Making Mathematics Fun, Accessible, and Challenging using *Mathematica* 6

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Abstract: The theme of this year's conference is "Making Mathematics Fun, Accessible and Challenging through Technology". Version 6 of *Mathematica*, released in May 2007, is instantly interactive, allowing teachers and students to easily construct live interfaces to illustrate and investigate mathematical concepts at all levels [1]. Over 2000 examples of these interactive capabilities, including a number written by high school students, are freely available at the Demonstrations Project website [2]. Other new features of Version 6, relevant to this conference, include high-quality adaptive visualization of functions and load-on-demand data of mathematical properties such as graphs, knots, lattices, and polyhedra.

1. Introduction

Version 1 of *Mathematica* was introduced in 1998 as a "System for doing Mathematics by Computer", combining numerics, symbolic computation, and graphics [3]. Almost 20 years later, *Mathematica* 6 was released, now consisting of seven tightly integrated components:

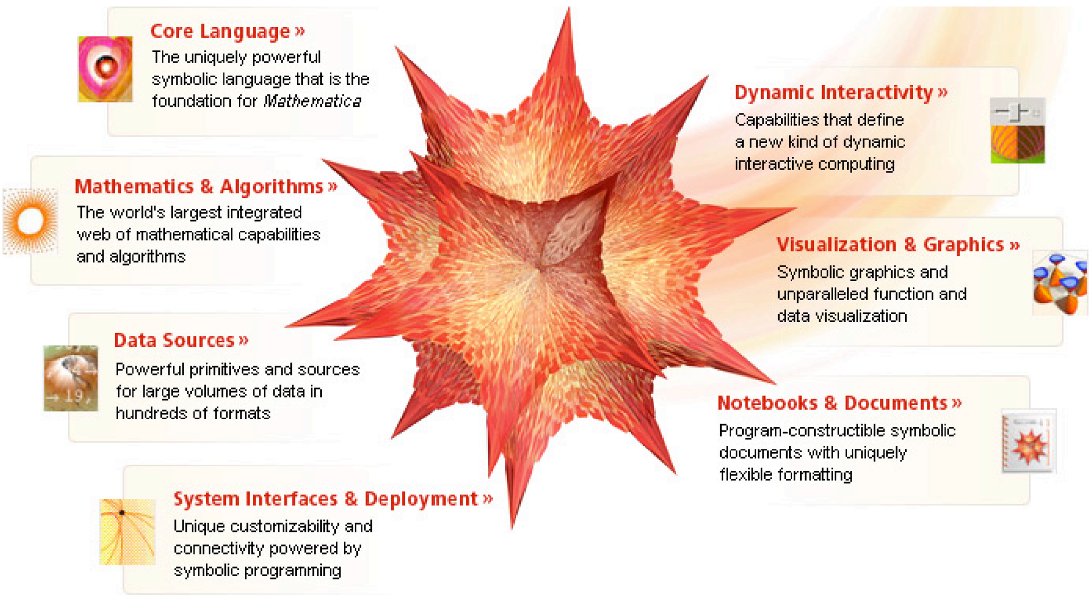


Figure 1 Overview of *Mathematica* 6.

Although there are a huge number of new features in *Mathematica* 6, from the perspective of ATCM, the most important ones are:

1. dramatic improvements in visualization and graphics;
2. sources for large volumes of data in hundreds of formats, including load-on-demand data of mathematical properties such as graphs, knots, lattices, and polyhedra;
3. a new fully integrated symbolic-dynamic interface for creation of complete dynamic interfaces;
4. the Demonstrations Project website [2].

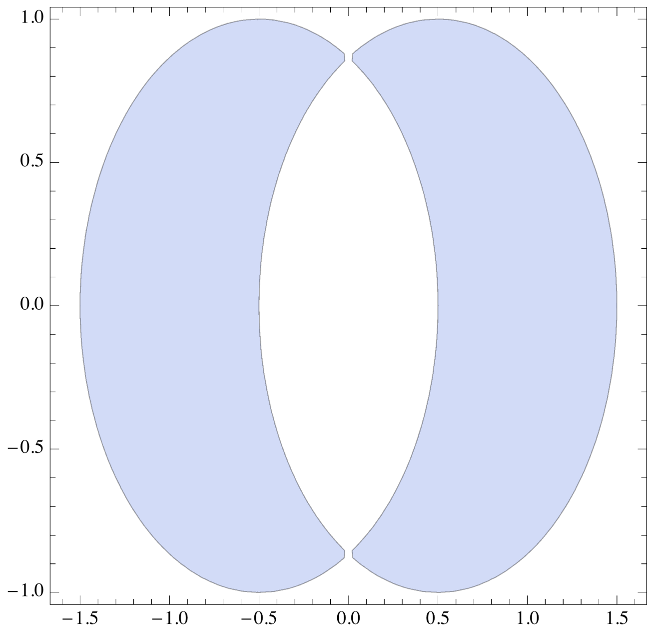
This paper is itself a *Mathematica* Notebook [4] and includes many interactive examples. Unfortunately, there is no easy way to transfer this interactivity to paper. However, even if you do not have access to *Mathematica*, you can run the interactive examples using the freely available *Mathematica Player* [5].

2. Visualization and Graphics

Here are a few examples of the enhanced visualization and graphics.

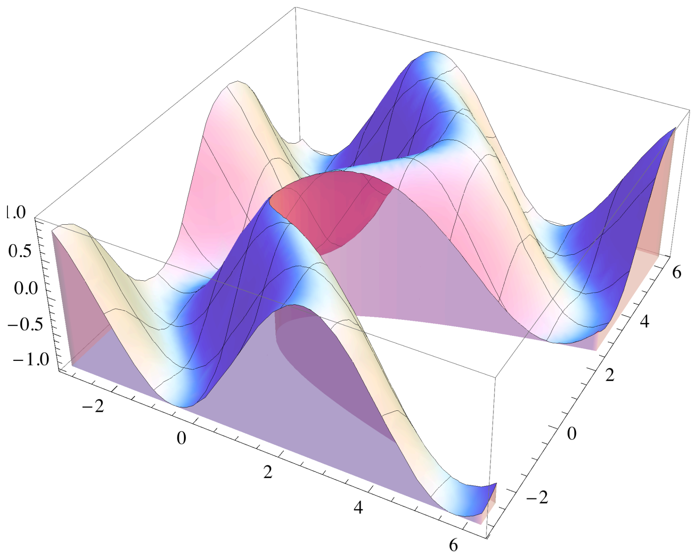
Use `RegionPlot` to display the region in which $\left(x+\frac{1}{2}\right)^2+y^2\leq 1 \veebar \left(x-\frac{1}{2}\right)^2+y^2\leq 1$.

```
RegionPlot[(x + 1/2)^2 + y^2 ≤ 1 ⊻ (x - 1/2)^2 + y^2 ≤ 1, {x, -1.6, 1.6}, {y, -1, 1}]
```



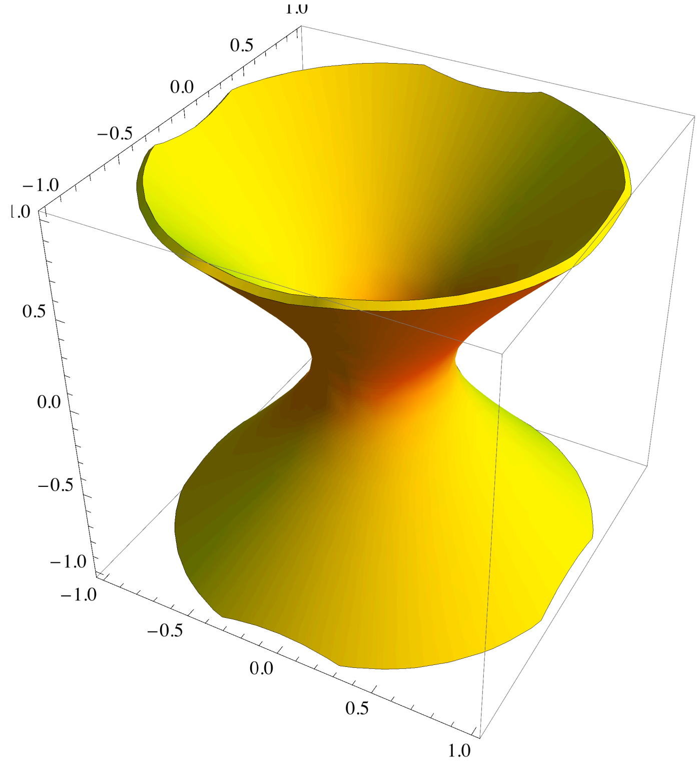
Plot $\sin(x+\cos(y))$ over $[-\pi, 2\pi]\times[-\pi, 2\pi]$ restricted such that $x < y^2$ using `RegionFunction`.

```
Plot3D[Sin[x + Cos[y]], {x, -π, 2 π}, {y, -π, 2 π},
 RegionFunction → Function[{x, y}, x < y^2], Filling → Bottom, Mesh → 8]
```



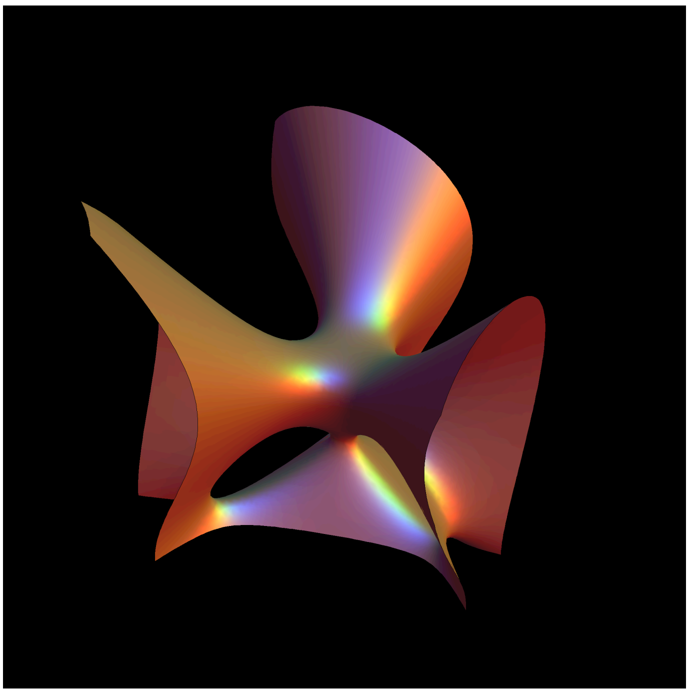
Produce a 3D Contour plot of $x^2 + y^2 - z^2 = 0.1$.

```
ContourPlot3D[x^2 + y^2 - z^2 = 0.1, {x, -1, 1}, {y, -1, 1},
 {z, -1, 1}, ContourStyle → {{Yellow, Thickness[0.04]}}, Mesh → None]
```



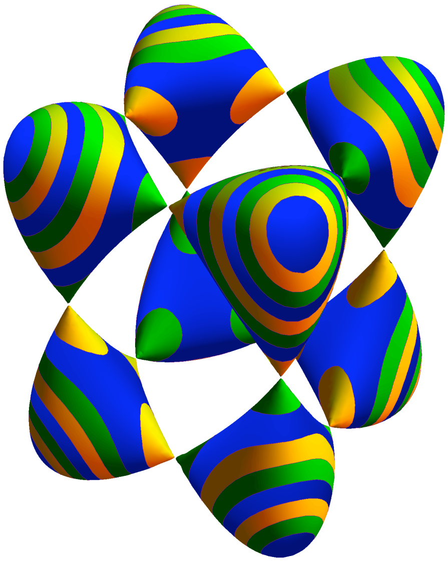
Smooth shading is generated via geometric or analytic vertex normal computation.

```
ContourPlot3D[16 x^3 + 16 y^3 - 31 z^3 + 24 x^2 z - 48 x^2 y - 48 x y^2 + 24 y^2 z - 54 √3 z^2 - 72 z = 1,
 {x, -2.5, 2.5}, {y, -2.5, 2.5}, {z, -2.5, 2.5}, Mesh → None, PlotPoints → 50,
 ContourStyle → {{Brown, Specularity[White, 20]}}, Background → Black,
 Boxed → False, Axes → False, SphericalRegion → True, Mesh → All]
```



Here is an example of an interesting algebraic surface.

```
ContourPlot3D[x^4 - 5 x^2 + y^4 - 5 y^2 + z^4 - 5 z^2 == -12.5, {x, -3, 3}, {y, -3, 3}, {z, -2.3, 2.3},
 PlotPoints → 50, MeshFunctions → {Function[{x, y, z}, x y z]}, MeshStyle → Purple,
 MeshShading → {Blue, Yellow, Green}, ContourStyle → Green, Boxed → False, Axes → False]
```

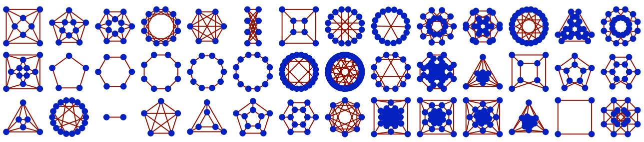


3. Data Sources

Next I present a few examples of the variety of data sources available in *Mathematica* 6.

Generate an array of Cayley graphs.

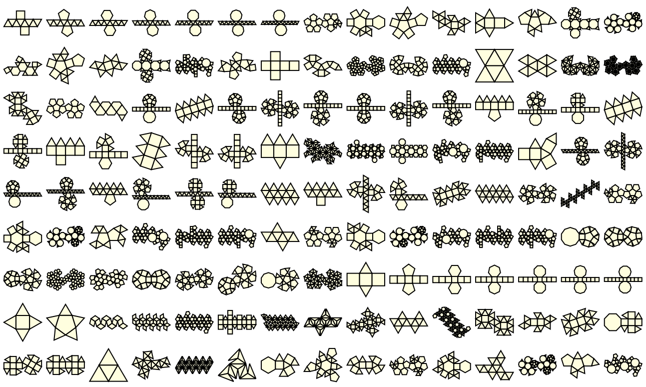
```
GraphicsGrid[Partition[GraphData /@ GraphData["CayleyGraph"], 14]]
```



If you take a polyhedron, cut along certain edges, and lay the whole thing flat the result is called a polyhedron net.

Here is a table of net images, indexed using `Tooltip`.

```
GraphicsGrid[Partition[
  (Graphics[Tooltip[PolyhedronData[#1, "NetImage"][[1]], PolyhedronData[#1, "Name"]]] &) /@
   PolyhedronData["NetImage", "Defined"], 15]]
```

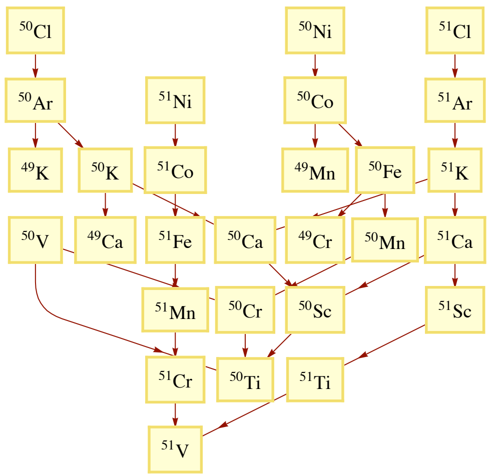


Look-up those nuclear isotopes with mass-number A in the range $50 \le A \le 51$ and then draw a decay network.

```
is = Select[IsotopeData[], 50 ≤ IsotopeData[#, "MassNumber"] ≤ 51 &]
```

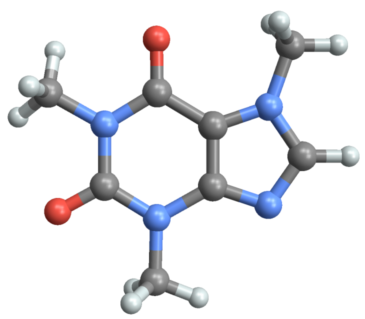
{Chlorine50, Chlorine51, Argon50, Argon51, Potassium50, Potassium51, Calcium50, Calcium51, Scandium50, Scandium51, Titanium50, Titanium51, Vanadium50, Vanadium51, Chromium50, Chromium51, Manganese50, Manganese51, Iron50, Iron51, Cobalt50, Cobalt51, Nickel50, Nickel51}

```
LayeredGraphPlot[
 Table[(IsotopeData[#, "Symbol"] & /@ (i → #)) & /@ IsotopeData[i, "DaughterNuclides"],
   {i, is}] // Flatten, DirectedEdges → True, VertexLabeling → True]
```



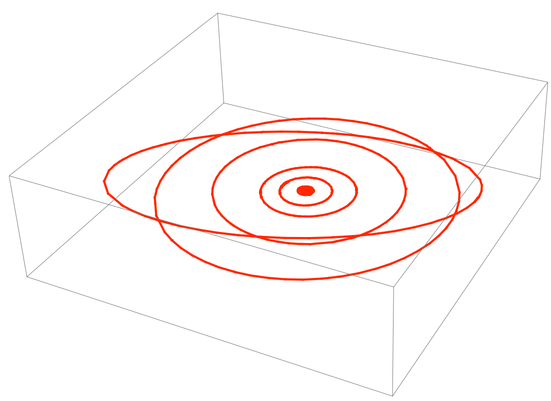
Visualize the caffeine molecule in 3D.

```
ChemicalData["Caffeine", "MoleculePlot"]
```



Generate a graphic of solar system orbit paths.

```
Graphics3D[{Red, AbsoluteThickness[1],
  AstronomicalData[#, "OrbitPath"] & /@ AstronomicalData["Planet"]}]
```



Generate a world map using a Mollweide projection.

```
CountryData["World", {"Shape", "Mollweide"}]
```

Since ATCM 2007 is held in Taiwan, find the country's population and show the shape of its electrical plug.

```
CountryData["Taiwan", #] & /@ {"Population", "ElectricalGridPlugImages"}
```

{23 036 087, {}}

4. Dynamic Interactivity

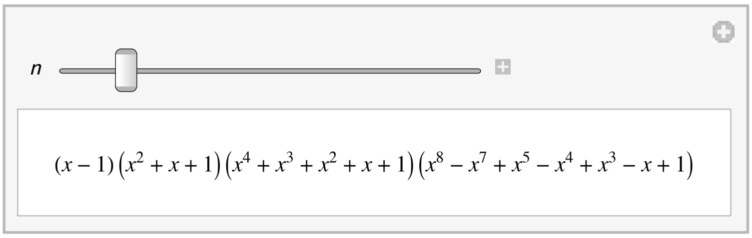
Version 6 includes a range of capabilities that define a new kind of dynamic interactive computing.

4.1 Manipulate

`Manipulate` is a new function that enables users to create dynamic user interfaces as easily as they might create a table or a plot. `Manipulate` is flexible and is tightly integrated with *Mathematica* notebooks, typesetting, and graphics.

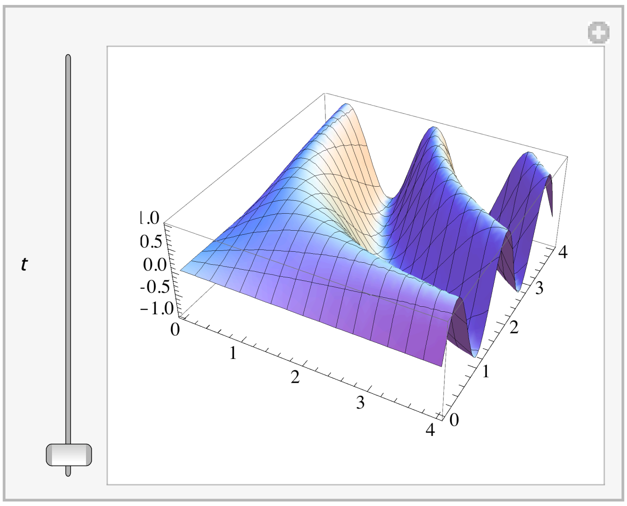
Show the factors of $1 - x^n$ for integer parameter $1 \le n \le 120$.

```
Manipulate[Factor[x^n - 1], {n, 1, 120, 1}]
```



Animate a three-dimensional plot of $\sin(x\,y + t)$ over $[0, 4] \times [0, 4]$ as t varies from 0 to 2π.

```
Manipulate[Plot3D[Sin[x y + t], {x, 0, 4}, {y, 0, 4}, ImageSize → 200],
 {t, 0, 2 π}, ControlPlacement → Left, ControlType → VerticalSlider]
```



4.2 LocatorPane and ClickPane

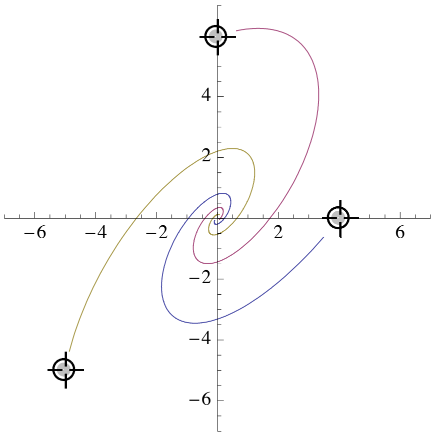
Here are two examples of interactive graphic solutions to matrix differential equations of the form

$$\dot{\boldsymbol{x}}(t) = A \cdot \boldsymbol{x}(t), \tag{1}$$

for `Dynamic` values of the initial condition, $\boldsymbol{x}(0)$, provided by the location of a mouse click.

