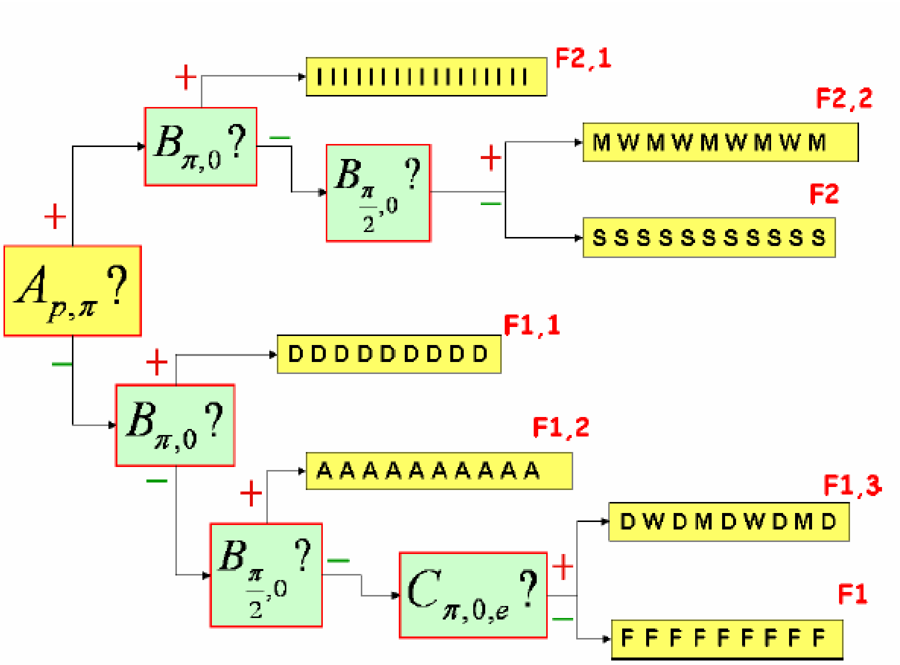

Fig. 1 *Classification of friezes and elements of their symmetry groups*

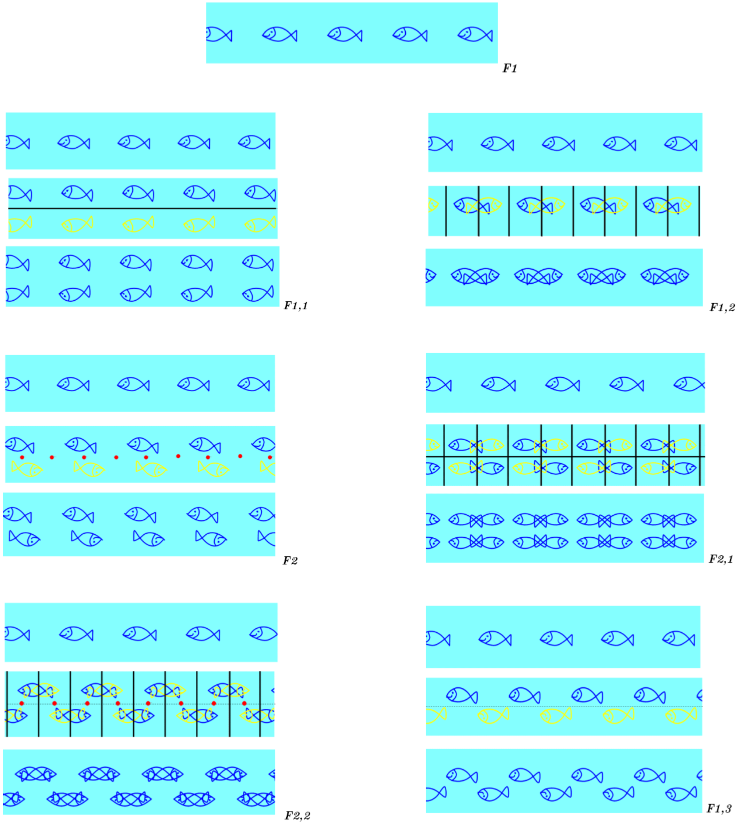

Fig. 2. *Creation of friezes by means of group generators*

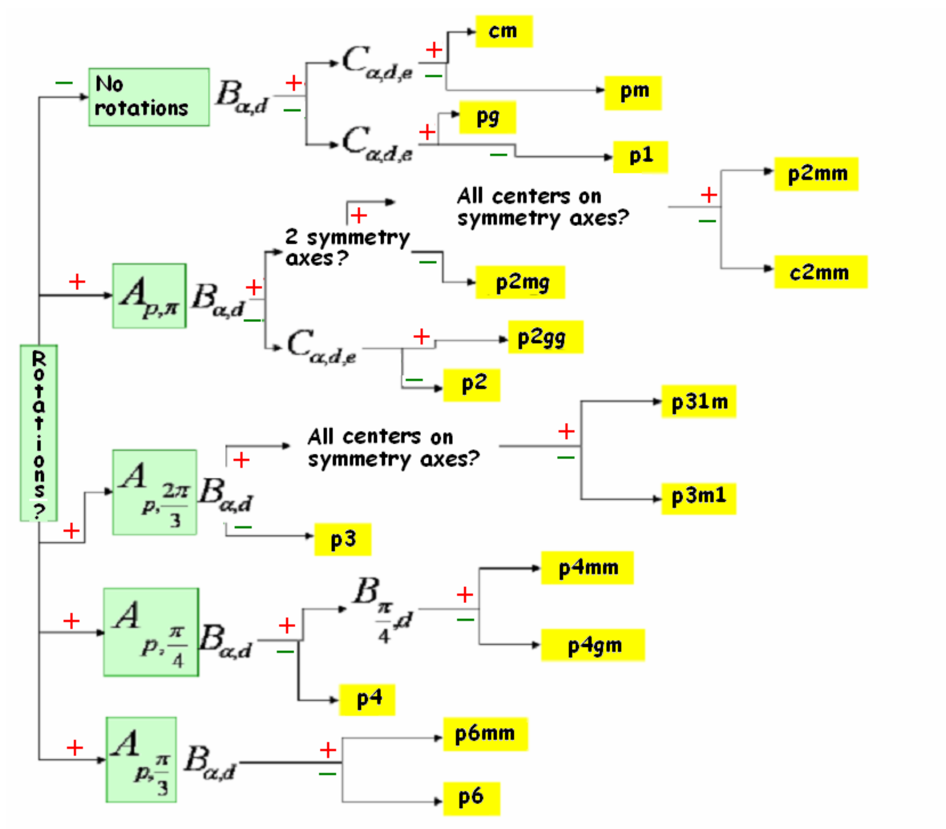

Fig. 3. *Classification of wallpapers and elements of their symmetry groups*