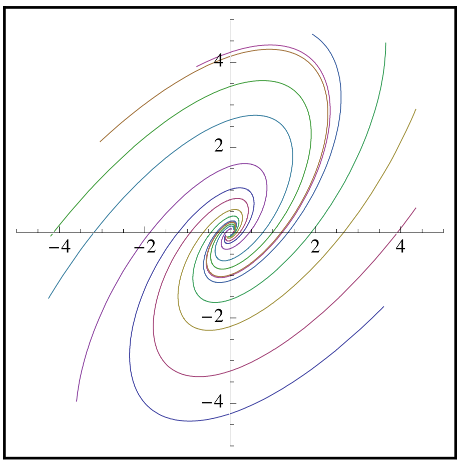
Visualize solutions using `LocatorPane`.

```
DynamicModule[{pt = {{4, 0}, {0, 6}, {-5, -5}}}, LocatorPane[Dynamic[pt],
  Dynamic@ParametricPlot[Evaluate[MatrixExp[({{-1.1, 0.9}, {-1.4, 0.3}}) t, #] & /@ pt],
   {t, 0, 10}, PlotRange → 7, ImageSize → 200]]]
```



Keep track of all solutions as you go using `ClickPane`.

```
DynamicModule[{s = {}},
 ClickPane[Framed@Dynamic@ParametricPlot[s, {t, 0, 10}, PlotRange → 5, ImageSize → 200],
  (AppendTo[s, MatrixExp[({{-1.1, 0.9}, {-1.4, 0.3}}) t].#]) &]]
```



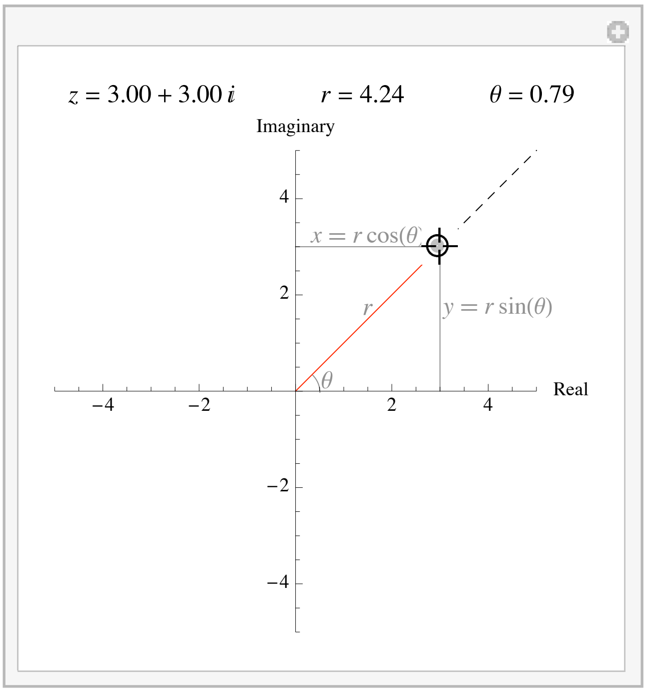
5. Teaching Examples

I have just started using this new interactive functionality in my teaching and would like to share with you some examples.

5.1 Complex Numbers Applied to Electric Circuits

Using complex numbers one can generalize Ohm's law so as to handle alternating current (AC) circuits containing resistors R, inductors L, and capacitors, C. First one needs to make clear the cartesian and polar representations of complex numbers.

The imaginary axis is perpendicular to the real axis. The point $\{x, y\} \equiv r\{\cos(\theta), \sin(\theta)\}$ is represented by $z = x + i\,y = r\,e^{i\,\theta}$ and is indicated by a `Locator`.



Now consider the circuit in Figure 2.

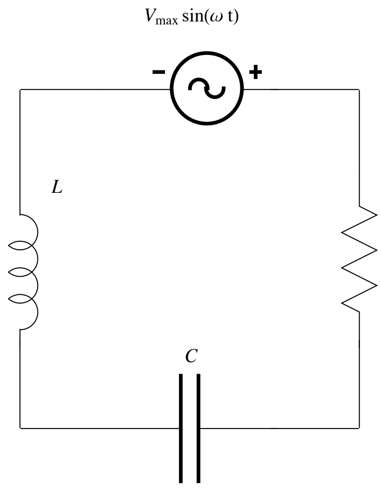


Figure 2 Series *LCR* circuit.

The complex impedance of this circuit is

$$Z = Z_L + Z_C + Z_R = R + \frac{1}{i\,\omega\,C} + i\,\omega\,L = R + i\left(\omega\,L - \frac{1}{\omega\,C}\right), \tag{2}$$

where $\omega = 2\,\pi\,\nu$ is the angular frequency of oscillation of the power source. Usually we are only interested in the magnitude and phase of $Z = |Z|\,e^{i\,\phi}$:

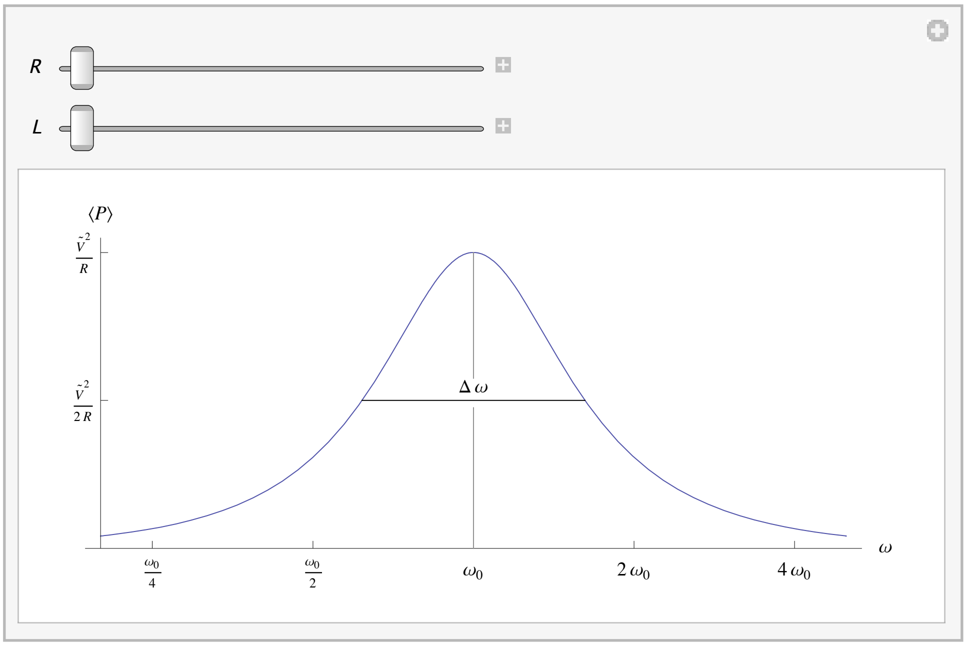
$$|Z| = \sqrt{R^2 + \left(\omega\,L - \frac{1}{\omega\,C}\right)^2}, \quad \tan(\phi) = \frac{y}{x} = \frac{\omega\,L - \frac{1}{\omega\,C}}{R}. \tag{3}$$

The average power $\langle P \rangle$ supplied by the power source in Figure 2 is

$$\langle P\rangle = \tilde{I}^2 R = \frac{\tilde{V}^2}{Z^2} R = \frac{\tilde{V}^2 R\,\omega^2}{\omega^2 R^2 + L^2\left(\omega^2 - \omega_0^2\right)^2} = \frac{\tilde{V}^2}{R} \Bigg/ \left(1 + \frac{L^2}{R^2}\omega^2\left(1 - \frac{\omega_0^2}{\omega^2}\right)^2\right), \tag{4}$$

where $\omega_0 = 1\big/\sqrt{L\,C}$ is the **resonant frequency**. Plots of $\langle P\rangle$ versus ω are called **resonance curves**. Here is a demonstration that shows how resonant curves (4) depend upon the resistance R and the inductance L.

Log-linear plot of $\langle P\rangle$ versus ω for the series LCR circuit in Figure 2. The line marked with $\Delta\omega = R/L$ joins the two **half-power points** and is called the **full-width at half maximum** (FWHM). The (dimensionless) **quality factor** is $Q = \omega_0/\Delta\omega$.



Once the behavior of simple circuits is understood, one can move to more complicated circuits such as that shown in Figure 3, where we measure the output voltage $\tilde{V}_{\text{out}}$ across R_2.

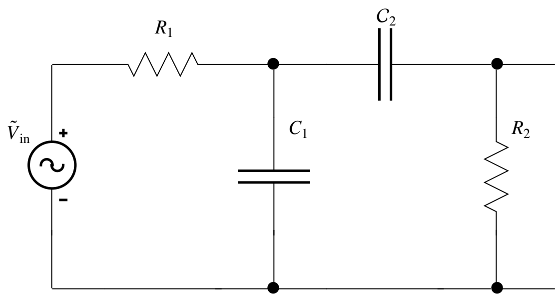


Figure 3 Filter circuit.

The total impedance of this circuit is complicated. However, we can effectively consider it as being built from two filter circuits as that shown in Figure 4.

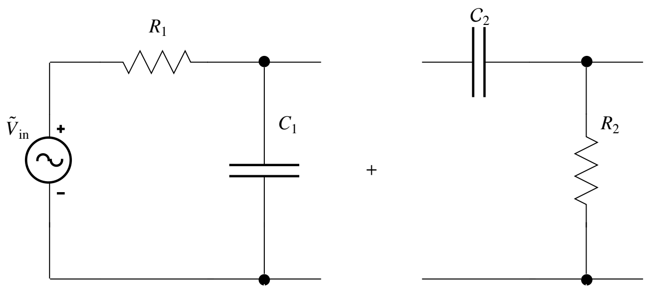
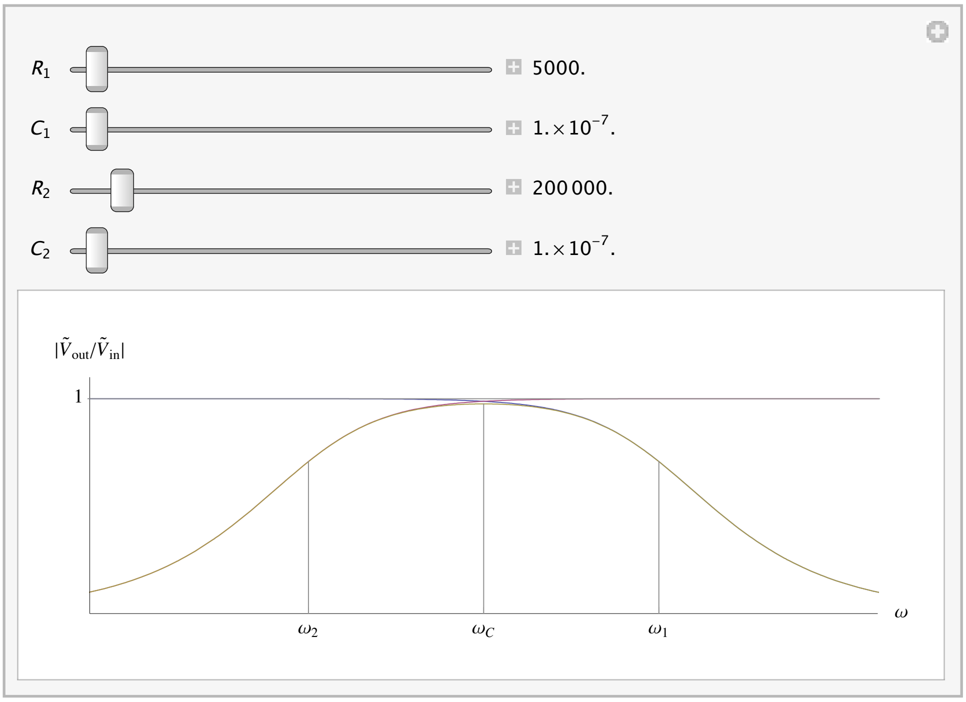


Figure 4 Simplified filter circuit.

Considering the output, $\tilde{V}_{\text{out}}$, of the first filter as the input, $\tilde{V}_{\text{in}}$, to the second filter, one sees that the transfer function of this combination is, approximately, just the *product* of the transfer function of each filter:

$$\left|\frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}}\right| \approx \frac{1}{\sqrt{1+(\omega/\omega_1)^2}}\,\frac{1}{\sqrt{1+(\omega_2/\omega)^2}}, \text{ where } \omega_i = \frac{1}{R_i\,C_i}. \tag{5}$$

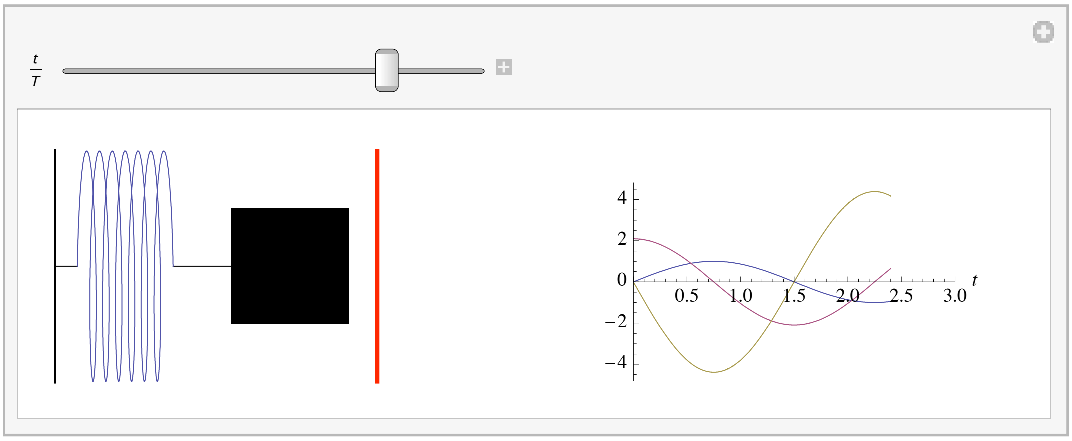
Log-linear plot of the *approximate* transfer function $\left|\tilde{V}_{\text{out}}/\tilde{V}_{\text{in}}\right|$ versus ω for a band-pass filter. The **centre frequency** is the **geometric mean** of the two half-power frequencies, $\omega_C = \sqrt{\omega_1\,\omega_2} \approx 316\,\text{rad}\cdot s^{-1}$.



5.2 Forced Oscillations and Resonance

Consider horizontal motion on a frictionless surface of a mass-spring system. The spring is loosely wound and can be compressed as well as stretched with force $F = -k\,x$, where k is the spring constant and x is the displacement from the equilibrium position.

Graphs of x (—), v (—), and a (—) for a mass-spring system.



The acceleration is always toward the equilibrium position ($x = 0$) and proportional to distance from the that point.

Now introduce a damping force, $F_d = -b\,v$, that opposes the motion and is proportional to the instantaneous velocity. Students should have an *intuitive feel* for the behaviour of this system; if the spring is stretched and then released, the mass will undergo damped simple harmonic motion with the amplitude reducing over time. And if the damping is sufficiently large, the mass will not oscillate at all.

From Newton's second law, the differential equation describing the motion of the mass, m, is

$$m \frac{d^2 x(t)}{d t^2} + b \frac{d x(t)}{d t} + k\, x(t) = 0, \tag{6}$$

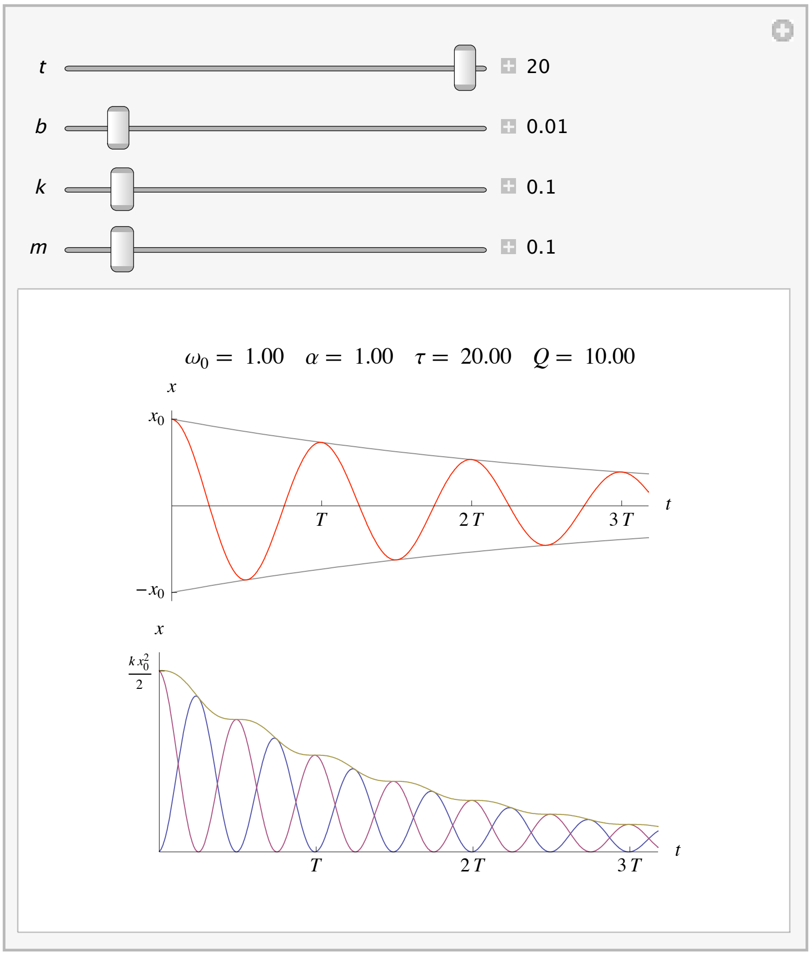
and students should therefore have a 'feel' for the behavior of solutions to this differential equation, namely exponential decay of harmonic motion, that is

$$x(t) = x_0\, e^{-t/(2\tau)} \cos(\alpha\, t), \tag{7}$$

where $x_0 = x(0)$ is determined by the initial condition and, to satisfy the differential equation, the parameters are $\tau = 1/\Delta\omega = m/b$ and $\alpha = \omega_0 \sqrt{1 - 1/\left(4\, Q^2\right)}$, where $\omega_0 = \sqrt{k/m}$ is the **resonant frequency** and the (dimensionless) **quality factor** is $Q = \omega_0 / \Delta\omega$.

An interesting question is what happens to the total energy. Energy is dissipated only when the mass is *moving*. For this reason, the total energy decays exponentially only on *average*.

The top plot displays the solution to the differential equation of the form $x_0\, e^{-t/(2\tau)} \cos(\alpha\, t)$. The **envelope** of the oscillation (—) is $\pm x_0\, e^{-t/(2\tau)}$. The bottom plot shows the kinetic energy (—), $K = \frac{1}{2} m\, v^2$, spring potential energy (—), $U = \frac{1}{2} k\, x^2$, and total energy (—), $E = K + U$, for the damped oscillator.



For an oscillatory driving force the differential equation reads

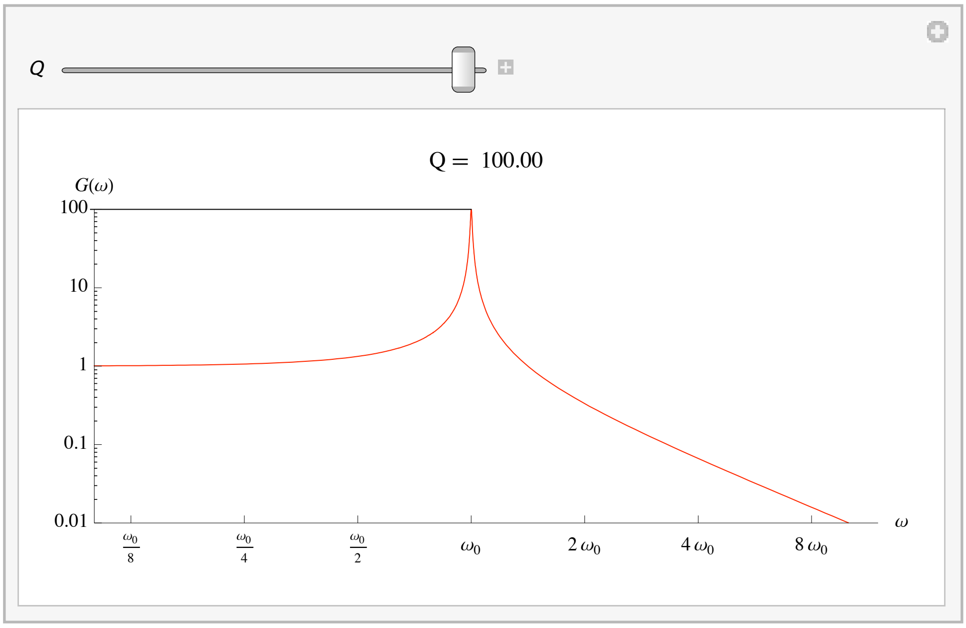
$$m \frac{d^2 x(t)}{d t^2} + b \frac{d x(t)}{d t} + k\, x(t) = F_0 \cos(\omega\, t), \tag{8}$$

where ω is the angular frequency of the driving force. By analogy with the series *LCR* circuit, the amplitude of a forced mechanical oscillator is

$$\Rightarrow \frac{x_0}{F_0} = \frac{1}{k} \frac{1}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{\omega}{Q\, \omega_0}\right)^2}} = \frac{1}{k} G(\omega), \tag{9}$$

where $G(\omega)$ is the (dimensionless) **transfer function**.

Plot of $G(\omega)$ as a function of ω for various values of Q. On resonance $G(\omega_0) = Q$.



At low frequency the response, $x_0 = F_0 / k$, depends only on the spring. On resonance the response depends on the Q. At high frequency the response is $x_0 \to F_0 / k\, (\omega_0 / \omega)^2 = F_0 \big/ \left(m\, \omega^2\right) \to 0$, which is the same as for a free mass.

5.3 Solitons

The Korteweg de Vries (KdV) equation, describing water waves in shallow channels, is given by

$$\frac{\partial u}{\partial t} - 6\, u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{10}$$

where $u(x, t)$ is the wave amplitude.