2.1 Examples of wallpapers and their group generators

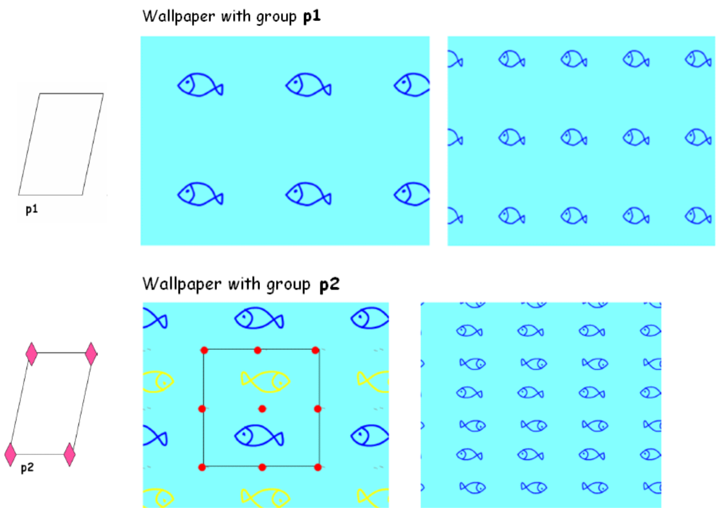

Fig. 4a *Examples of wallpapers with groups p1, p2 and their generators*

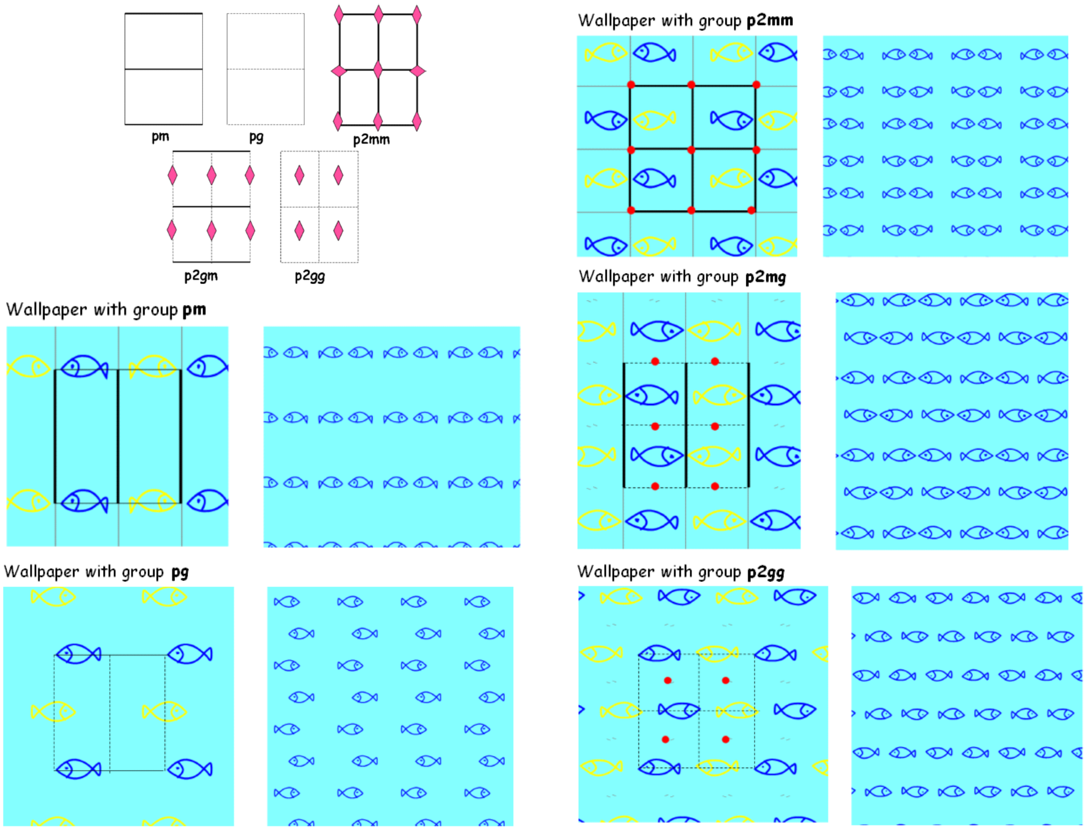

Fig. 4b *Examples of wallpapers with groups pm, pg, p2mm, p2gm, p2gg and their generators*