The initial value of the n-soliton solution is $u_n(x, 0) = -n(n+1)\operatorname{sech}^2(x) = 2\,\mathcal{V}_n(x)$. If $\mathcal{V}_n(x)$ is interpreted as the *potential* of the Schrödinger equation, that is

$$\frac{-1}{2} \frac{\partial^2 \psi(x)}{\partial x^2} + \mathcal{V}_n(x)\, \psi(x) = \mathcal{E}\, \psi(x), \tag{11}$$

where

$$\mathcal{V}_{\text{n\_}}(\text{x\_}) = -\frac{n\,(n+1)}{2} \operatorname{sech}^2(x);$$

the (unnormalized) solution is

$$\psi_{\text{n\_,m\_}}(\text{x\_}) = P_n^m(\tanh(x));$$

where $P_n^m(z)$ is the associated Legendre function, and the energy is

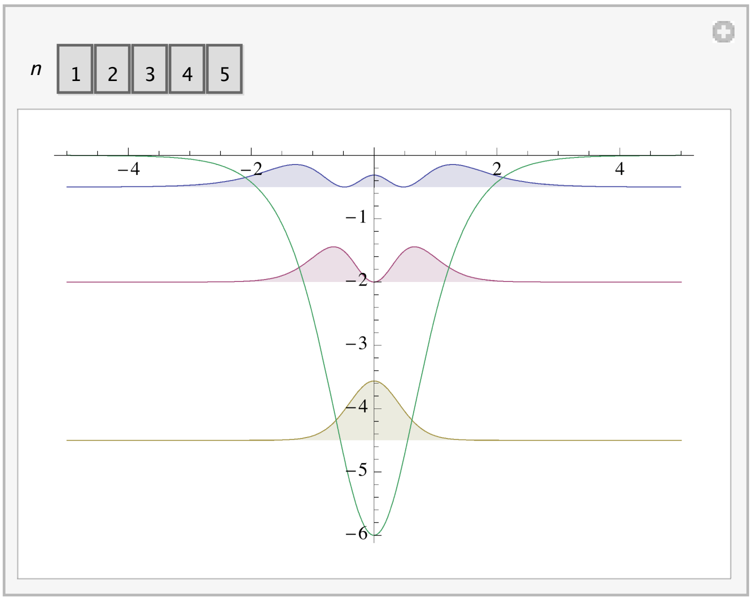
$$\mathcal{E}_{\text{m\_}} = \frac{-m^2}{2};$$

For each n, there exists n bound states with $1 \le m \le n$. First we compute and save the normalization constants $\mathcal{N}_{n,m}$ determined by requiring $\mathcal{N}_{n,m}^2 \int_{-\infty}^{\infty} \psi_{n,m}(x)^2\, dx = 1$.

$$\mathcal{N} /: \mathcal{N}_{\text{n\_,m\_}} := \mathcal{N} /: \mathcal{N}_{n,m} = 1 \Big/ \sqrt{\int_{-\infty}^{\infty} \psi_{n,m}(x)^2\, dx}$$

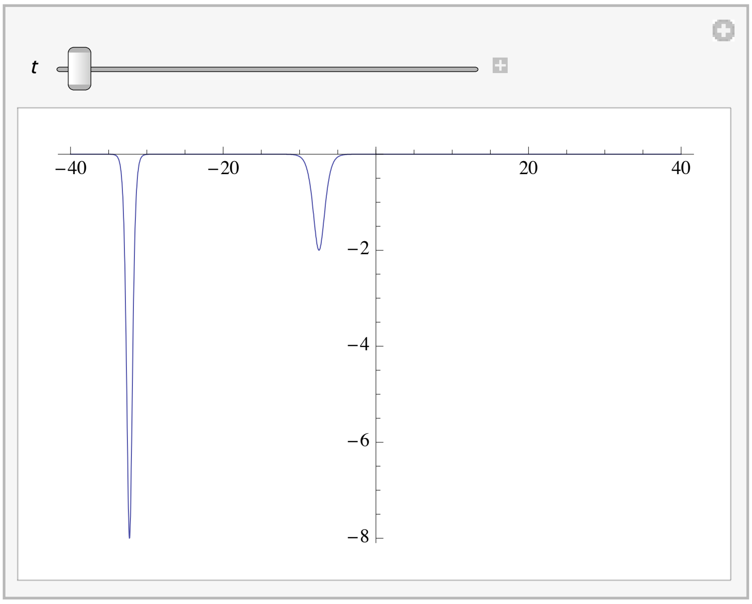
On one plot, display the potential $V_n(x)$ together with the normalized eigenfunctions, shifted downwards by the associated energy, $\mathcal{N}_{n,m}^2\,\psi_{n,m}(x)^2 + \mathcal{E}_m$, for $1 \le n \le 5$ and $1 \le m \le n$.

Manipulate[Plot[Evaluate[Join[Table[$\mathcal{N}_{n,m}^2\,\psi_{n,m}(x)^2 + \mathcal{E}_m$, {$m$, n}], {$\mathcal{V}_n(x)$}]], {x, −5, 5}, Filling → Table[$m \to \mathcal{E}_m$, {m, n}]], {n, Range[5]}, SaveDefinitions → True]



Here we visualize the time evolution of the "two-soliton" solution to the KdV equation (10). It is clear that there are now two waves; one slow and one fast.

Manipulate[Plot[$-\dfrac{12\,(4\cosh(8\,t-2\,x)+\cosh(64\,t-4\,x)+3)}{(3\cosh(28\,t-x)+\cosh(36\,t-3\,x))^2}$, {$x$, −40, 40}, PlotRange → {−8.1, 0.1}], {t, −2, 2, 0.25}]

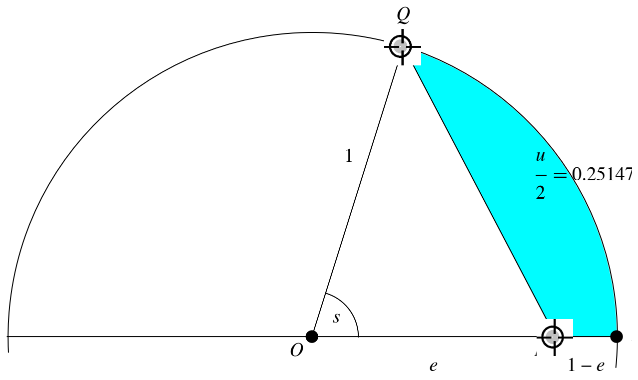


5.4 Kepler Equation

The *Kepler equation*, $s = u + e\sin(s)$ — critical in celestial mechanics — relates the mean anomaly u (a parameterization of time) of a body in an elliptical orbit of eccentricity e to the body's eccentric anomaly s (a parameterization of polar angle) [6]. Computing s is a commonly-used intermediate step to the calculation of planetary position as a function of time.

Consider the following related geometrical example: Pick an arbitrary point F inside the unit circle centered at O. Let P be the point on the circle closest to F and pick another point Q elsewhere on the circle. The location of F and Q are dynamic and can be modified by moving the `Locator` associated with each point. Define s and e as pictured below and let u be twice the area of the shaded sector PFQ.

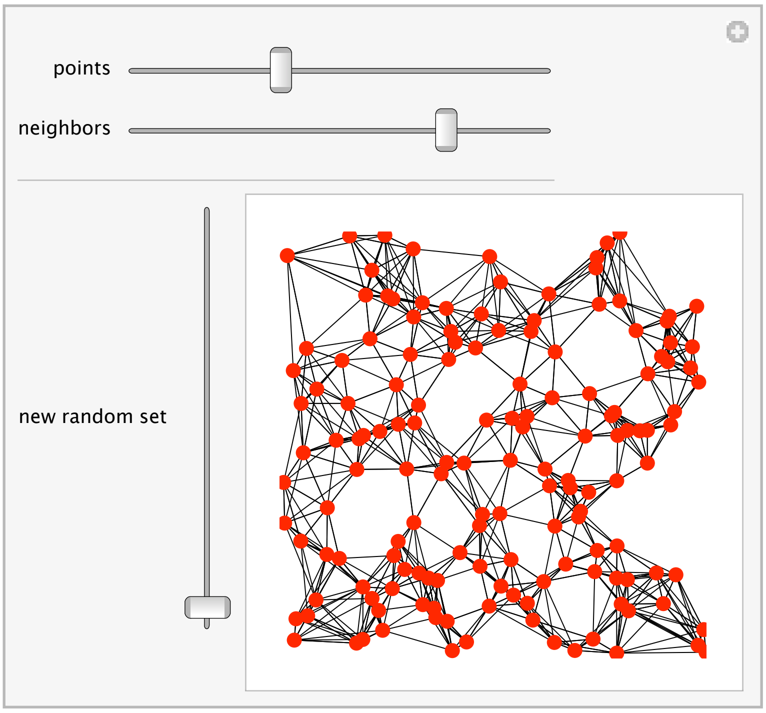
Observe that $\frac{1}{2}\,u$ = Area(sector $P0Q$) − Area($\Delta F0Q$) = $\frac{1}{2}\,s - \frac{1}{2}\,e\sin(s)$.



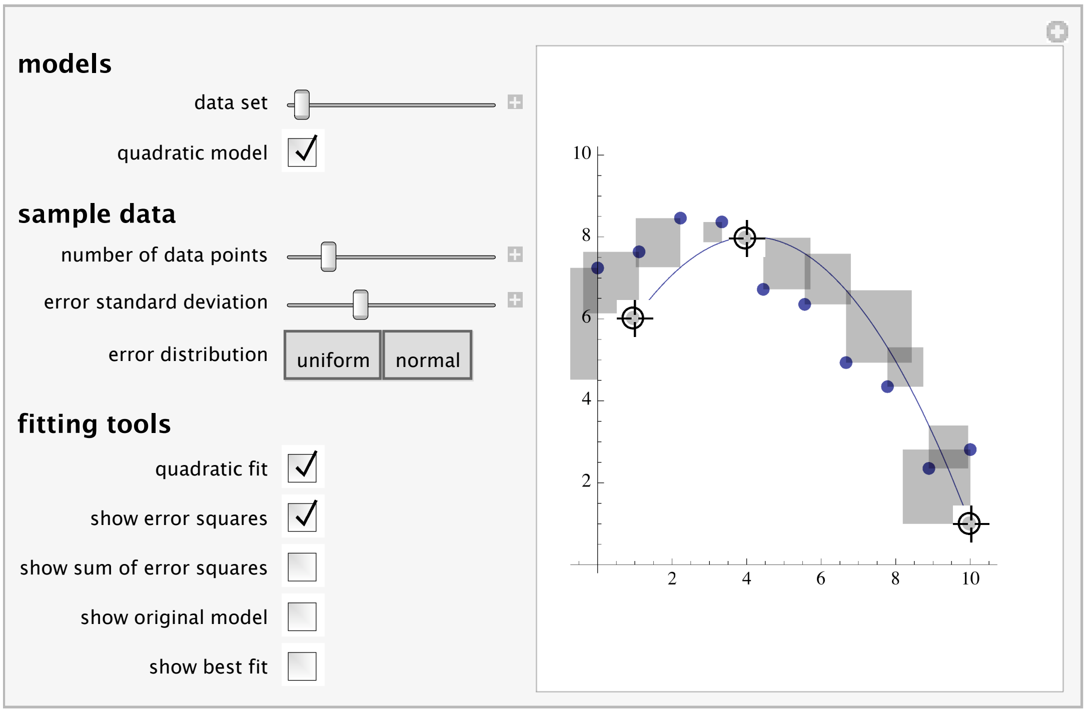
6. Demonstrations

Here are a few more advanced examples of the functionality of `Manipulate` freely available for download from the Demonstrations Project website [2].

Produce a random set of up to 200 points and join each point to a specified number of nearest neighbors, computed using the `Nearest` function [7].

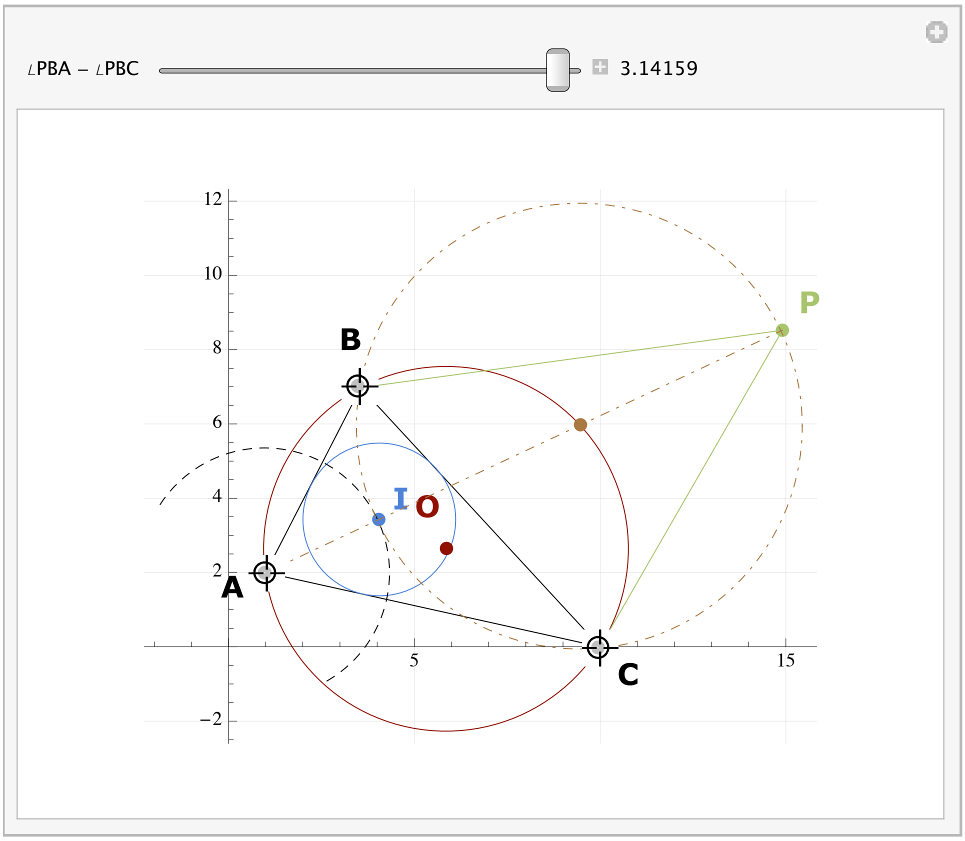


Here is a very nice demonstration of least-squares curve fitting [8].



The following problem was presented at the International Mathematical Olympiad (IMO) of 2006 in Slovenia [9]: Let ABC be a triangle with incentre I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that AP ≥ AI, and that equality holds if and only if $P = I$.

Observe that P moves along the brown circle with center at the intersection of the circumcircle and the bisector of the angle A.



7. Conclusion

In this talk I demonstrated several of the new capabilities in *Mathematica* Version 6, showing how much fun one can have when the technology is high-level, robust, portable, and easy to use.

References

[1] http://www.wolfram.com

[2] http://demonstrations.wolfram.com

[3] http://www.wolfram.com/products/mathematica/history.html

[4] Available at http://physics.uwa.edu.au/pub/ATCM/2007 as a *Mathematica* Notebook

[5] http://www.wolfram.com/products/player/

[6] http://mathworld.wolfram.com/KeplersEquation.html

[7] http://demonstrations.wolfram.com/NearestNeighborNetworks/

[8] http://demonstrations.wolfram.com/LinearAndQuadraticCurveFittingPractice/

[9] http://demonstrations.wolfram.com/AnIMOTriangleProblem/

Spreadsheets in Mathematics: Accessibility, Creativity, and Fun

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Abstract: A spreadsheet, such as *Microsoft Excel*, provides educators with a creative tool for the study and teaching of mathematics, mathematical modeling, and mathematical visualization. It enables students to gain mathematical insights into a diverse range of interesting and significant applications in an engaging setting while they simultaneously acquire practical skills in using the principal mathematical tool of the workplace. This paper illustrates novel ways to use this powerful and accessible tool and its outstanding graphic features creatively in teaching a surprising number of mathematically oriented topics. Illustrations come from such disciplines as the physical and social sciences, statistics, mathematics, computer sciences, and the arts. The paper describes how *Excel*'s graphics can create eye-catching animated graphic displays and inject more fun into the study of mathematics.

1. Introduction

The computer spreadsheet is nearly 30 years old. During its lifetime, applications of this creative software have diversified and spread widely from the original uses in financial fields into all areas of human endeavor. Today it is the principal mathematical tool of the workplace. It is readily available on virtually every computer, with the most common version being *Microsoft Excel*. It also increasingly appears in the teaching and communication of mathematics at all levels. In this paper, we provide some visual glimpses into a broad range of applications of spreadsheets in mathematics education.

The spreadsheet offers many advantages for learning mathematical concepts. It is an easily accessible and creative tool whose basic operations are familiar to most students. Educators can use spreadsheets so that the actual process of creating a spreadsheet model itself teaches and reinforces mathematical concepts. In addition, the spreadsheet design often allows students to study successfully numerous topics ordinarily considered as being too difficult for them. Its use also provides students with valuable practical experience in working with a tool that they will use in their future jobs. Finally, the spreadsheet helps teachers and students to find the study of mathematics to be a fun experience.

Our approach to the use of spreadsheets contains an emphasis on designing effective graphics to promote the development of visualization skills. To do this, we present illustrative examples that describe creative ways to use spreadsheets to produce interactive animated graphics. Our examples come from a wide range of applications and disciplines. Each of these examples incorporates the use of mathematics.

We have designed this paper to give a visual overview of our approach. We create our examples using *Microsoft Excel 2003*. Anyone interested in the technical details of spreadsheet operation, how to create scroll bars and simple macros for animated graphics, and ways to use the Solver and Data Table tools, can consult [8] or download a paper and *Excel* application files from [2].

A spreadsheet lends itself to a variety of educational uses. First, it provides educators with a natural way to implement mathematical algorithms and models and to create interactive graphs for use in student assignments and activities. In the latter case, it provides a way for students to work in groups on more substantial projects. Second, it enables teachers to prepare original and effective classroom demonstrations to illustrate mathematical ideas. In addition, it allows teachers to create visual models for most textbook topics, including those in algebra, calculus, statistics, numerical analysis, and linear algebra. Third, it can provide an avenue for the professional development of educators, opening opportunities for them to give professional presentations of new approaches to teaching and research. This applies not only to mathematics, but also to virtually any other discipline. Finally, it is an excellent avenue for teaching continuing education courses, and for communicating with the public and with colleagues from other disciplines. The author has used spreadsheets successfully in presenting mathematics to wide variety of adult learners in both developing and technologically advanced nations. Additional discussions of teaching uses of spreadsheets may be found in [1], [3] and on the Web [9], [10].

2. Mathematical Modeling

Originally, the business community was the principal user of spreadsheets, employing them to create interactive financial mathematical models that they interrogated in a “What if …?” manner. Over the ensuing years, many additional mathematical applications for spreadsheets arose, causing their usage to expand into the design of mathematical models from diverse fields. As a result, today we use spreadsheets in many classrooms to study a wide and complex range of interesting mathematical models. Many topics that ordinarily we might regard as too advanced are now accessible to our students. The spreadsheet provides students with a tool that encourages creativity, and enables them to have fun while investigating significant mathematical ideas.

As our first example, in Figure 2.1 we present an ecological model that involves the study of the interaction of two competing species of animals. We base it on a more traditional presentation found in pages 41-45 of [7]. Here we consider two species of squirrels, gray and red, whose populations grow exponentially if there were no competition between them. However, since there is an interaction effect that reduces population growth, this also is included in the model.

For our spreadsheet descriptions, we use the arrow notation of [8]. In Figures 2.2-2.3, we first enter the population sizes and growth rates, and then form a table that gives the sizes of the populations in successive time periods. After reproducing the initial populations in the top row of our table, we compute the number of new grey species as the product of their population and the growth rate. We indicate that the growth rate is an absolute reference by placing a pin in the source cell. We next compute the amount of grey population reduction that occurs due to competition between the species by assuming that it is proportional to the number of possible interactions between individuals (the product of the populations). We assume that studies have approximated the value of the parameter for the reduction rate. We then find the next period’s population of the grey species in the following row. We treat the red species similarly. We then simply copy the formulas.

	Initial	New	Interact			Period
	Population	Growth	Effect		N	20
Grey	145	0.24	0.003		Grey	41.3
Red	79	0.3	0.002		Red	204.9
Period	Grey	Red	New G	Loss R	New G	Loss G
1	145.0	79.0	34.8	34.4	23.7	22.9
2	145.4	79.8	34.9	34.8	23.9	23.2
3	145.5	80.5	34.9	35.2	24.2	23.4

Figure 2.1. Species Competition Output

	Initial	New	Interact	
	Population	Growth	Effect	
Grey	145	0.24	0.003	
Red	79	0.3	0.002	
period n	Grey Pop	Red Pop	New Grey	Grey Loss
1	145	79	*	* *
2				

Figure 2.2. Species Formulas I

	Initial	New	Interact	
	Population	Growth	Effect	
Grey	145	0.24	0.003	
Red	79	0.3	0.002	
period n	Grey Pop	Red Pop	New Grey	Grey Loss
1	145	79	34.8	34.365
2	+ -			

Figure 2.3. Species Formulas II

In order to produce an image to help us in visualizing the results, we create the *xy*-graph in Figure 2.4. It shows the population sizes over a sequence of periods. We see that with our assumptions the red species eventually drives out the grey one. Students can also discover that there are initial population values that will result in equilibrium, but one that is extremely unstable. The slightest changes cause the extinction of one of the species. Our second graph in Figure 2.5 gives us a different way to visualize the results. The randomly placed markers represent the numbers of each species in a given period. We further enhance our model by inserting scroll bars (see [8], [2]), and move their sliders to see the effects of varying the number of periods or the model's parameters. In the second graph, the original mixture of red and grey gradually becomes entirely red.