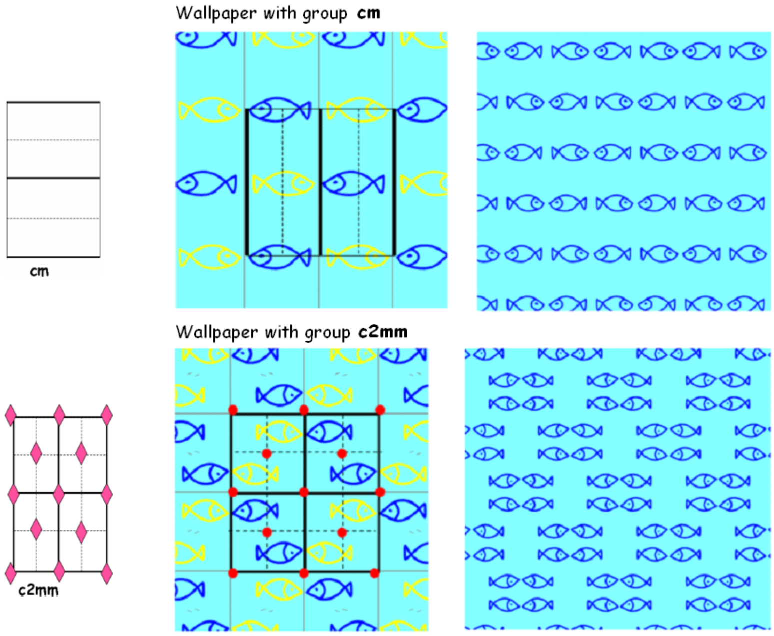

Fig. 4c *Examples of wallpapers with groups cm, c2mm and their generators*

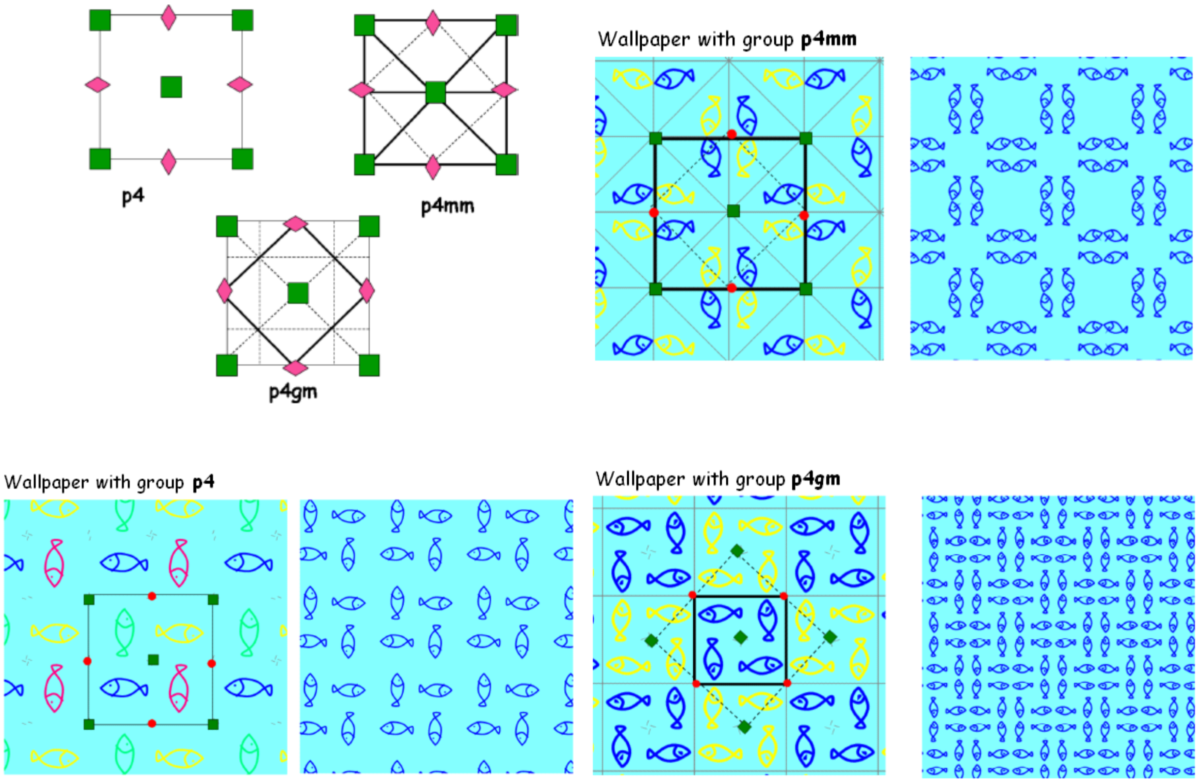

Fig. 4d *Examples of wallpapers with groups p4, p4mm, p4gm and their generators*

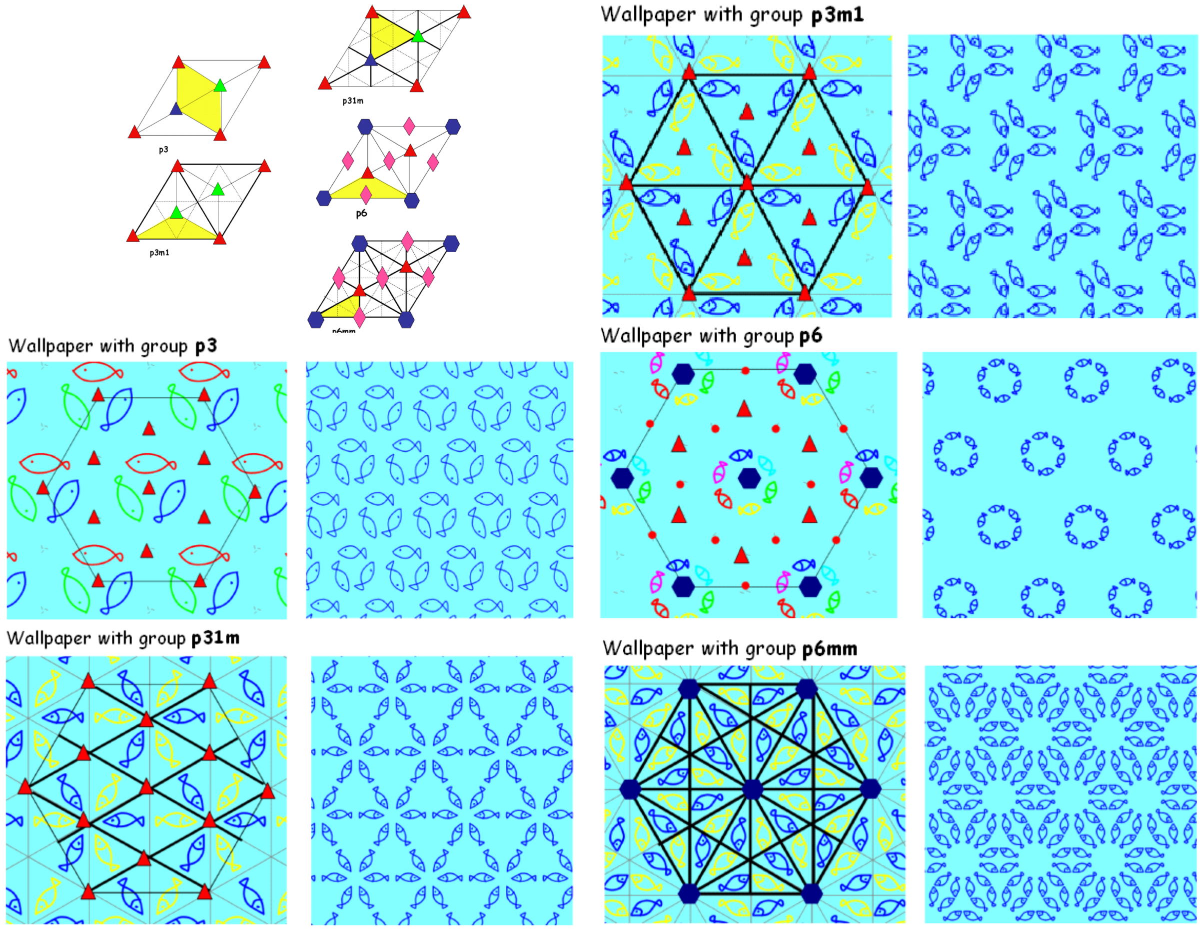

Fig. 4e *Examples of wallpapers with groups p3m1, p3, p6, p31m, p6mm and their generators*

3 From friezes and wallpapers to decorated rods

By extending our situation to the case $k = 3$, where $G \cap Tr_3 = \langle \tau \rangle$ and $G \diagup G \cap Tr_3$ is finite, we jump into the beautiful, however more complicated world of spatial isometries.

They were already known to Leonhard Euler in 1776 when the classification theorem of spatial isometries was given.

Theorem (see [6]) An isometry in the Euclidean space is exactly one of the following:

translation	reflection
rotation	glide reflection
screw	rotary reflection

How to approach this classification problem for the above groups G? And paralelly in such a way that for each group G we can find a corresponding pattern on a rod or, if we assume a rod to be empty inside, on a tube. In the literature on symmetry it was dealt with already in 1940 by Shubnikov (see [7]).