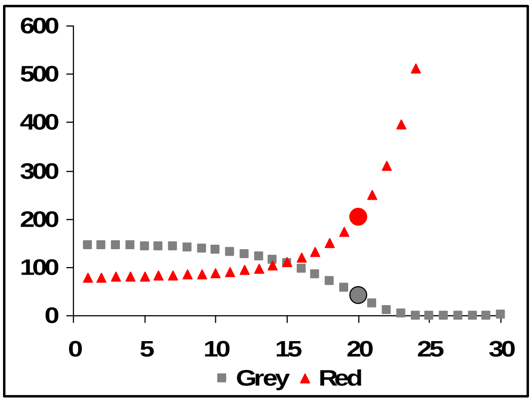

Figure 2.4. Species Graph I

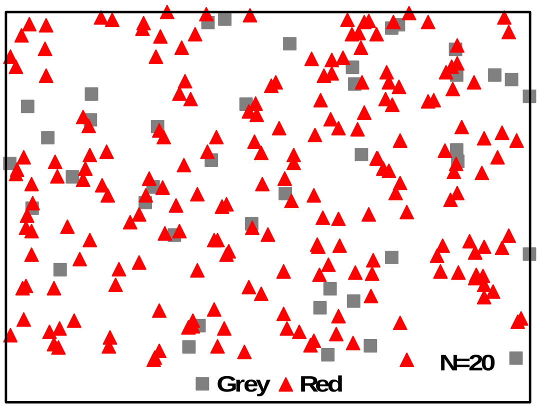

Figure 2.5. Species Graph II

Similar models and visualization techniques appear in [8], with investigations of such topics as population growth, resource harvesting, epidemics, predator-prey interactions, genetics, medicine dosage, financial analyses, projectiles, heat flow, apportionment, and planetary motion. Other possible topics include pollution, cooling, drug testing, and many more. These topics provide interesting group projects, allowing students to design creative implementations and graphics.

3. Graphing Functions

Spreadsheets are also effective tools for creating animated graphs of functions. Here we graph the polar equation $r = \cos(t/c)$. In the model of Figures 3.1-3.2, the parameter, *c*, is set to 2. The left column counts degrees. We convert these to radians using *Excel*'s built-in RADIANS function in the second column, enter the formula for r in the third one, and compute the values of x and y in the next two by $x = r\cos t$, $y = r\sin t$.

c	2		N	4		
step	1					
n	t	r	x	y	X	Y
0	0	1	1	0	1	0
1	0.017	1	1	0.017	1	0.017
2	0.035	1	0.999	0.035	0.999	0.035
3	0.052	1	0.998	0.052	0.998	0.052
4	0.07	0.999	0.997	0.07	0.997	0.07
5	0.087	0.999	0.995	0.087	0.997	0.07

Figure 3.1. Graphing Output

c	2		N	4	
step	1				
n	t	r	x	y	X
0	0	1	1	0	IF(<= , ,)

Figure 3.2. Graph Formulas

We use the 4th and 5th columns to create the complete *xy*-graph in Figure 3.3. However, it is easy to animate the tracing of the curve to appear much as we would draw it manually in class. We generate only the first *N* points of the curve by employing the IF…THEN…ELSE structure in the two rightmost columns. These expressions reproduce the (x, y) values when $n \le N$, and otherwise copy the value from the cell above. We then link a scroll bar to *N*. As we move it, we see the curve being traced out. Many additional graphing applications appear in [4] and [8].

Students can use mathematical concepts to have fun in enhancing the model. In Figure 3.4, we have created a simple image of a butterfly, and then scaled, translated, and rotated it so that it moves at the trace point and remains tangent to the path. Using a few more mathematical ideas and spreadsheet techniques, students can color in the wings, cause them to flap periodically, draw parametric curves in the shape of flowers along the curve, and create a simple macro to continually update *N*, thereby creating a movie of the butterfly moving from flower to flower.

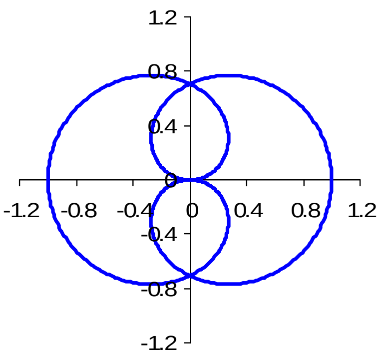

Figure 3.3. Static Graph

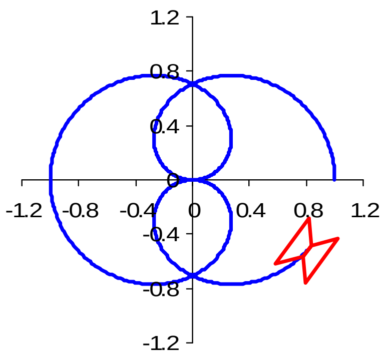

Figure 3.4. Animated Trace Graph

4. Numerical Algorithms

We also implement the algorithms of numerical analysis effectively in a spreadsheet, including those for finding zeroes, numerical integration, solving differential equations, fitting curves, solving linear systems, and computing eigenvalues. We can also create animated versions of virtually every static diagram that appears in a text. These creations provide students with new ways to visualize the effects of changes in parameters, to observe possible divergence, and to explore other aspects.

Teachers and students can also create animated demonstrations of algorithms. In Figures 4.1-4.3, we implement Newton's Method in a table format to find a zero of $f(x) = x^2 - 2$. We first enter 0.4 as an initial estimate. We enter formulas to compute $f(x)$ and $f'(x)$ in the cells to the right. In the second row, we obtain the next approximation of the zero by $x - f(x)/f'(x)$. We copy these expressions down their columns. We can observe the rate of convergence, experiment with the initial estimate, and change functions. By designing other animated graphics as in [8], we can better illustrate functions in which the algorithm does not converge or is sensitive to the initial estimate. Such graphics generally attract more student attention than do traditional techniques.

Newton's Method			
n	x	y	y'
0	0.4	-1.84	0.8
1	2.7	5.29	5.4
2	1.72	0.96	3.441
3	1.441	0.078	2.883
4	1.414	7E-04	2.829
5	1.414	7E-08	2.828
6	1.414	0	2.828

Figure 4.1. Newton

Newton's Method			
n	x	y	y'
0	0.4	^2-2	2*
1+			

Figure 4.2. Formulas I

Newton's Method			
n	x	y	y'
0	~~0.4~~	-1.84	0.8
1	- /		

Figure 4.3. Formulas II

Additionally, teachers can design demonstrations that lead students through an algorithm in a step-by-step manner. Two steps in an animated teaching construction that we update by clicking on a button appear in Figures 4.4-4.5.

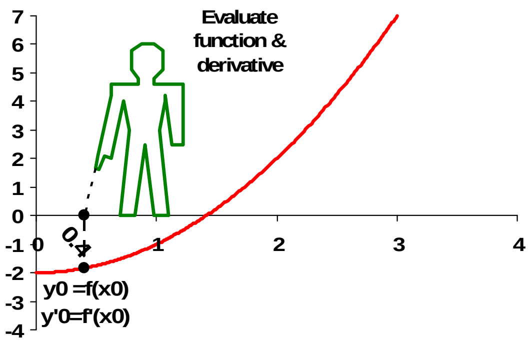

Figure 4.4. Instructional Graphic I

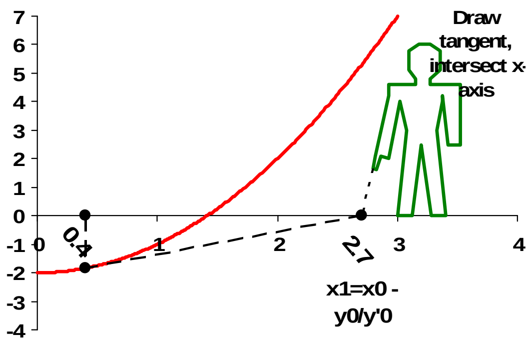

Figure 4.5. Instructional Graphic II

5. Linear Algebra and Vectors

Excel provides built-in matrix functions for multiplication, inversion, and determinates (see [8], [2]). We can use these and other standard features of *Excel* to create interactive animated displays for a great range of ideas from the linear algebra of $\Re^2$, including such topics as the grids for various bases, eigenvalues, linear programming, pivoting, and the visualizations of linear transformations. We employed some of these techniques in creating the butterfly graph above and the reflection transformation shown in Figure 5.1.

In Figure 5.2, we use the ideas of vectors in illustrating pursuit problems, enhanced by forming *xy*-images of airplanes. We first parameterize the path of one airplane, and then use vectors and a rotation matrix to cause another airplane to pursue the first as it moves directly toward the first in small discrete increments. We set velocities of the planes as parameters, and use a scroll bar to vary time, causing the planes to trace out curves. We observe the resulting path of the pursuit plane and when, or if, it overtakes the first. Students can also discover and investigate other pursuit strategies.

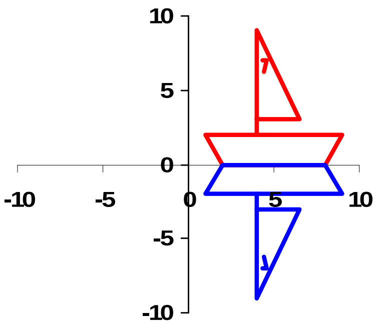

Figure 5.1. Matrices (reflection)

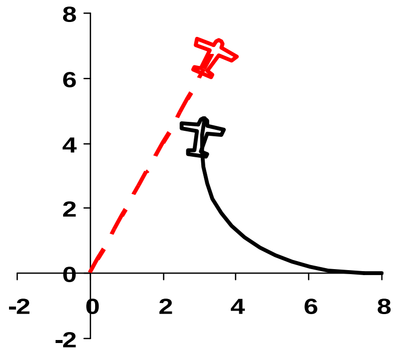

Figure 5.2. Matrices (rotation)

6. Computer Science

We can study a broad range of computer science concepts naturally on a spreadsheet. In addition to using the spreadsheet's built-in database features, students can discover creative ways to study such data structures topics as algorithms for sorting, searching, and stacks with this widely used computing tool. The act of creating an implementation enhances a student's understanding of the concepts. We also can create different ways to visualize traditional computer science topics. In Figures 6.1-6.2, we show two steps in the iterative solution of the classical Towers of Hanoi problem [5], while Figures 6.3-6.5 illustrate the continuous morphing of a butterfly into a rabbit.

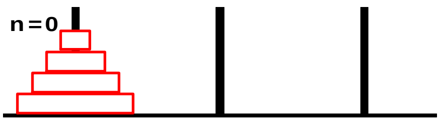

Figure 6.1. Towers Hanoi Graphic I

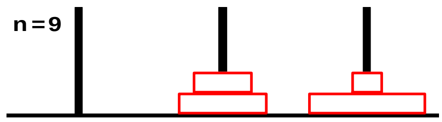

Figure 6.2. Towers Hanoi Graphic II

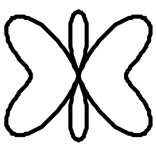

Figure 6.3. Morphing I

Figure 6.4. Morphing II

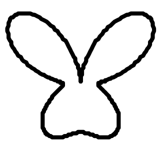

Figure 6.5. Morphing III

7. Art and Culture

Another fascinating source for interest, fun, and enjoyment comes from examining the arts and cultures of different societies. For example, symmetry, geometry, and other mathematical ideas often appear in the design of national flags and traditional art. A few spreadsheet creations of these appear in Figures 7.1-7.4. A challenge to students is to animate the flags to simulate waving in the breeze. Doing this requires both inventiveness and mathematical insights. Other sources of cultural projects come from traditional arts and crafts, such as the design of quilts, needlepoint, string-and-nail art, bilums (net bags) of Papua New Guinea, and kilims (rugs) of Turkey. Using a spreadsheet to design these cultural images provides many students who otherwise may not be attracted to mathematics with an attractive pathway to encounter the subject through its geometrical aspects by the creative application of a popular and accessible technological tool.

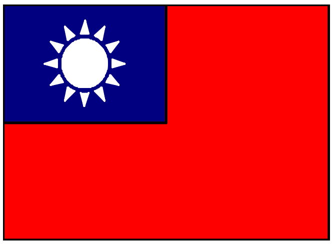

Figure 7.1. Flag Design I

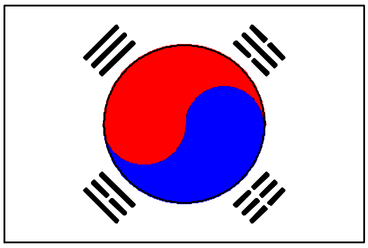

Figure 7.2. Flag Design II

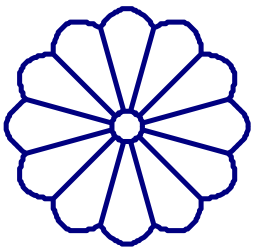

Figure 7.3. Cultural Design

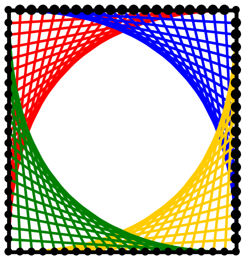

Figure 7.4. Nail and String Art

8. Statistics

Despite shortcomings in some of its statistical features, *Excel* is a valuable tool for learning the fundamental concepts of statistics. It invariably supplies us with several levels at which to approach a topic. Thus, we can pursue a concept such as the standard deviation either directly from definitions or by using built-in functions. In either case, we can supplement our models with effective original graphic visualizations.

One inventive way to display data in a map, as in Figure 8.1, is by using a bubble graph, where the geographical border is composed of tiny circles, and the circles for cities are proportional to their populations. This also is an excellent way for students to create attractive scatter diagrams for topics of particular interest to them. One of the author's earliest statistical memories is of a magazine's graphic that showed the relative strengths of U.S. university football teams via scaled footballs in a map.

Although we enter the coordinates of this map manually, the Web provides some data sets that supply (x,y) coordinates of many cartographic boundaries, and freeware such as *PlotDigitizer* give us a more convenient way to generate more easily coordinates to digitize a wide variety of images.

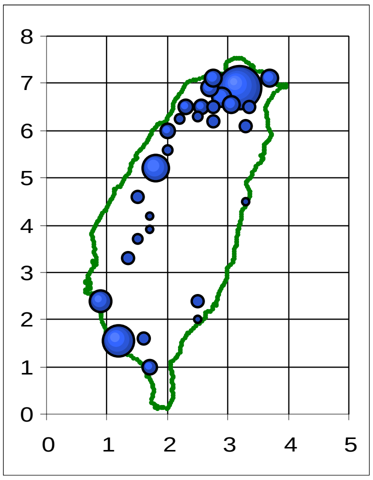

Figure 8.1. Scatter Display via Bubble Graph

Another versatile spreadsheet tool is the Solver. In Figures 8.2-8.4 we first employ the RANDOM function to generate 100 (x,y) points over a chosen range, and use the CORRELATION function to observe that the correlation coefficient, r, is nearly 0. We then convert the values into constants and use the Solver to set the correlation coefficient to a chosen value ($r = 0.8$) by having the Solver change x and y values. The spreadsheet makes the adjustments and produces a graph of a distribution giving the desired value of r. If we use 1 or –1 for r, then we will obtain a straight line.

r	0.019	
n	x	y
1	-0.06	0.185
2	0.468	-0.68
3	-0.52	-0.05
4	-0.11	0.872

Figure 8.2. Random Data

r	0.8	
n	x	y
1	0.032	0.16
2	0.129	-0.43
3	-0.53	-0.29
4	0.316	0.805

Figure 8.3. Correlation Data

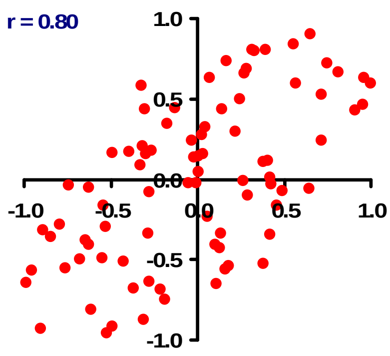

Figure 8.4. Correlation Graph

9. Historical Topics

During the author's university studies and his early years of teaching, the principal computational tool was the slide rule. Most students today are completely unfamiliar with its operation and its use of logarithms. This and other historical ideas provide wonderful projects for making classes interesting and fun. In the display below, we use scroll bars to move the center rule and the crosshair. The display of Figure 9.1 also provides the underlying numbers to better show the ideas involved. Other historical topics for student projects include the abacus, Galileo's sector compass, Napier's bones, and the Galton board of statistics.

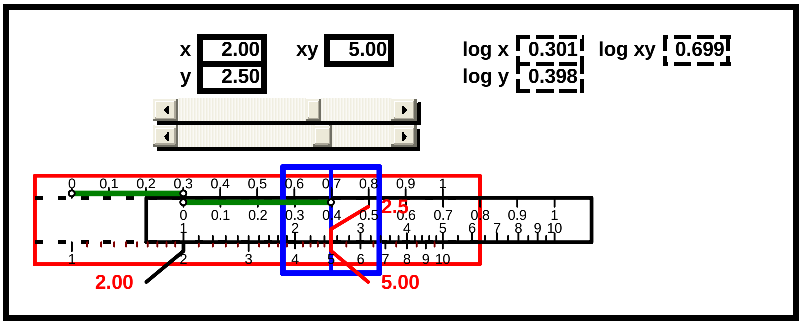

Figure 9.1. Animated Slide Rule Display

10. Creative Fun

Finally, students are generally quite adept at finding interesting and fun things to use in illustrating mathematical concepts. We show two examples in Figures 10.1-10.2. One of these is a carnival ride, called a Ferris wheel in the U.S. We create it using our earlier graphic techniques. We then use a rotation matrix to rotate the wheel through *k* degrees, and build a macro containing a loop to iterate the process many times, causing the wheel to rotate. The larger the value of *k*, the faster the wheel moves. Using a negative value for *k* reverses the direction of rotation.

Another interesting project is creating a traditional analog clock by using mathematics to insure that the second, minute, and hour hands advance appropriately. We then link the time to the computer's clock. Students can also draw a building in which to display the clock, or modify the design for a "backward" moving clock, such as the historical Josefov clock in Prague.

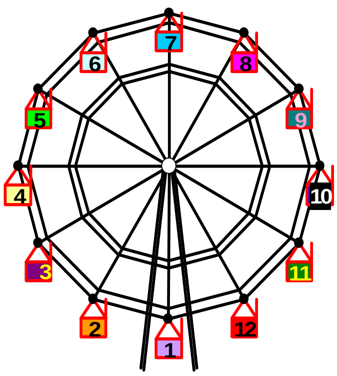

Figure 10.1. Animated Recreations

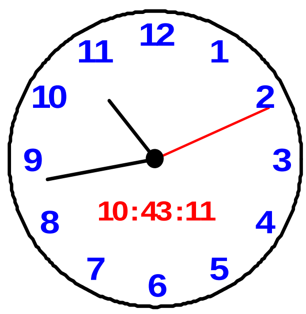

Figure 10.2. Animated Clock

Another creative application comes from psychology, where optical illusions help in the analysis of visual perception. Students create designs using a spreadsheet, and manipulate then through sliders, spinners, and buttons. In Figure 10.3, the left and right red segments appear not to be in line. However, when we click a button to expose connecting line segment in Figure 10.4, we see that, in fact, they are. In Figure 10.5, we use a scroll bar to move the center arrow to try to divide the line into two equal parts. We seldom will be correct. Once again, we can see the correct location after pushing a button to display the correct location. In many illusions, color plays a vital role, and the spreadsheet provides us with a good tool for this. Consult [5] to see a vast array of examples of optical illusions to implement.

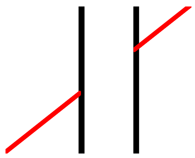

Figure 10.3. Illusion Ia

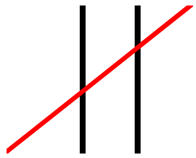

Figure 10.4. Illusion Ib

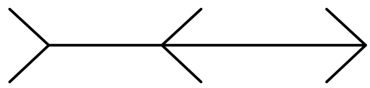

Figure 10.5. Optical Illusion II

Note: While we have created our images using *Excel 2003*, the models created in this version run on well on *Excel 2007*, which contains almost all of the features of the earlier version. However, some of these may be more difficult to find (sliders are on the Developer ribbon under Insert; graphing is on the Insert ribbon), and unfortunately the drag-and-drop technique for graphing has been eliminated, although we can accomplish the same thing with a copy and paste special option. F

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Interactive Geometry and Critical Points

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Abstract: Interactive geometry programs make it possible to treat a broad subject like critical points of functions in a unified way, across a wide range of courses at different levels. In this article, we show how color-coding of graphs of families of functions of one and more variables makes it possible to convey geometric information that is usually missing from introductory presentations of this material.

Introduction

How does an interactive geometry program make it possible to treat a broad subject like critical points of functions in a unified way, across a wide range of courses at different levels? In this article, we show how color-coding of graphs of families of functions of one and more variables makes it possible to convey geometric information that is usually missing from introductory presentations of this material. We introduce progressive elaborations of the fundamental ideas of critical point theory to motivate and illustrate more and more complicated geometric phenomena.

1. Graphs of Functions of a Single Variable.

By the simple device of coloring red all segments of positive slope in a polygonal approximation of a function, we can immediately see where the function has local maxima and minima by observing the color changes that occur as we go from the left-hand endpoint of an interval to the right-hand endpoint. Specifically if the end segments have the same color, the number of critical points in the interval is even and if the end segments have opposite colors, the number of critical points in the interval is odd.

For the graph of a differentiable function of one variable f(x), this means that the number of points c where f'(c) = 0 in the interval [a,b] is even if f'(a) and f'(b) have the same sign and this number is odd if the signs are different.

Since the normal line is perpendicular to the tangent line to the function graph, we color the graph red near any point with upward unit normal having angle lying in the quarter-circle from $\pi/2$ to π.