It is not an easy task since it covers lots of cases. One possible approach is to follow the same method as when classifying the wallpaper groups. However, there, we, in principle, are fighting with ten cases of possible finite groups J. Here, the number of cases increases dramatically,since the number of orthogonal subgroups of Iz_3 includes 32 types (see [4]).

And the tools are not so handy as they involve quaternions which serve for the description of isometries in space (see [5]).

The difficulties arise as situations like the following may occur.

Let $G \cap Tr_3 = \langle \tau \rangle$. Then there may exist a screw γ in G such that $\gamma^r = \tau^s$ for some $r, s \in \mathbb{Z}$, $\gcd(r, s) = 1$.

And even "worse" things can occur, like the existence of two screws:

$\gamma_1 = \rho\tau_1$ and $\gamma_2 = \rho^{-1}\tau_1$, where ρ is the rotation component and τ_1 is the translation component of the screw γ.

One may think of another approach which produces quite a number of patterns on rods. If we have a decorated rod, the pattern can be viewed as a pattern on a tube.

Now, we can make a cut and flatten the tube on the Euclidean plane. We get a single patterned stripe. At first, we may assume that the tube has circumference equal to 1 and is flattened with one edge on the real axis of the complex plane. Then we can consider a procedure like taking a covering space of this stripe and extending the pattern from one stripe to the whole plane.

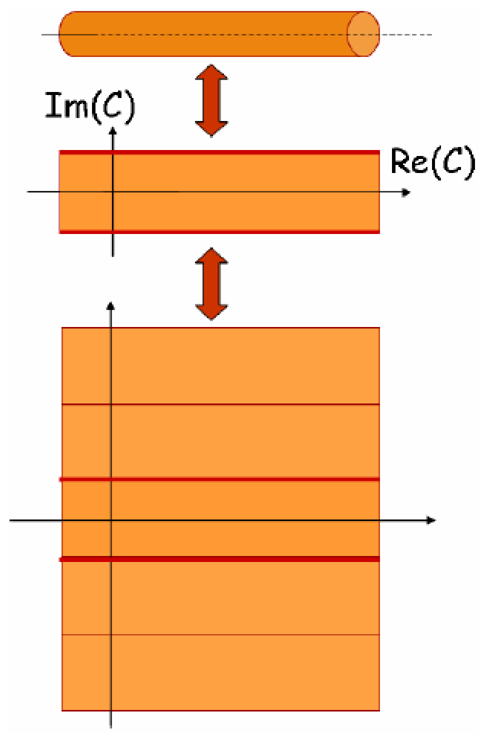

Fig. 5. *Tube, its flattening and covering*

Definition We say that a pattern on a plane E is a covering of a tube T if there exists a stripe S which is a flattening of the tube T and $E = \bigcup_{k \in \mathbb{Z}} \mu^k S$, where μ is a translation whose vector is perpendicular to the edges of S and of length equal to the distance between edges of the stripe S.

Now, building an interpretation dictionary between spatial isometries of a tube and plane isometries of its covering arises naturally (see appendix 1).

Now, we can look again at our familiar patterns of friezes and wallpapers to produce a whole range of patterns on tubes. We simply can stick a frieze on a tube. This can be done by sticking it parallel to the main axis or winding it on a circle perpendicular to the main axis or winding it spiral like.

The second case is out of our interest, since then $Tr_3 \cap G = \{id\}$, contrary to our initial assumption that $Tr_3 \cap G = \langle \tau \rangle$. In the third case we consider only such windings that if the screw has a presentation $\gamma = \rho\tau_1$ then there exists such a number $m \in \mathbb{Z}$ that $\rho^m = id$. When such a number does not exist, again, $Tr_3 \cap G = \{id\}$ .It is also possible to wind more friezes on one tube, i.e. one producing a screw $\gamma = \rho\tau_1$ and the other one producing a screw $\gamma' = \rho^{-1}\tau_1$. In this procedure a process of dessymmetrization may occur (see [9]). We may be loosing certain symmetries of a frieze if they are not becoming symmetries of a tube in the space. For example, if we are winding some frieze then, depending on the way of winding, we may obtain different patterns on a tube (represented by its flattening).

Up to this moment we were sticking or winding friezes. Now, we may wind wallpapers on a tube and in this way produce patterns on a tube and determine its symmetry groups.

This process is equivalent to finding a stripe on a wallpaper such that this wallpaper is a covering space for the tube related to this stripe. In this way from one wallpaper we may produce a whole family of patterns on a tube.