How does interactive graphics change the way we deal with this kind of presentation? With interactive graphics, we can carry out an online investigation of a family of functions depending on one or more parameters.

As examples of such families, we can see what happens as we change one coefficient of a polynomial. For linear functions, f(x) = mx + b, changing m will just rotate the line about its y-intercept, and changing b will alter the y-intercept while preserving the slope. For quadratic functions $f(x) = ax^2 + mx + b$, transforming the constant term moves the graph vertically, and changing the coefficient of the squared term from positive to negative will change the graph from a parabola that opens upward to a parabola that opens downward. The effect of changing the coefficient of the linear term is more complicated, and it is a good exercise for students to experiment and to record their observations. When we keep a and b fixed and change m, the basic shape of the curve will not be altered, but the position of the critical point will change.

2. The Family of Cubic Equations

What about a cubic equation $f(x) = x^3 + ax^2 + mx + b$? Once again, changing the constant term b only alters the y-intercept. In this case, changing the coefficient a will not alter the shape of the graph, but changing the slope m makes a big difference. Students can investigate the functions $f(x) = x^3 + mx$ and observe the numbers of critical points of the graph. It is immediate to conjecture that when m is positive, there are no critical points for the function and when m is negative, the number of critical points is two. This conjecture can then be proven by showing that the derivative $f'(x) = 3x^2 + m$ will never be zero if m is positive and that there are two values of x for which $f'(x) = 0$ when m is negative. The intermediate case, with $m = 0$, has one "degenerate" critical point at the origin, where $f'(0) = 0$ so the tangent line is horizontal.

We can color the points on the graph by a darker color when the slope of the tangent line is negative, so we have a visual record of the places where the slope changes from positive to negative (the local maxima of the curve) and where it changes from negative to positive (at the local minima).

Note that $f''(x) = 6x$ no matter what m is, so the concavity of the graph changes from convex downward to convex upward as x goes from negative values to positive values in the domain. By making the graph thick when the second derivative is positive and thin where it is negative, we can observe the inflections points of the graph, where the second derivative changes sign and the concavity of the graph changes.

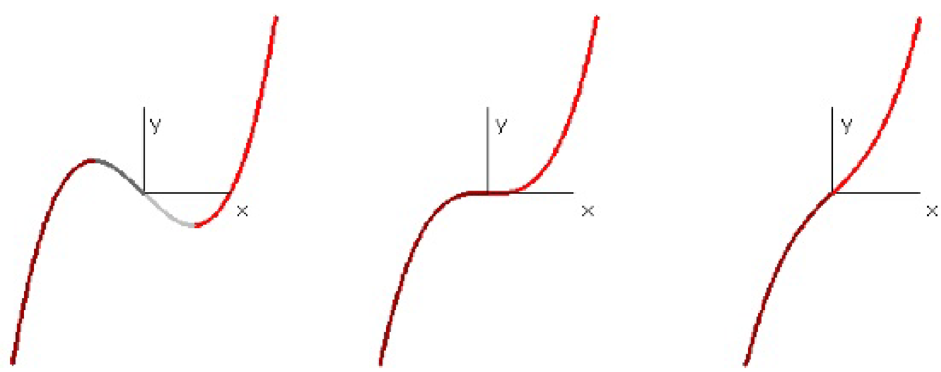


Figure 1: The family of cubic equations

3. The Family of Quartic Equations

More interesting is the case of a quartic, or fourth degree polynomial $f(x) = x^4 + cx^3 + ax^2 + mx + b$. As in the case of the quadratic and the cubic, changing b only shifts the y-intercept up and down without changing the shape of the curve. Also, changing the coefficient of the second term only moves the graph of the function from side to side without changing the shape of the graph, and in particular the number of critical points of the function. We are left with two coefficients to consider, $f(x) = x^4 + ax^2 + mx$, and changing the coefficients a and m can result in very different kinds of graphs.

Once again, we can invite students to explore the "two-dimensional control space" by choosing various values of a and m and recording the shapes of the graphs that result. Sometimes the number of critical points is 1, for example when m = 0 and a is positive, and other times there are 3 critical points, for example when m = 0 and a is negative. What happens if m is not zero? When will we get 3 critical points and when will we get 1? Will we ever get exactly 2 critical points? If m = 0, then we observe that for any negative number a, the graph will have three critical points, two minima and one local maxima. We can show this algebraically by computing $f'(x) = 4x^3 + 2ax = 2x(2x^2+a)$, which will equal zero for at x = 0 and at two other values of x if a is negative, and only at x = 0 if a is positive.

If we choose a = -1 so that $f(x) = x^4 - x^2 + mx$, then for some values of m near zero, the number of critical points is still 3, but for some value of m, the number of critical points changes to 1. We can observe that this change occurs at the value of m for which the function has a horizontal inflection point, where $f'(x) = 0$ and $f''(x) = 0$. Thus we must have both $f'(x) = 4x^3 - 2x + m = 0$ and $f''(x) = 12x^2 - 2 = 0$. From the second equation, x is plus or minus the square root of 1/6, so $m = \pm(4/3)\sqrt{(1/6)}$. For general a, we obtain the equations $f'(x) = 4x^3 - 2ax + m = 0$ and $f''(x) = 12x^2 + 2a = 0$ so $m = \sqrt{(-1/6)}$ and $27m^2 = -8a^3$. This curve in the "control space" of all possible choices of m and a will separate the functions with 3 critical points from those with 1 critical point.

We can use the interactive graphics system to construct a collection of function graphs that exhibit the different configurations of critical points for different choices of a and m,

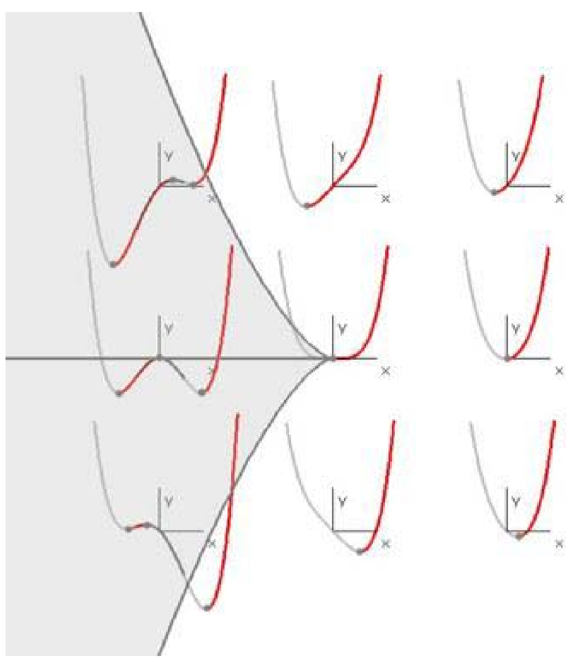


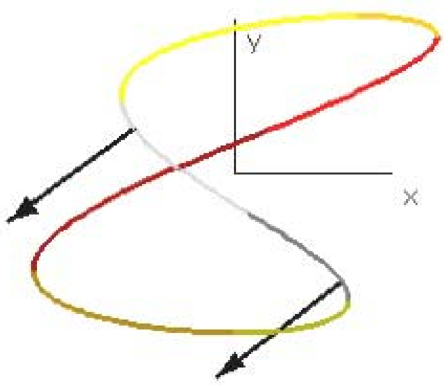
Figure 2: the family of quartic equations

Mathematicians familiar with catastrophe theory will recognize that these two families of functions, the cubic curves and the quartic curves, are the most important starting examples in that powerful theory. Originally arising in the sciences of optics and structural design, catastrophe theory has significant applications not only in physics and engineering, but also in biological and social sciences as well as in differential geometry and other parts of mathematics. It is noteworthy that interactive computer graphics makes it possible to introduce this subject to students just beginning the study of calculus and analytic geometry for functions of one variable.

4. Critical Points for Parametric Curves

There are two natural ways to extent this construction. Instead of considering only the graphs of functions, we may consider parametric curves in the plane, and then we can consider graphs of functions of two variables, in three-dimensional space, and eventually parametric surfaces.

For a closed curve in the plane given in parametric form by (x(t),y(t)) with t going from a to b, we color each segment according to the position of its normal vector: red for the second quadrant, orange for the third and yellow for the fourth. We can then read off the critical points of the horizontal coordinate function x(t) by finding points where the color changes from yellow to white or from red to orange, or conversely. The critical points of other coordinate function y(t) on the other hand occur where the color changes from white to red or from orange to yellow or conversely. The number of color changes for each of the two coordinate functions will be even and the total number will be even. This is true whether or not the curve intersects itself.



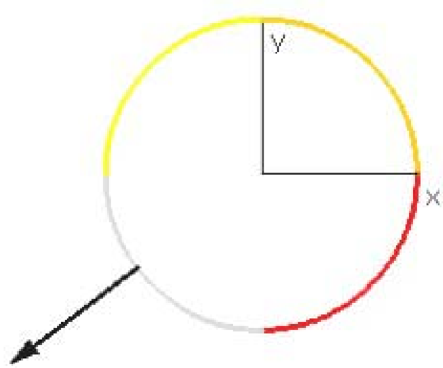


Figure 3: Color-labeled parametric curves in the plane

Using an interactive graphics program, we can display a parametric curve in the plane and compare the segments of the graph to the segments of a counterclockwise circle. If the curve is given by a pair of differentiable functions x(t) and y(t), then the velocity vector at each point has coordinates (x'(t),y'(t)) and rotating this vector by one-quarter turn in the counterclockwise direction gives the normal vector (-y'(t),x'(t)). The color of the curve is given by the color of the point on the circle where the outward normal vector points in the same direction as (-y'(t),x'(t)).

As the point (x(t),y(t)) moves along the curve, the corresponding unit normal vector on the unit circle is either moving counterclockwise or clockwise, and we can indicate this by making the curve thin in the first case and thick in the second case. The points of the curve where the direction changes are the inflection points of the curve. Students can investigate various curves, and observe the number of points on the curve that correspond to any fixed point on the circle. How does the number of corresponding points change as we move the position on the circle? What happens as we pass a point on the circle that corresponds to an inflection point on the curve? The investigation of curves in the plane leads to the theory of tangential degree of a curve, an important concept in the differential geometry of curves in the plane.

5. Families of Parametric Curves

Just as we considered one- and two-parameter families of function graphs, we can explore families of closed parametric plane curves such as the cardioid family ((-(u+cos(t))(-cos(t)),(u+cos(t))sin(t)). Students can discover that the shapes of these curves depend on u and in particular, that the shape changes dramatically near u = 1 and -1. As we watch an animation of this family as u runs from u = .8 to u = 1.2, we can observe that the curve has a loop for u less than 1, then a cusp at u = 1 and then a pair of inflection points. Students can also record what happens near other significant point, when u = -2, -1, 0, 1, and 2. Similar behavior will occur for other families of curves in the plane, a subject of importance in differential geometry and in the topology of plane curves.

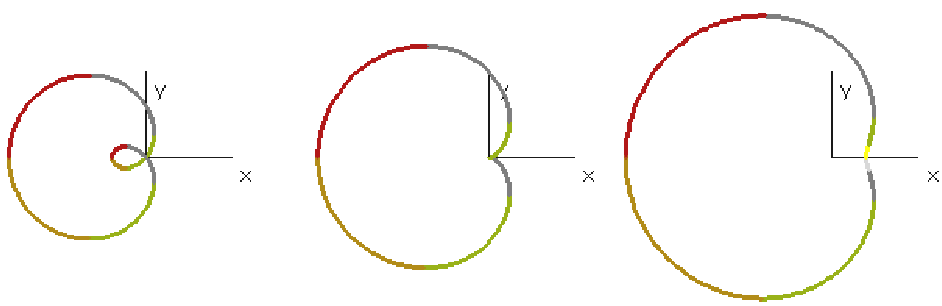


Figure 4: The cardioid family, color-coded

6. Critical Points and Functions of Two Variables

We can generalize the study of curves that are graphs of functions of one variable in the plane to the study of surfaces that are the graphs of functions of two variables in three-space. As in the case of functions of one variable, we want to find the range of a function, namely the smallest cylindrical box over the domain that will contain all of the function values for points in the domain. The top and bottom planes of such a cylinder will be horizontal tangent planes to the function graph in the case where the function is differentiable, or it might be that the highest or lowest point occurs at a boundary point of the domain of definition, and in that case the tangent line to the boundary curve will be horizontal at the local or global maxima or minima.

In addition to the global and local maxima and minima of functions of two variables, there are other points where the tangent plane is horizontal, for example the origin in the saddle-shaped graph of $f(x,y) = x^2 - y^2$, defined either over a square domain or a circular disc domain centered at the origin. This is a typical "saddle point" for the graph of the function. Finding the critical points of a function includes finding all local maxima, local minima, and saddle points. An important theorem of Marston Morse states that, for almost all functions, the only critical points of a function of two variables are of this form.

Two particular examples give a good idea of the critical point behavior for functions of two variables. Both are named for geographical features: Twin Peaks and Crater Lake. Both of these surfaces have been described in the author's Scientific American Library volume "Beyond the Third Dimension" [1].

7. Twin Peaks the Geometry of Peaks and Passes

Twin Peaks is the graph of the function $f(x,y) = -x^4 + 2x^2 - y^2$ over the domain $-1.5 \leq x \leq 1.5$, $-1 < y < 1$. It is straightforward to show algebraically that $f(x,y) \leq 1$ and that equality occurs at two "peaks" (1,0) and (-1,0). Between the two peaks there is a saddle point at (0,0), and there are no other critical points for the function. The level set for $z = 1$ consists of the two points (1,0) and (-1,0). For $0 < z < 1$, the level set consists of two closed curves. The level set for $z = 0$ is a self-intersecting "figure eight" curve, and for $z < 0$, the level set the portion of a single closed curve inside the rectangular domain.

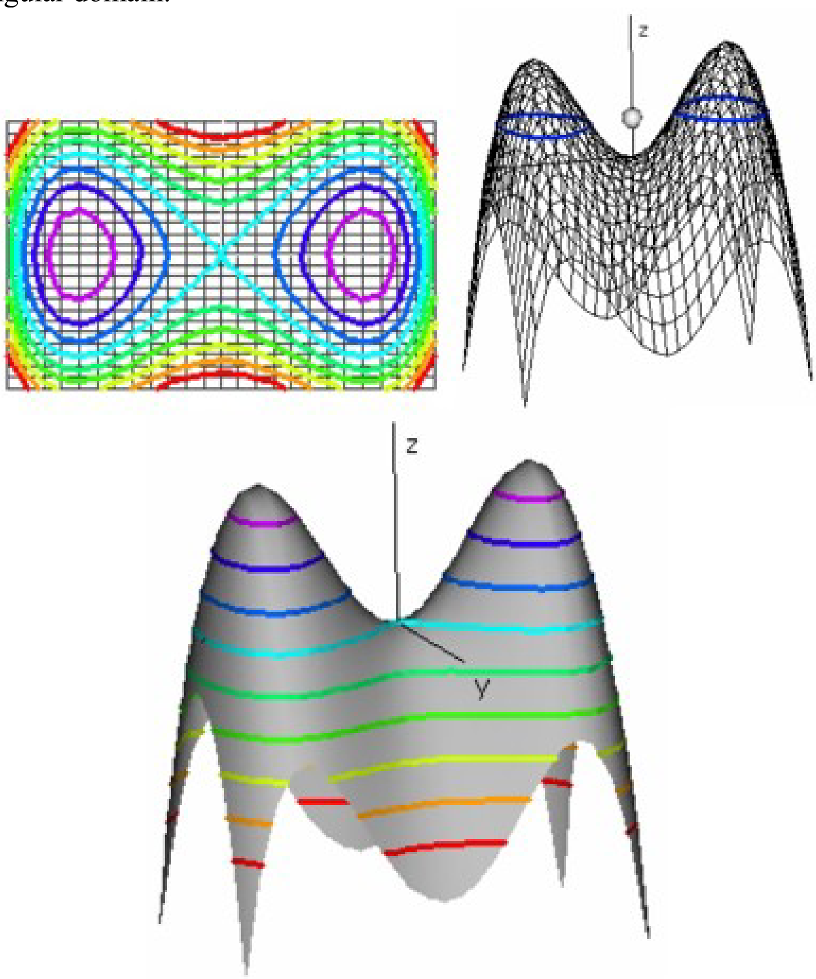


Figure 5: Graph and slices of Twin Peaks

This kind of "level set analysis" is crucial in an interactive graphics approach to graphs of functions of one or more variables. Students can investigate function graphs and report their observations on worksheets or by online messages. They can make conjectures about the reasons various configurations arise in graphs of certain kinds of equations, and then go on to find algebraic reasons for these observed phenomena.

For Twin Peaks, we can observe that the fact that there are two different global maxima is a sort of accident that can be removed by a slight perturbation. Just as adding a small linear term to the function $f(x) = -x + 2x^2$ to get $f(x) = -x^4 + 2x^2 + mx$ changed the heights of the global maxima, the same thing happens here. If there is an "earthquake" that shears the configuration by mx, we have the graph of $f(x,y) = -x^4 + 2x^2 + mx - y^2$, with two peaks at different heights, at least if m is small enough in absolute value. For an extreme earthquake, the number of peaks goes from 2 to 1 and the saddle point disappears. Students can determine by observation where that crucial changeover occurs.

8. Crater Lake and the Geometry of Pits and Passes

Crater Lake exhibits a different kind of unstable situation. The graph of $g(x,y) = -(x^2 + y^2)^2 + 2(x^2 + y^2)$ is a surface of revolution about the z-axis with profile curve $–x^4 + 2x^2$. The maximum value of this function is 1, and there are infinitely many global maxima, lying over the circle $x^2 + y^2 = 1$ in the domain. There is one local minimum at the origin. Once again an earthquake shear will perturb this situation to give the graph of $g(x,y) = -(x^2 + y^2)^2 + 2(x^2 + y^2) + mx$, which will still have one local minimum, but the circle of maxima will break up into one global maximum and one saddle point. The level set at the local minimum will be an isolated point surrounded by a single level curve. The level set at the global maximum will be a single point, and the level set at the saddle point will be a "double loop with a single crossing point". Near the saddle point, the level set looks like a pair of intersecting lines, as in the case of the figure-eight curve for Twin Peaks. Between the level of the maximum and the saddle, the level set is a single curve. Between the saddle and the local minimum, the level set consists of two curves, one inside the other (as opposed to next to each other as in the case of the figure-eight). Below the local minimum, the level set is again the part of a single curve lying in the domain.

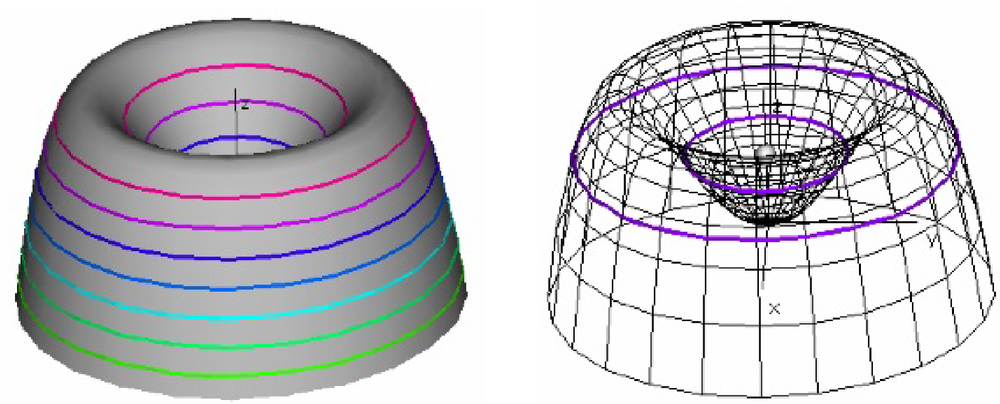


Figure 6: Graph and slices of Crater Lake

This description of the critical points configuration assumes that the earthquake has not been too severe. If the number m becomes large enough then we obtain a function graph with one global maximum and no other critical points. The local minimum and the saddle point have come together and disappeared. The water in Crater Lake has spilled out. Students can find this point by observation (and compare the value with the m for the crucial earthquake in the case of Twin Peaks).

9. The Critical Point Theorem for Graphs of Functions of Two Variables

Both of these topographical examples suggest that it will be possible to obtain other configurations of critical points, say with n+1 local maxima and n saddle points in between, then m local minima at the bottom of craters, each with one saddle point. This will produce a function with n+1 maxima, n + m saddles and m minima, so in particular the number of maxima plus the number of minima is one greater than the number of saddles, and the total number of critical points will be an odd number.

We can collect these observations in a conjecture: For an "island" with one level shoreline curve and everything on the island above sea level, there will be at least one global maximum, and if all critical points are local maxima, local minima, and ordinary saddles, then the number of critical points is odd and moreover #maxima - #saddles + #minima = 1. This conjecture epitomizes fundamental results in Critical Point Theory, of immense importance in global geometry and analysis over the last 80 years. It is also of fundamental importance for the geometry of surfaces, leading to a modern proof of the crucial Gauss-Bonnet Theorem as well as properties of knotted "strings" in contemporary molecular biology and theoretical physics.

The mathematician who popularized critical point theory was Marston Morse, who developed the subject eighty years ago. He enjoyed giving popular lectures on the topic for students of all levels, and two of his previously unpublished presentations have just been printed in the November 2007 issue of The American Mathematical Monthly [2]. On a visit to Providence RI in the 1970's, he came to the computer graphics laboratory at Brown University where my computer scientist colleague Charles Strauss and I were developing a program for analyzing the geometry of surfaces in three and four dimensions. Although the graphics programs were primitive by today's standards, and very slow, Marston Morse immediately appreciated the potential of such approaches for teaching, research, and public exposition of geometric ideas arising in critical point theory.