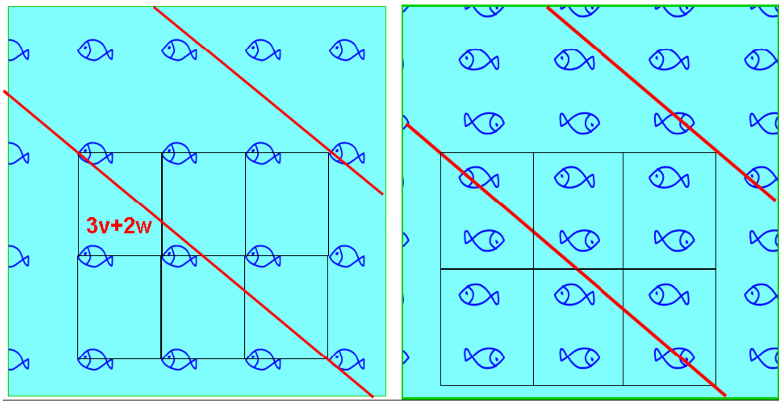

Fig. 7. *An essential observation is that the edges of stripes may be paralel to vectors* $l_1v_1 + l_2v_2$, *for arbitrary* $l_1, l_2 \in \mathbb{Z}$.

As the first on the list there are one-parameter families in which one of the vectors v_1 corresponding to minimal translation τ_1 is parallel to the edge of a stripe. The parameter l corresponds to the rank of rotation derived from the other minimal vector, i.e. $lv_2 = i$.

However, there are patterns on tubes that can be obtained neither by sticking a frieze nor by winding a wallpaper.

We do not aim here at giving a full classification of patterns on tubes (and equivalently classification of rod groups) but at indicating how using computers aids to visualize possibilities one might not think of. Full classification is a tedious job and may be derived from a series of scientific papers listed in (see [9]).

4 From decorated rods and tubes to circular and spiral mosaics

Once we have a whole load of examples we may think of transferring our knowledge to another world - the world of inversive transformations. We may use either of their presentations

$$Inv(\mathbb{C}) = \langle M, \sigma_{\pi,0} \rangle = \left\langle P, \sigma_{C(0,1)} \right\rangle$$

where M denotes the group of Möbius transformations of the Gaussian plane defined by $z \rightarrowtail \frac{az+b}{cz+d}$, with $ad - bc = 1$, $a, b, c, d \in \mathbb{C}$, and $\sigma_{\pi,0}(z) = \overline{z}$; P denotes the group of similarities of the Gaussian plane and finally $\sigma_{C(0,1)}$ is the inversion in the unit circle. To be consistent in terminology recall:

Theorem (see [6]) A nonidentity similarity of the Euclidean plane is exactly one of the following: isometry, homotethy ($\varkappa_0^r$), homotethy followed by rotation ($\rho_0^\alpha \varkappa_0^r$) (called spiral homotethy), homotethy followed by reflection ($\sigma_l \varkappa_0^r$,$0 \in l$).

Inversive transformations allow us to describe the so called circular and spiral mosaics. They are invariants of circular and spiral groups.

Definition (see [8]) Let Ξ_0 be the group of homotethies centered at 0. *Group G is said to be circular* if its pattern has a fix point 0, $G \cap \Xi_0 = \langle \varkappa_0^r \rangle$ for some $r \in \mathbb{R} \setminus \{-1, 0, 1\}$, $G / G \cap \Xi_0$ is finite and in G there is no spiral homotethy different from $\rho \varkappa_0^r$, where $\varkappa_0^r$ is a generator of Ξ_0. Moreover, for arbitrary $z \in (\mathbb{C} \cup \{\infty\}) \setminus \{0\}$ there exists $\varepsilon > 0$ such that $C(z, \varepsilon) \cap Gz$ is a finite set, where $C(z, \varepsilon)$ is a circle centered at z with radius ε.

Definition (see[8]) Let Ξ_0 be the group of homotethies centered at 0. *Group G is said to be spiral* if its pattern has a fix point 0, $G \cap \Xi_0 = \langle \varkappa_0^r \rangle$ for some $r \in \mathbb{R} \setminus \{-1, 0, 1\}$, $G / G \cap \Xi_0$ is finite and in G there is a spiral homotethy different from $\rho \varkappa_0^r$, where $\varkappa_0^r$ is a generator of Ξ_0. Moreover, for arbitrary $z \in (\mathbb{C} \cup \{\infty\}) \setminus \{0\}$ there exists $\varepsilon > 0$ such that $C(z, \varepsilon) \cap Gz$ is a finite set, where $C(z, \varepsilon)$ is a circle centered at z with radius ε.

Let us observe that in both cases an inversion $\iota(0, r)$ with center in point 0 may be a member of G. Moreover, $\iota(0, r_1)\ \iota(0, r_2) = \varkappa_0^{(\frac{r_1}{r_2})^2}$.

Now we can establish an interpretation dictionary between symmetry groups of tubes and circular or spiral groups (see appendix 1). In this case the interpretation proceeds by transitioning the problem at first to symmetries of the covering of a flattened tube on a complex plane and next applying the complex function $z \longmapsto e^z$ which transforms each flattened tube in the covering to an annulus.