10. Color-Coding Graphs of Functions and Partial Derivatives

How does interactive computer graphics help to illustrate the geometry of critical points for graphs of functions of two variables, or families of such functions? For a differentiable function there is a well-defined tangent plane at each point of the graph and an upward-pointing normal perpendicular to that plane. If f_x(x,y) and f_y(x,y) denote the first partial derivatives of f with respect to x and y respectively, then (-f_x(x,y),-f_y(x,y),1) will be a normal vector. We color the surface white if f_x(x,y) is positive and red if it is negative, and we color it white if f_y(x,y) is positive and blue if it is negative. We can use both colors additively to color the surface purple where it is both red and blue. Thus we color a point white, red, purple, or blue as the point (-f_x(x,y),-f_y(x,y)) lies in the first, second, third, or fourth quadrant in the plane.

A point will be a critical point if all four colors meet at the point. We can also express this condition by saying that the point is in the intersection of the locus f_x(x,y) = 0 and the locus f_y(x,y) = 0. Usually those conditions are expressed and dealt with only algebraically, but interactive computer graphics provides a direct visual display of the geometric properties of these partial derivative functions.

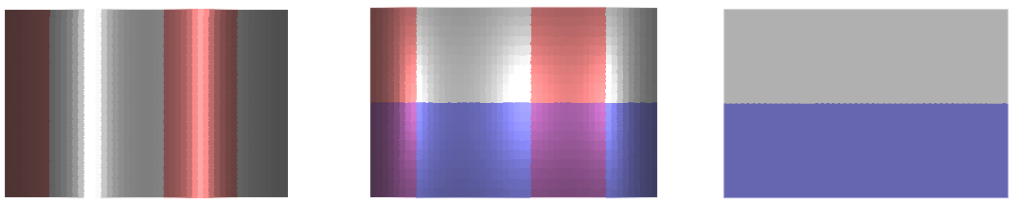


Figure 7: Perturbed Twin Peaks, color-coded domains

We obtain a bonus from this method of coloring because we can tell whether a critical point represents a maximum or minimum on one hand or a saddle point on the other. In the first case, the four regions at a point are red, purple, blue and white in counterclockwise order, while in the second case, the cyclic order is red, white, blue, and purple. The difference between these two orderings is the basis of the sign of the Gauss spherical image mapping defined on the surface.

This method of visualizing critical point configurations is especially helpful when we are exploring one-parameter families of functions, for example the perturbations of Twin Peaks.

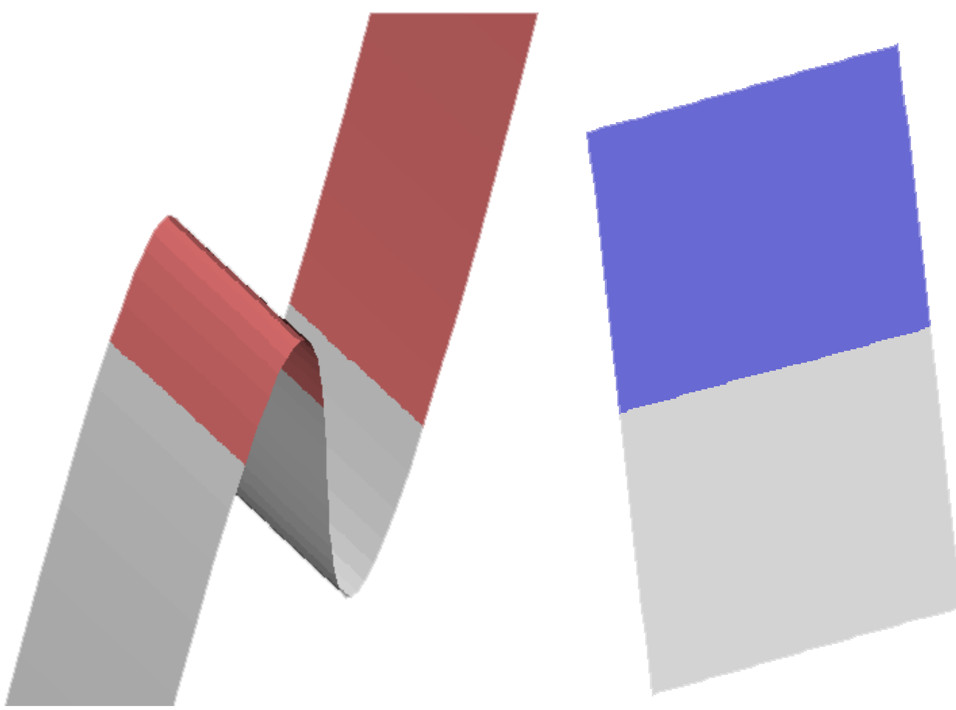


Figure 8: Color-coded partial derivative graphs for Twin Peaks

The picture on the left shows the graph of the first partial derivative of f(x,y) with respect to x, colored to indicate where the value of that function is positive. The colored region is separated from the uncolored region by curves indicating where the f_x(x,y) = 0. The picture on the right shows the corresponding graph for the first partial derivative with respect to y. The colored region indicates where f_y(x,y) is positive, and that region is bounded by the curve where f_(x,y) = 0.

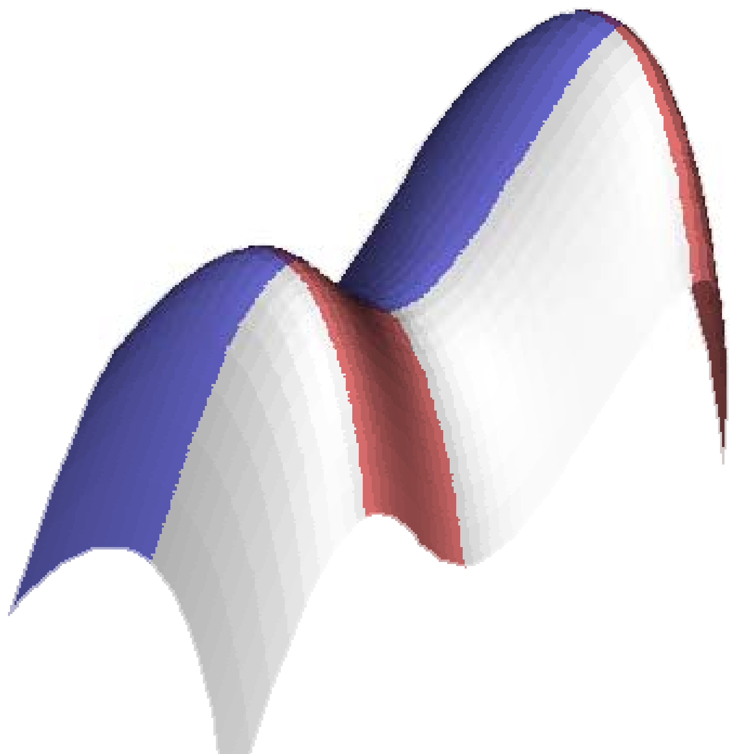


Figure 9: Color-coded graph of perturbed Twin Peaks

The graph of the perturbed Twin Peaks function colored according to the signs of the two first partial derivatives has four different colors near any critical point of the function, where the curves corresponding to f_x(x,y) = 0 and f_y(x,y) = 0 intersect.

11. Perturbations of the Graph of Crater Lake

We can carry out the same analysis for the Crater Lake function to show the graphs of the two first partial derivatives. In the domain, we have a pair of curves where f_x(x,y) = 0 and a pair of intersecting curves where f_y(x,y) = 0. The intersections of these two pairs of curves yield three critical points, one global maximum, one ordinary saddle, and one local minimum.

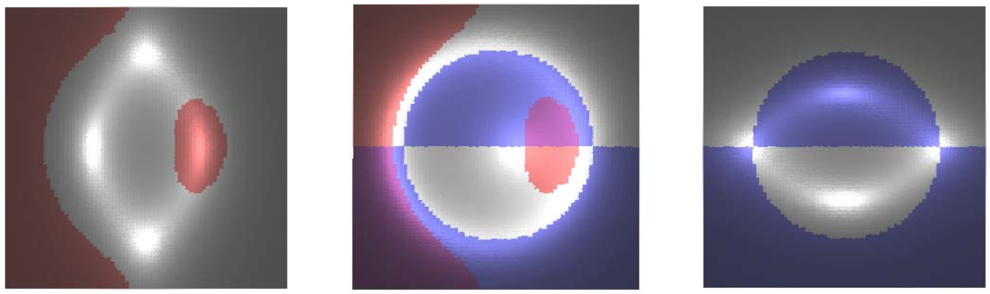


Figure 10: Perturbed Crater Lake, color-coded domain

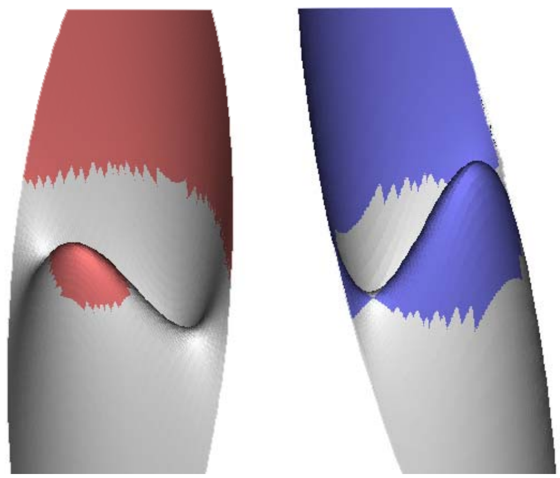


Figure 11: Color-coded partial derivative graphs for Crater Lake

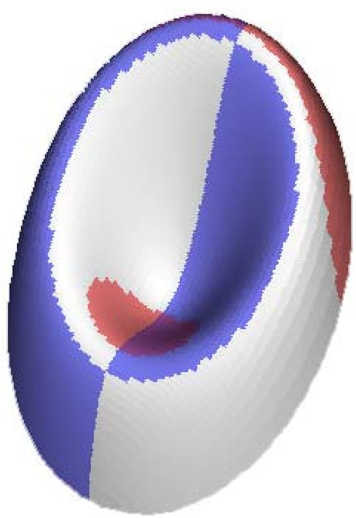


Figure 12: Color-coded graph of perturbed Crater Lake

The graph of perturbed Creater Lakes function colored according to the signs of the two first partial derivatives has four different colors near any critical point of the function, where the curves corresponding to f_x(x,y) = 0 and f_y(x,y) = 0 intersect.

11. Parametric Surfaces in Three-Dimensional Space

Just as we went from graphs of functions of one variable to closed parametric curves in the plane, we can go from graphs of functions of two variables to parametric surfaces in space. For such surfaces, the outer normal vector can point into the upper hemisphere where the coloring is white, red, purple, and blue, or into the lower hemisphere, for which directions the color is yellow, overlaid as appropriate with red to make orange, or blue to make green, or purple to make brown. The eight quadrants of the unit sphere are then color-coded in such a way that a small polygonal region on a surface receives the color of the octant within which its outward normal vector lies.

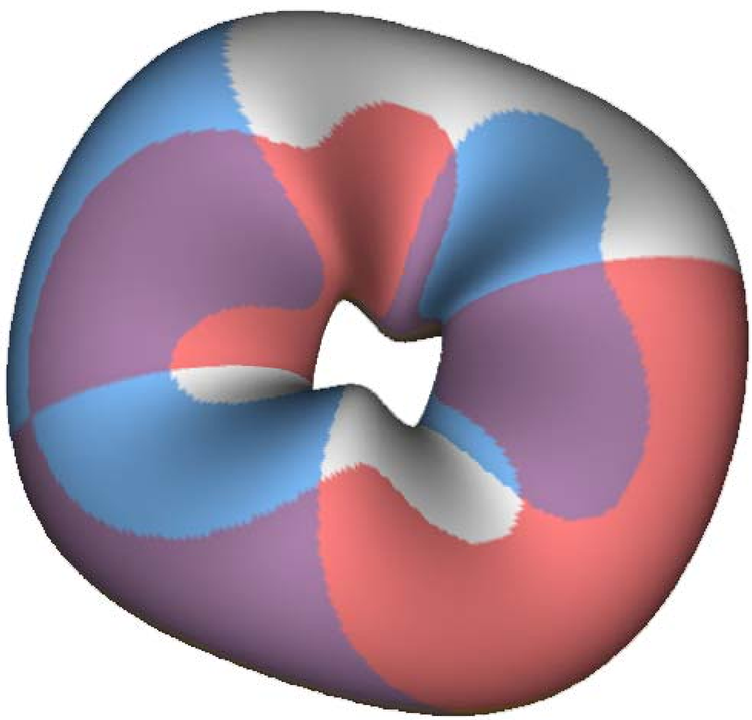


Figure 13: Color-coded parametric warped torus, top view

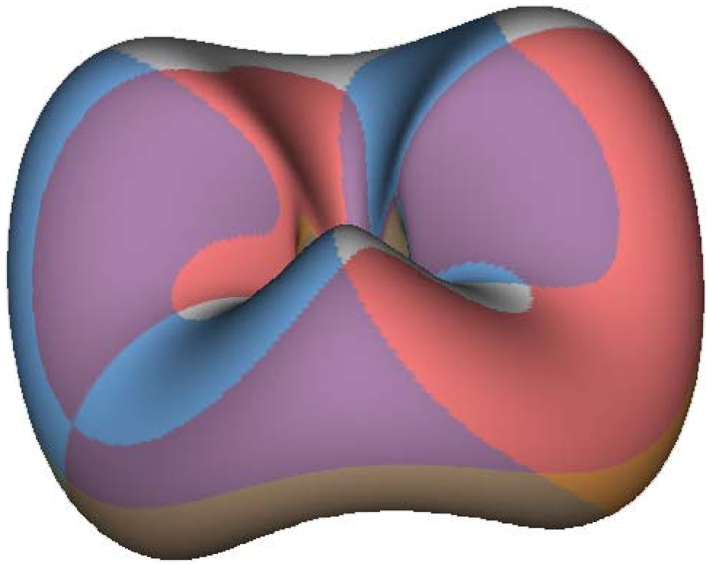


Figure 14: Color-coded parametric warped torus, side view

We can exploit this color visualization to give an interpretation of the Hopf degree theorem for embedded (non-self-intersecting) surfaces and a modern proof of the extrinsic form of the Gauss-Bonnet Theorem, the most important theorem in the differential geometry of curves and surfaces. This will be the subject of a subsequent paper on interactive differential geometry.

Conclusion

Interactive computer graphics makes it possible to investigate phenomena connected with critical points of functions in the plane and in three-dimensional space, starting with elementary calculus and proceeding to theorems in differential geometry and topology. These techniques have great potential for engaging students and general audiences as well as providing fruitful areas for research, in pedagogy as well as geometry and topology.

Acknowledgements

Special thanks are due to Michael Schwarz, who wrote the Java demonstrations used for the illustrations in this article. The software that produced the Java applets was created at Brown University by David Eigen.

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How to motivate teachers to want to use technology

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Abstract: This paper will argue that mathematics teachers need as much help with ICT issues out of the classroom as in, and will discuss the TSM (Technology for Secondary/College Mathematics) model, which always mixes courses on 'productivity' tools with sessions on dynamic software. Only with increased confidence in both preparing lessons and delivering them can teachers be motivated to make ICT work for them.

Introduction

TSM Workshops(1) have been taking place since 1994 with an increasingly international team of tutors, either at Oundle School, or on the premises of schools and colleges in the UK and abroad. The aim of the workshops is to help those mathematics teachers who want to use ICT in their work, but who generally:

1. have little idea how to create mathematical expressions in *Word*, even more so now that the interface for MS Office 2007 is so different.

2. have little idea how to use Word's *Drawing* facilities to create accurate and realistic mathematical diagrams

3. lack confidence in finding and incorporating effective web resources into their lesson plans using hyperlinks

4. are generally mystified by the mathematical possibilities of *Excel*, and are certainly worried when it appears to go wrong

5. lack confidence in planning lessons that incorporate dynamic software.

Experience from these workshops suggests that these skills should be part of the ICT toolkit for all mathematics teachers, both in their initial training and as part of their on-going in-service training. With this in mind, a 3-hour summary of these ideas has been presented successfully for the past few years at a number of PGCE Teacher Training courses in England.

There is of course a clear difference between presenting these ideas to teachers, however attentive they may be, and conducting a hands-on workshop. Even half-day or whole day courses can leave the teachers confused and overwhelmed, and asking for more time to practise. With this background, the 3-day residential TSM workshop was born, and it is now in its seventh sell-out year!

This paper will outline the contents of the workshops, and touch on some evidence which supports the need for the workshops to last for more than a day.

1. Mathematical expressions in Word

Making *Word* mathematically friendly is a popular strand in the TSM workshops. The sessions often starts with asking the teachers to put ‘π’ into a document. Many will want to use the *Symbol* font; some will want to use the *Equation Editor*. The TSM approach is to introduce teachers to the *Unicode* font extensions which can produce a wide variety of single-line expressions, and which are, importantly, also font-independent:

eg: $\mathbf{y = x\sqrt{(1 - x^2)}}$ $\mathbf{y = \pi \pm \sin^{-1}(\tfrac{1}{2}x)}$

$\mathbf{y = e^{\sin^2 x}}$ $\mathbf{\int \sin^2\theta d\theta = \int \tfrac{1}{2}(1 - \cos 2\theta)d\theta = \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta + c}$

$\mathbf{\chi^2 \sim N(\mu, \sigma^2)}$ $\mathbf{\int \sin^2\theta \cos\theta d\theta = \tfrac{1}{3}\sin^3\theta + c}$

Teachers are also shown how to make use of the on-screen keyboard that is supplied with Autograph[(2)], which allows users to enter a wide variety of mathematical symbols as text.

The very useful off-keyboard, but sub-Unicode, characters for ‘squared’, ‘cubed’ and ‘minus’, amongst others, allow quadratic and cubic equations to be entered trivially and without superscripts, not only in Word but in any other environment, including HTML:

eg: $y = x^3 - 2x^2 + 1$

and teachers are shown how to use the ALT-0xxx shortcuts on the keyboard keypad for characters:

eg: °, ±, ², ³, –, €, •, ½ and ÷.

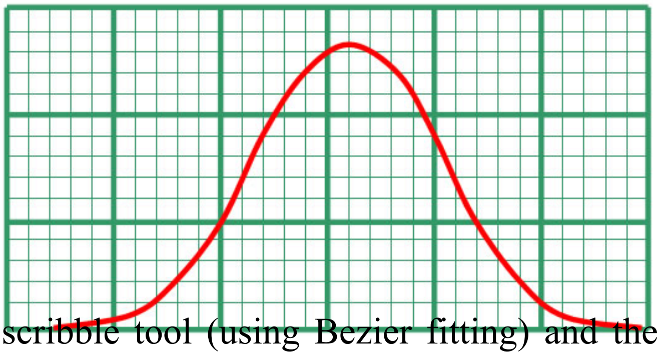
The *Equation Editor*, produces graphics, not text, and is best used when one-line expressions fail,

eg: $\boxed{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$ and $\boxed{s^2 = \dfrac{1}{n}\sum fx^2 - \overline{x}^2}$.

Teachers are often surprised how quickly they can master this, enabling them to enter professional looking expressions, including 'proper' fractions, eg $7\frac{2}{3}$.

All equation objects are (at first) placed as 'in-line with text' objects, and can therefore be given simple shortcuts in the "Autocorrect" feature in Word (eg 'var' to enter the variance formula above, or '2/3' to enter $\frac{2}{3}$).

2. Mathematical Diagrams in Word



The drawing facilities offer many possibilities for the creation of mathematically correct diagrams, including regular polygons, and shapes.

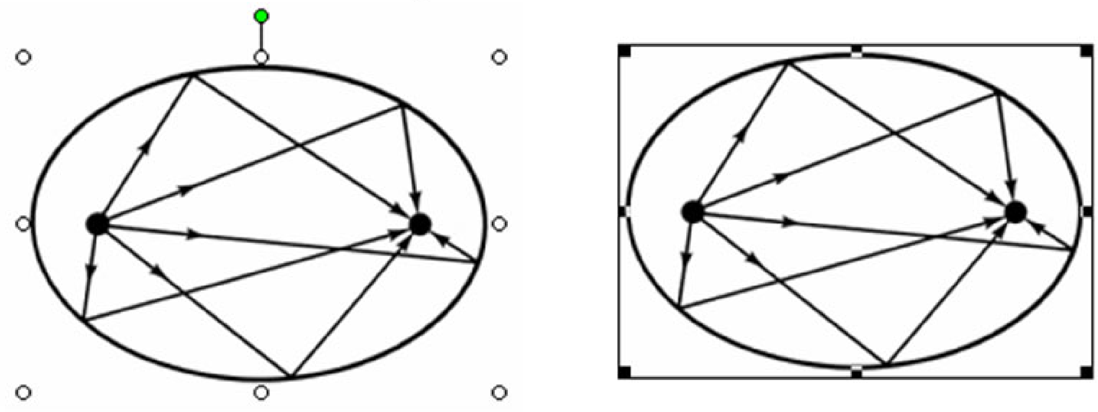
A few special tricks are popular, and are quickly picked up, including the judicious use of Shift and Ctrl. The scribble tool (using Bezier fitting) and the undocumented Ctrl-D which performs intelligent duplication (eg to create the graph paper) always give pleasure to most teachers.

Teachers who are discovering how to create expressions and diagrams this way are now able to generate high quality worksheets, and can be highly motivated by this.