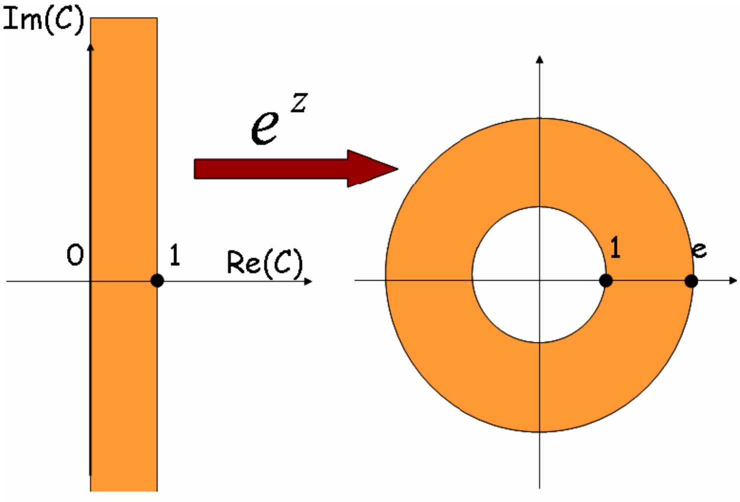

Fig. 8 *Function* $x \longrightarrow e^z$ *transforms the flattened tube to an annulus*

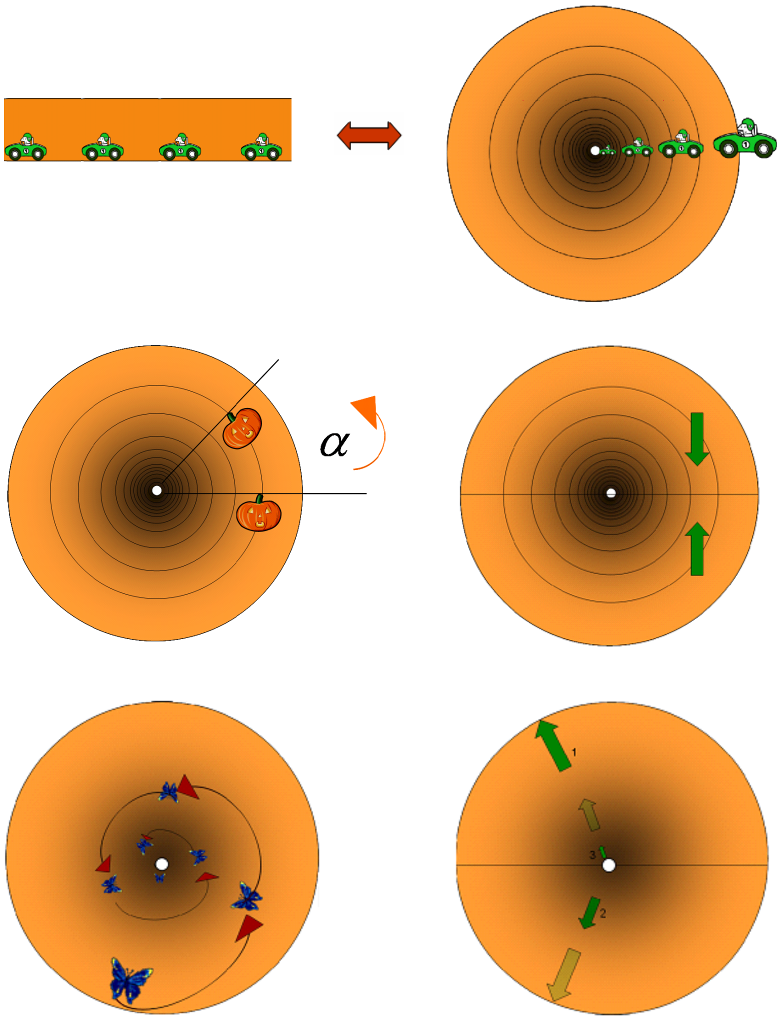

Fig. 9. *Inversive transformations to be translated by the reader as an exercise*

One may now start playing with producing circular and spiral mosaics.