3. Finding and incorporating effective web resources

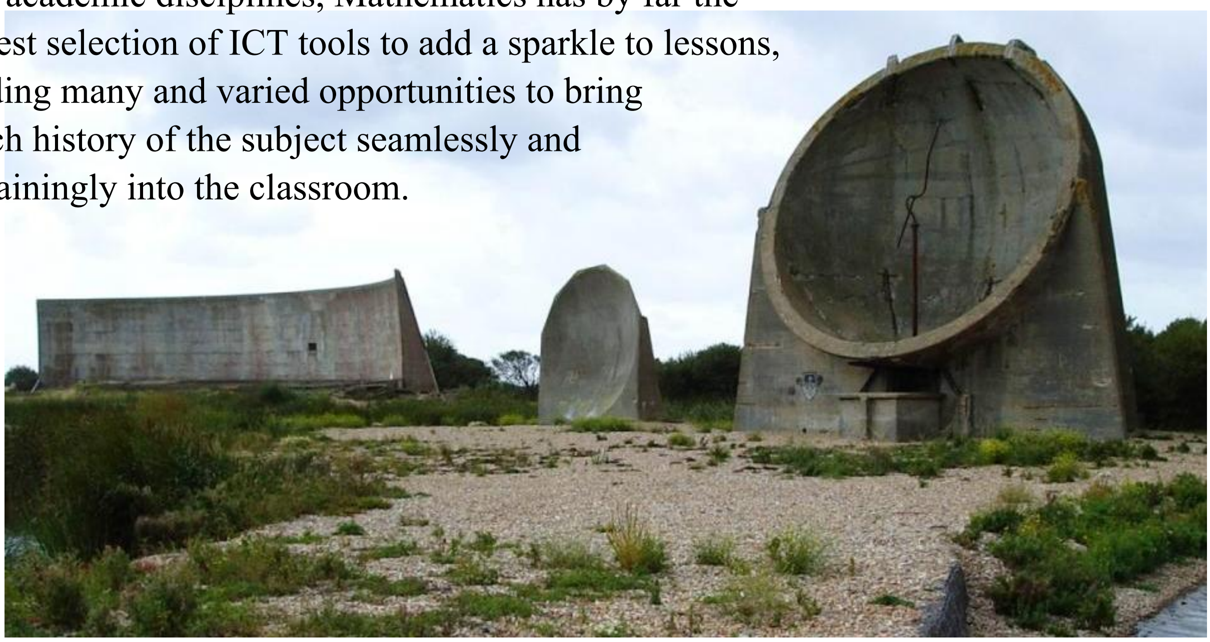
To accompany the TSM workshops, an extensive site of carefully researched web resources has been assembled at the TSM-Resources web site (3).

Firstly, teachers are shown how easy it is to copy and paste images from a web page into a resource they are creating in *Word*. Most mathematical documents in Word are likely to be a mix of text and graphics, and it is unfortunate that many teachers do not know how to control such a mix. The TSM courses emphasise the importance of how to manipulate images, using the "Picture" controls, together with the concept of "in-line with text" and other wrapping options.



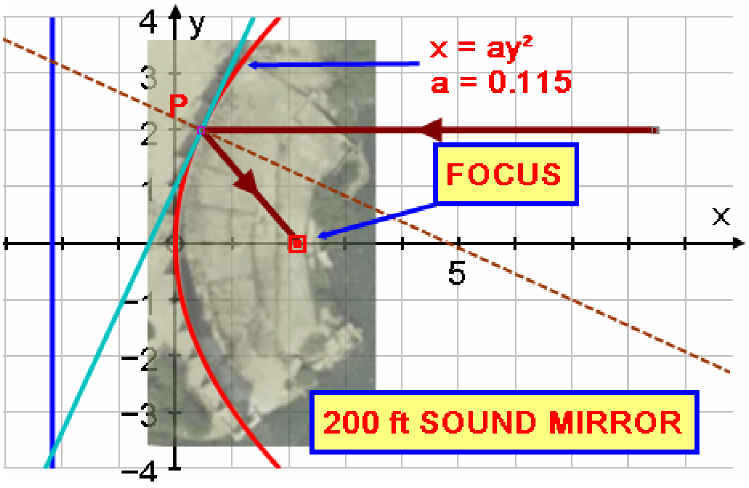
Two selected graphics in Word: 'floating', allowing wrapping (left) and 'in-line-with- text' (right). This, incidentally, is a good example of a web resource that can show how mathematics is applied to different areas: here the properties of the ellipse can be used in non-invasive surgery, using an elliptical mirror to focus the radiation.

Of all academic disciplines, Mathematics has by far the broadest selection of ICT tools to add a sparkle to lessons, including many and varied opportunities to bring the rich history of the subject seamlessly and entertainingly into the classroom.

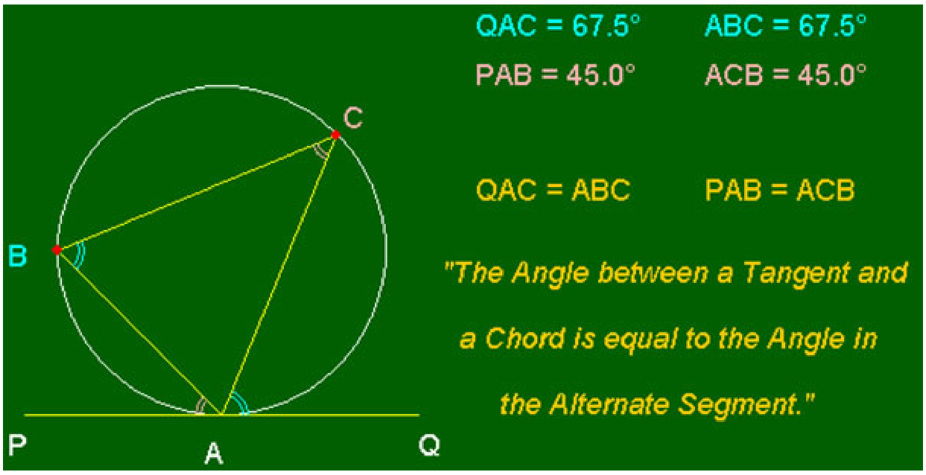


New tools such as Flash Earth(4) enable large, mathematically interesting objects to be studied easily.

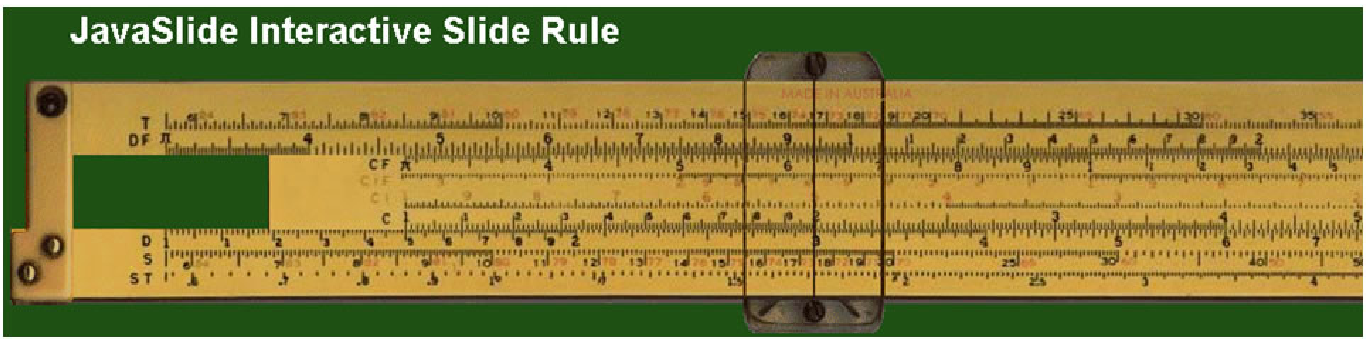
Illustrated here is one of the two remaining giant concrete mirrors (used as pre-radar aircraft detection), which can be shown to have a parabolic surface, and its focus found.



There are two main types of web site: 'passive' (like the example above, and the NRICH's amazing PLUS Mathematics magazine (5)) and 'active'. Active sites mainly use Java applets, offering teachers a safe taste of dynamic software which will always work, but which usually has a limited objective. 'YouTube' videos(6) are also a great source of material, and even a place where pupil's serious work can be lodged.



One of the suite of "Waldo" Java Applets from Ron Barrow (7)
Here the Alternate Segment Theorem can be animated by moving 'B' or 'C'

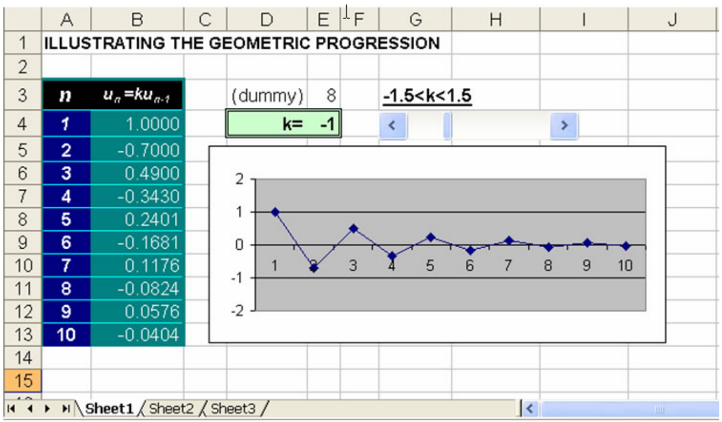


A lesson on logarithms can be brought to life with a Java-based slide rule (8).

4. The mathematical possibilities of Excel

TSM courses always include a section on spreadsheets, trying to help teachers to realise the potential of putting *Excel* to work for them. Initially some of the pitfalls are pointed out, conscious of the damage they can do to a teacher's confidence:

- *Excel* does not know about Degrees, forcing the use of the RADIANS key word.
- *Excel* only works to 15 decimal places, so avoid setting it higher (eg to display π!)
- Formula results can be truncated, eg in the next example, 'k' should be –0.7!



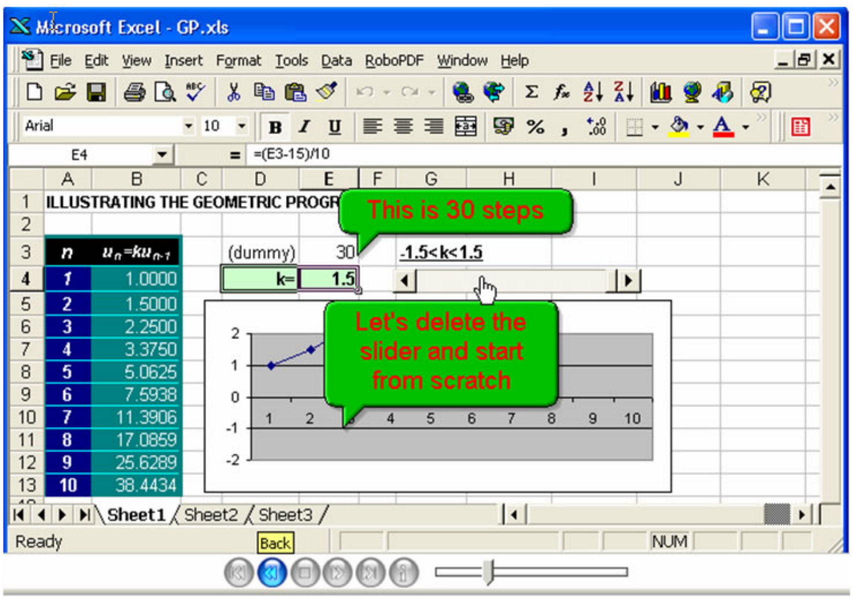
TSM Courses usually include *Excel* topics at different levels, but for the beginners the following are often included:

- creating series
- elementary statistics (including random sorting)
- how to set up a Slider Bar in Excel (as in the above illustration of a GP).

Invariably teachers will forget a detail from earlier in the session. With this in mind, a number of Flash Tutorials have been created (eg on "How to add a Slider Bar in *Excel*"), using the excellent discrete recording software *Turbo Demo*(10), and placed on the associated TSM page "Useful Files for Mathematics Teaching" (11).

In a Turbo Demo tutorial, the creator can place instructions and comments during the editing process.

Playback controls allow users to stop, start and re-play. The result is "authored once" for "learning anywhere".



5. Hardware considerations for ICT-driven teaching

TSM courses also discuss which technology works best, and the benefits of using ICT in a whole-class situation. For example teachers are invited to discuss the relative merits of:

- fixed (ceiling-mounted) data projectors: many now regard these as a tool of the trade, and they are becoming at the same time brighter and more affordable. Bulb replacements though remain a serious budget cost.
- interactive whiteboards (IBW): these can add many effective 'tactile' opportunities, both for pupils and teachers
- the emerging use of the Tablet PC laptop and also the Bluetooth Tablet slate: either of these could supplant the IWB as they solve the problem of the awkward shadow, the teacher being 'in the way' and also in some sense 'trapped' at the front.

With the Tablet or slate the ICT elements of a lesson can be run from anywhere in the room. However with the Tablet PC the essential wireless link to the projector is still rather slow, preventing videos and fast animations from working.

6. Procedural considerations for ICT-driven teaching

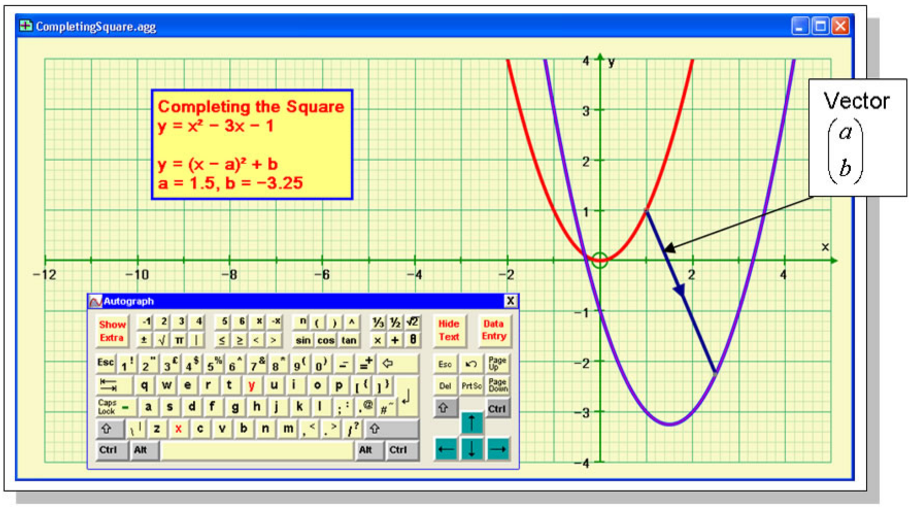
Teachers also need to consider how to conduct the lessons including ICT in a way that excites and motivates the students.

(a) There a need to ensure that the pupils are fully engaged and not in any sense just watching a 'show'
(b) Ideally for every step in which ICT is involved, the teacher needs to try to get the students to PREDICTED what is about to happen. This way they are involved at each stage, and become curious to see what the outcome will be.

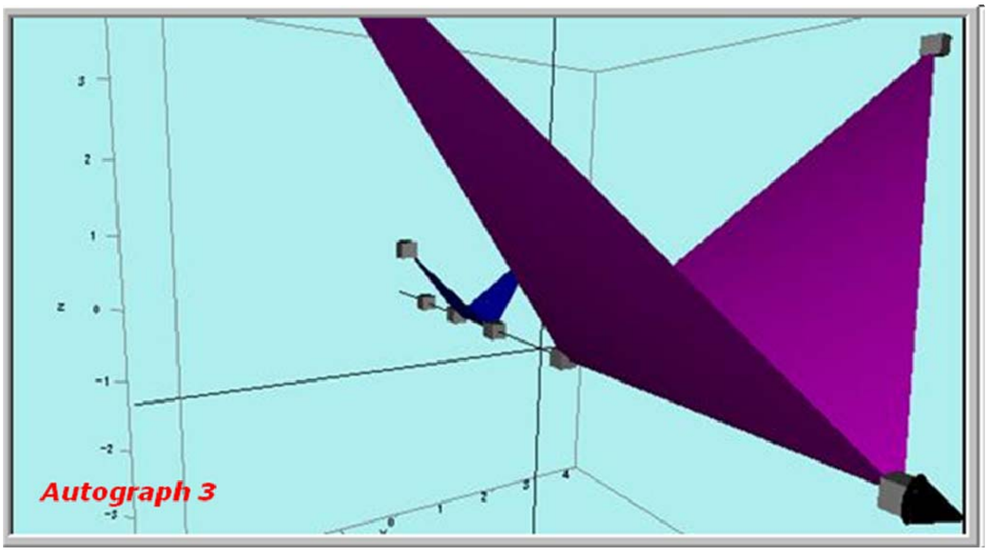
To help with planning lessons which incorporate dynamic images, TSM offers courses on *Sketchpad*, *Cabri* and *Autograph* to show how basic concepts can be covered more effectively. It is not easy to represent this dynamic approach in a static paper, so readers are invited to look at the Turbo Demo tutorials given on the TSM "Useful Files" page (11).

It is also important to ensure that a variety of approaches is adopted, and to be absolutely sure that what excites *you* as the teacher also excites the pupils.

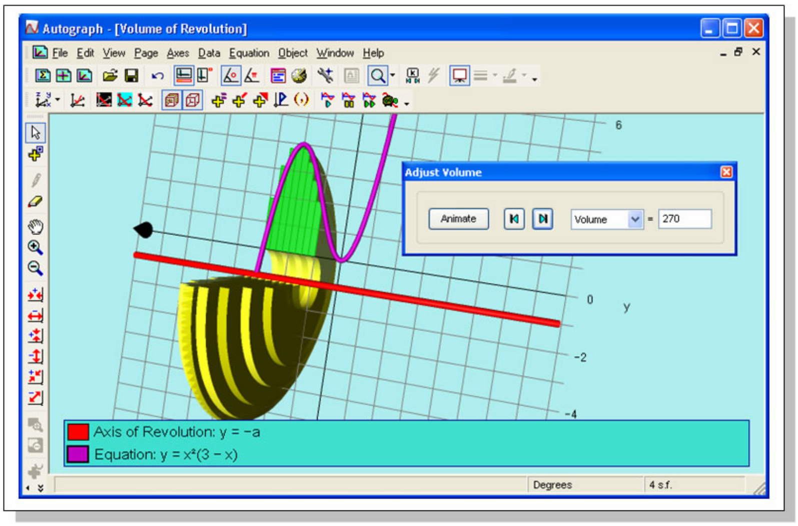
7. Some examples of lesson ideas using dynamic software



Here 'Completing the Square', which is usually exclusively an algebraic topic, is given a coordinate geometry treatment, showing dynamically that this example is equivalent to a translation of $y = x^2$ by the vector [a, b].



While studying enlargement in 2D, why not extend to 3D?
Here is a simplification of the image used at the start of most sci-fi movies where a space station moves towards, and past the camera.

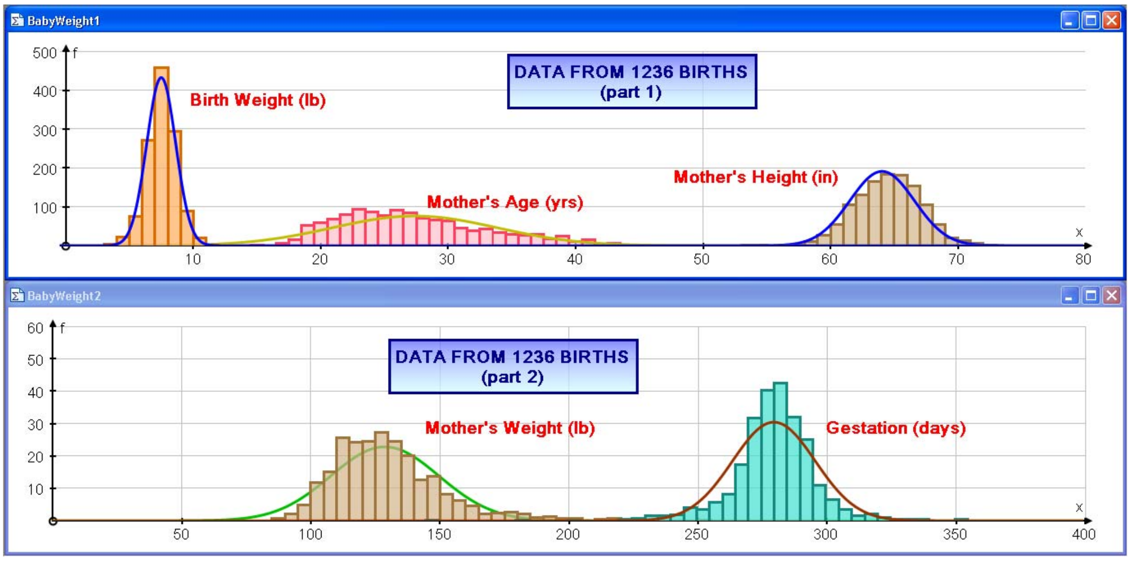


Many mathematics topics have been dropped from main-stream post 16 teaching because they were hard to visualise, and just at a time when technology is enabling students to make sense of them. 'Volume of Revolution' is a good example of this, along with most of the 3D coordinate geometry syllabus.

8. The visualisation of Probability and Statistics

Another area particularly suited to the dynamic approach is data handling. Several countries are in the process of enlarging the statistical content of their secondary curriculum, including South Africa and Australia (New South Wales). Many teachers did not learn this subject themselves at school, so are still learning the mathematics, as well as the ICT!

Here, teachers can be really motivated to use ICT: they can enjoy the way large amounts of data can be seamlessly analysed, and presented visually on screen.



Using a large data set of baby births from Stanford University [12], analysed by Autograph

9. Conclusion

The TSM training programme has evolved over that last 15 years to address the needs of the mathematics teacher who is intrigued by all the technology and wants to gain the confidence to put it into practice, both in and out of the classroom. Feedback has been very positive, particularly from those teachers using ICT-based methods in their teaching for the first time.

The evidence suggests that this success can only really be achieved by hands-on workshops that last more than one or two days. This of course raises important and far-reaching issues concerning the funding of regular INSET time for teachers.