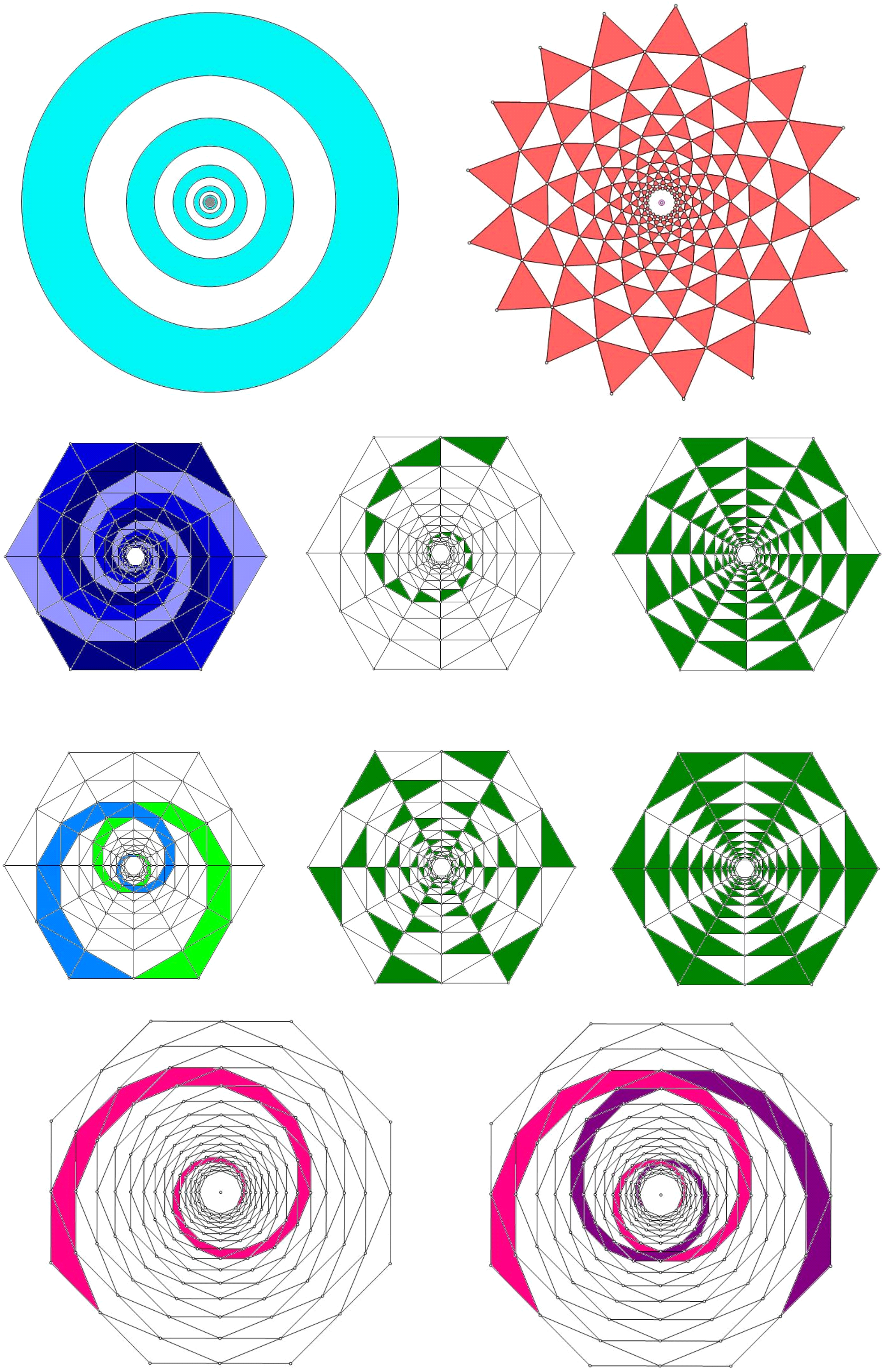

Fig. 10. *Circular and spiral mosaics depicted by Dynamic Geometry software Geometers Sketchpad*

4 References

[1] Allenby R.B.J.T., Rings, Fields and Groups, An Introduction to Abstract Algebra, Second edition, Edward Arnold, London , 1991.

[2] Armstrong M.A., Groups and Symmetry, Springer Verlag, New York, 1988.

[3] Berger M., Geometrie, CEDIC, Fernand Nathan, Paris 1977/78.

[4] Coxeter H.S.M., Moser W.O.J., Generators and Relations for Discrete Groups, 4th ed., Springer Verlag, New York, 1980.

[5] Coxeter H.S.M., Regular Complex Polytopes, Second edition, Cambridge University Press, Cambridge, 1991.

[6] Martin G.E., Transformation Geometry, An Introduction to Symmetry, Springer Verlag, New York, 1982.

[7] Shubnikov A.V., Simmetrija [in Russian], Izdat.Akad. Nauk. SSSR, Moskva, 1940.

[8] Zagorska E., Grupy dyskretne fryzów, tapet, rurek i mozaik spiralnych [in Polish], master thesis in mathematics supervised by B. Klemp-Dyczek, WMiI UMK Torun, 2008.

[9] http://www.emis.de/monographs/jablan/nind.htm (accessed on 2006-04-05)

Appendix 1 - Interpretation Dictionaries

	Symmetries of a tube in space	Symmetries of the covering of a flattened tube on the complex plane	Symmetries of a circular or spiral mosaics on the Gaussian plane
1	Translation of a tube	Translation by a real vector	Homotethy
2	Rotation about the main axis (minimal rotation)	Translation by a purely imaginary vector (minimal translation)	Rotation
3	Half turn about an axis perpendicular to the main axis	Half turn about a point	Inversion in a point followed by a reflection
4	Screw	Translation by a vector non parallel and non perpendicular to the real axis	Spiral homotethy
5	Reflection in a plane containing the main axis	Reflection in a line of the form $\mathrm{Re}(\mathbb{C}) + di, \quad d \in \mathbb{R}$	Reflection in a line
6	Reflection in a plane perpendicular to the main axis	Reflection in a line $\mathrm{Im}(\mathbb{C}) + d,$ $d \in \mathbb{R}$	Inversion in a point
7	Glide reflection (translation part parallel to the main axis)	Glide reflection (reflection is in a line parallel to the real axis)	Inversion in a point followed by a homotethy
8	Rotary reflection (reflection is in a plane perpendicular to the main axis)	Glide reflection (reflection is in a line parallel to the imaginary axis)	Inversion in a point followed by a rotation
9	Central symmetry	Glide reflection (translation vector is equal to $\frac{i}{2}$)	Inversion in a point followed by a half turn