References

[1] TSM Workshop:	www.tsm-resources.com/July08.html
[2] Autograph:	www.autograph-math.com
[3] TSM Resources	www.tsm-resources.com
[4] Flash Earth	www.flashearth.com
[5] YouTobe videos	www.youtube.com
[6] NRICH PLUS Magazine:	http://pass.maths.org.uk/
[7] Waldo's Java Applets:	www.waldomaths.com/
[8] Java Slide Rule:	www.taswegian.com/SRTP/javaslide/javaslide.html
[9] You Tube videos:	www.youtube.com
[10] Turbo Demo Tutorials:	www.turbodemo.com
[11] TSM Useful Files	www.tsm-resources.com/useful-files.html
[12] Baby data from Stanford University	www-stat.stanford.edu/~rag/stat141/exs/babies.dat www-stat.stanford.edu/~rag/stat141/06Lab4.pdf

Both Computer and Traditional Technology Are Inevitable for Mathematics Teaching:
Revisiting why we use technology

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Abstract: We use technology to improve mathematics education. The first part of this lecture, the logical inconsistencies for introducing technology come from the difference of society, curriculum and technology itself will be mentioned. Because inconsistencies logically existed, we should develop the judicious position of using technology. The judicious users will be teachers and students. The second part, I would illustrate the Japanese approach by focusing on teachers. The later half, I would addressed the importance of judicious using of technology if it is to be a necessity in enhancing students mathematical explorations and developing mental object in order to support their mathematization. The basic theories of mathematics education will be applied to explain this position. To illustrate the importance of both the traditional and computer technology, I would use the perspective drawing and mechanical motions as for the example.

1. Beyond the inconsistencies of using technology in classroom

Technology is a part of necessary tools for knowledge based society. It is true that technology has been pushing globalization and changing society. It is necessary to prepare children, and technology should be introduced into classroom in preparation of the changing society. On the other hands, we should be concerned about some of the logical inconsistencies based on the difference of society, aims of reform and technology itself.

First inconsistency is the mismatch between technology and society of academy and teachers. There is a simplistic motion of introducing technology into classroom because of the existence of some advanced countries in using certain special technology. Even if using technology itself is mathematically interesting, its usage in classroom is meaningful in some countries but not in some other countries. In 1990s, the reform movements of using of innovative technology had influenced the world. The reform of AP-Calculus in USA influenced the other countries. It spreads the use of graphing calculators to limited countries which shared the similar setting and target in their reforms but not with countries having different setting and target. In the case of East Asia, many students have good achievement in their mathematics curriculum without innovative technology. In these countries, teachers are reluctant to use technology even if they well recognize the significance of technology for instance the power of visualization and importance of exploration. In comparison to East Asia, there are many countries in which the mathematics teachers are not well prepared for teaching mathematics. It is not uncommon that mathematics educators who teach elementary mathematics to future prospective teachers do not have good experience in geometrical proof themselves. Judicious technology using (Lynda Ball, Kaye Stacy. 2005) is a general necessary expectation in the teaching content for the knowledge based society when we teach students both on how to use technology and thinking mathematically. On the other hands, if teachers use technology such as Dynamic Geometry Software in their classroom and does not well understand geometrical proof, their exploration with DGS will be limited hence the meaning of judicious using itself is not similar to other countries.

Second inconsistency is the mismatch between curriculum and technology. Introducing new technology into classroom sometimes means the change in the teaching content and aims of education. There is a simplistic motion to use technology as an alternative to paper and pencil approach. When students explore the free fall phenomena using the digitizing system of the motion of graphing calculator, the approximation by the four degree function is better than the quadrilateral function. Here, teachers are teaching students how to explore the phenomena with graphing calculator on its statistical meaning or its mathematical modeling but not teaching them to consider mathematical structure of the free fall phenomena. If we introduce technology as a necessary tool for learning, it changes the content of mathematics teaching itself. If calculus teacher want to use the free fall phenomena as a model of fundamental theorem of calculus, he hopes that pre-calculus teachers teach it as an example of the quadrilateral function. Many teachers who are teaching upper level of mathematics deny teachers who are teaching lower level mathematics, change the content with technology. Upper level mathematics usually uses lower level mathematics as the mental object for the base of constructing upper level mathematics. In some countries, the movement of introducing technology is ongoing with the regression of mathematics teaching content.

Third inconsistency is among innovations. Even if one learned how to use the innovative technology it gets out dated very fast. On the integrity and fair competition of technological innovation, new technology is tentatively new until new products come and governments have to spend too much for every revision.

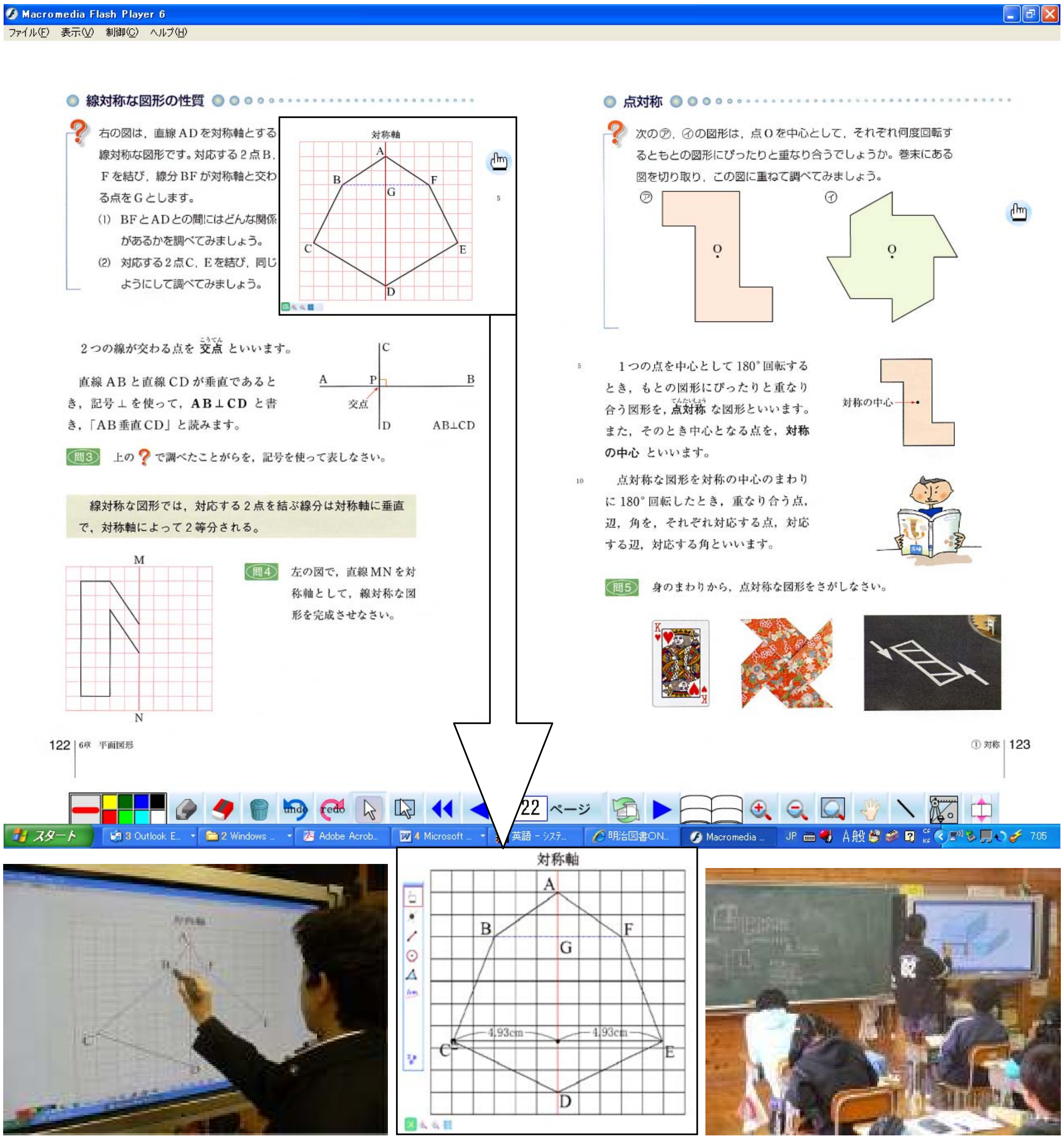


Figure 1. e-textbook and its interactive way of using a part of corkboard.

Against these inconsistencies, there are some approaches. In the case of Japan, for avoiding these inconsistencies, technology is being fixed by both teaching approaches and textbook based on curriculum standard. To introduce the latest technology into education in keeping tracks with the curriculum innovative ideas, e-textbooks with embedded mathematical software on traditional textbook have been developed in Japan. The feature of e-textbook is a part of teacher's guidebook and it almost means free because teachers usually buy teacher's guidebook even if they do not use it. Figure 1 is a sample page taken from the e-textbook which is the same as students textbook but could be projected in the classroom by flash player and demonstrate the way of using software such as DGS and graphing tool embedded in the e-textbook. On the screen in figure 2, parts of the textbook were displayed and used like an interactive chalkboard and integrating its use as a whole classroom activity.

The important feature of this Japanese approach is that it still keeps track with the same curriculum plan of the traditional textbook and Japanese way of teaching. Teachers do not need to change their ways of teaching (Figure 2). Students do not need to learn how to use technology even if teacher uses e-textbook. Teachers are expected to consider judicious technology using but not students because in this case the using of e-textbook is preferred to be used by teachers. This is a weakness but Japanese approach does not aimed in teaching how to use certain technology.

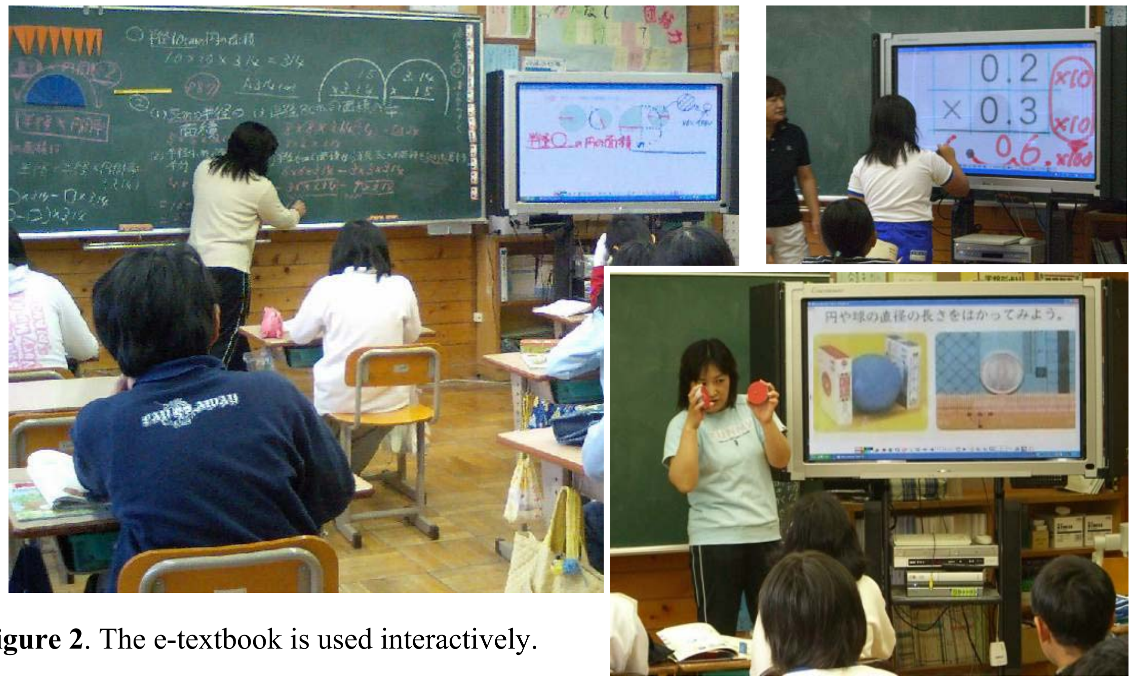


Figure 2. The e-textbook is used interactively.

2. Using Technology for Mathematical Exploration.

Besides those inconsistencies, there is a consistency for developing technology from the view point of mathematics. All technology in mathematics has been developed to enhance the using of mathematics and exploration of mathematics. These kinds of technology are useful and meaningful for students to be able to experience mathematical exploration.

People who use technology in mathematics know that technology is developed to explore mathematics and used for acquiring invariant in mathematics. For example, it is developed for enabling us to

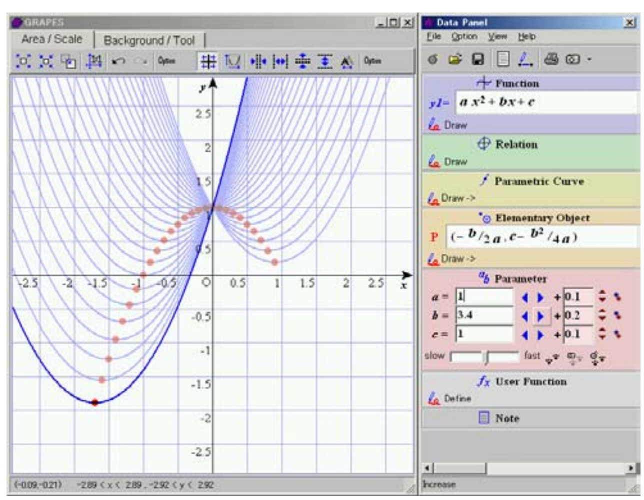


Figure 3. Visualized Invariant on GRAPES

visualize invisible object and increased our possibility of exploring mathematics. Figure 3 is a well known example. The approach in figure 3 is known by the visualization of the graphing tools. Changing parameters are an important approach for exploring invariant of the family of functions. Before graphing tools existed, only limited students can imagine those kinds of invariant. In traditional curriculum, most of students only learned quadratic function with the equation of $y = a(x-\alpha)^2 + \beta$ but not the meaning of parameters a, b and c on the equation of $y = ax^2 + bx + c$, even though they learned the meaning of a and b on $y = ax + b$.

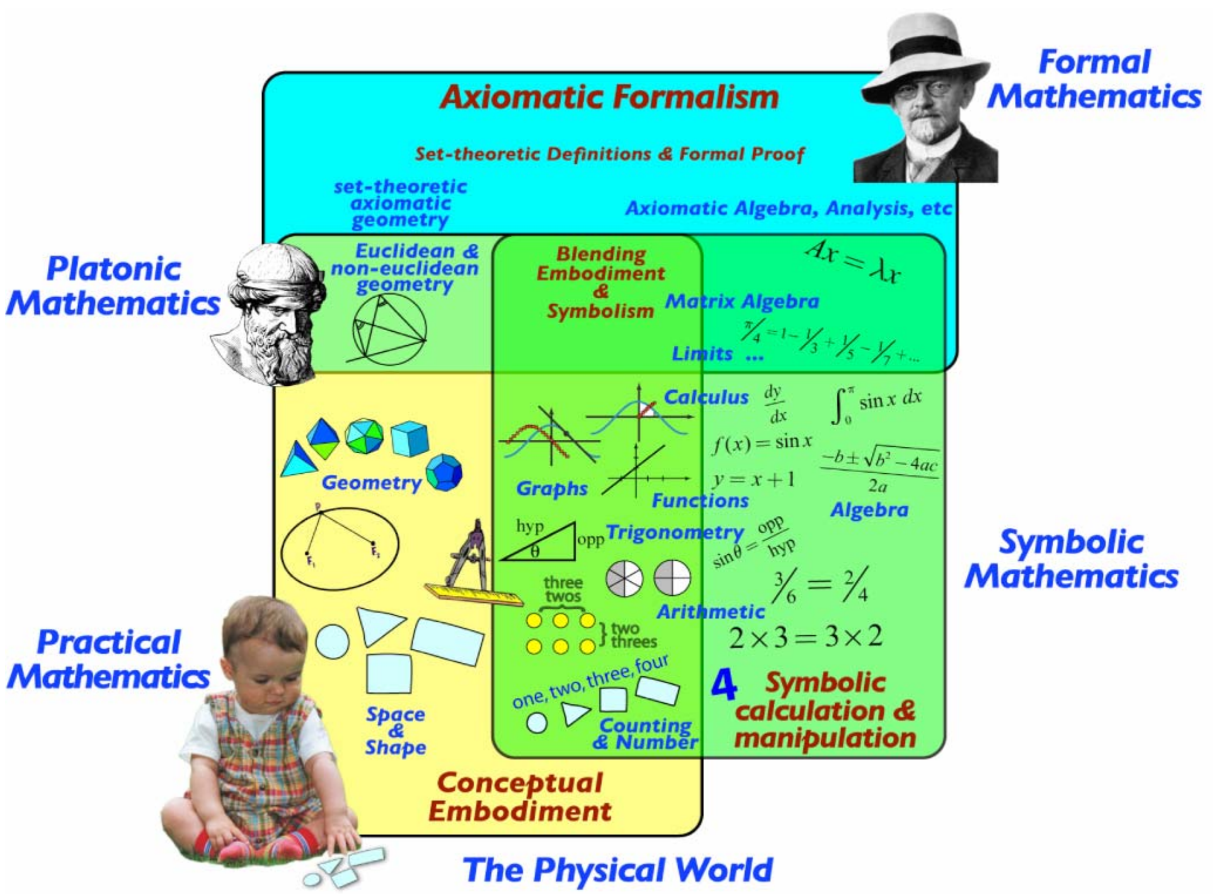


Figure 4. Conceptual Embodiment, Symbolic Calculation & Manipulation, and Axiomatic Formalism (David Tall)

Technology is developed to enable to understand mathematics in another way such as embodiment. Before the graphing tools, it is very difficult to imagine the graphs of $\{\, y = ax^2 + bx + c$; a=1, c=1, and b={…-3,-2,-1,0, 1,2,3,…}}. The graphing tools allow us to manipulate these parameters.

David Tall introduced the three world of mathematics in explaining the simple features of conceptual development in mathematics (Figure 4. see such as Tall & Isoda, to appear). There are many difficulties for conceptual development in mathematics hence we needs long learning processes in each area of mathematics. On the other hands, if we simplify the processes, there are

three major activities in mathematics: Conceptual Embodiment, Symbolic calculation & manipulation, and Axiomatic formalism. We can map technology into his diagram as follows.
CAS (Computer Algebra System): It enables us beyond hand calculation and the embodiment through visualization. There are some researches such as Michele Cerulli who developed CAS for Axiomatic Formation of Algebraic Calculation.
DGS (Dynamic Geometry Software): It enables us the manipulation of geometrical object. Some system includes automatic proof system.
Graphing Tools including calculator: Most cases used the Blending part. Some tools such as CBR enables us the embodiment in physical activity.

When we consider the way of using and developing innovative technology for mathematics, his map is also useful. For example, we can identify the limitation of current technology and the necessary development such as integrating different software and its mathematical difficulty. At the same time, we should also consider the limitation of figure 4. For example, it shows the existence of other worlds such as Physical World but he focused more on mathematical thinking within the three worlds. On his map, there are traditional technologies which showed that he does not only focus on current technology for conceptual development even though he was the earlier developer of graphing tools. We are not teaching mathematics for technology. For teaching mathematics, if it is necessary we prefer some necessary technology.

3. Technology for developing mental object

Judicious using of various tools and ideas is one of the key idea of mathematics education. There is no matheducator who only chose to use one technology. Depending on the aim of mathematics teaching, the meaning of judicious itself will change too. In this chapter we do not want to distinguish old, traditional, new and innovative technology because its preferences depend on the aim of education. For considering of how to develop mental object, we fix mathematics teaching to teach mathematics through Mathematization. Mathematization is a specialized idea of mathematical activities by Freudenthal (1973): organizing reality by mathematical means (methods) is called mathematizing. From the view point of phenomenology, he also enhanced the development of mental object (entity) in mathematics (Freudenthal 1983).

He argued that to teach mathematics as a given is an anti-didactic inversion, and that students should reinvent mathematics as well as mathematicians invented mathematics through organizing reality. Based on his mathematical experience and historian's experiences, he described the learning process through mathematization using some examples of van Hiele levels and emphasized the importance of reflection on experience at a lower level for overcoming the discontinuity between levels. This means that students should experience the mathematization of reality. People are accustomed to use tools in their life with mathematical reasoning and developed mathematics through the reflection of these experiences.

To illustrate the process of mathematization, I would like to use the example from the development of perspective drawing.

Today in art lessons, different perspective drawings using one-point, two-point or three-point perspective drawings are taught as a composition technique for drawing pictures. Some people misunderstand that this is the theory of perspective drawing.

Through the screen windows, painters recognized the way of drawing as mental object and researched the way of using the screen window to draw what they actually saw and tried to translate their geometrical experience onto their drawing board (see Figure 10). Figure 10 is explained by

Figure 11 using Dynamic Geometry Software. During that era, how to draw the depth of pictures was an essential problem. Based on the idea of figure 11, painters could know how to draw (contract) the depth in figure 10. The idea of figure 10 was expanded to the anamorphose through inclining screen windows shown in figure 12 and 13.

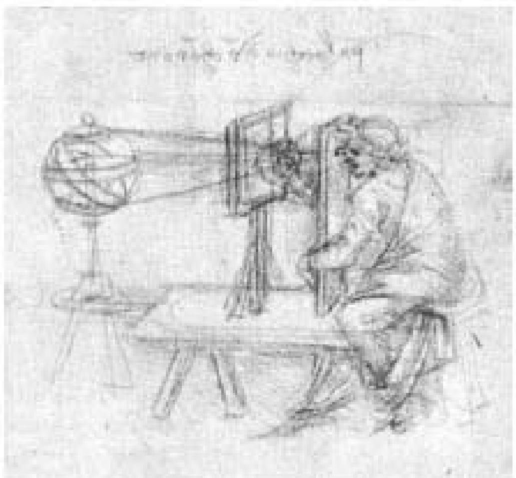


Figure 5. da Vinci

Figure 6. Düler

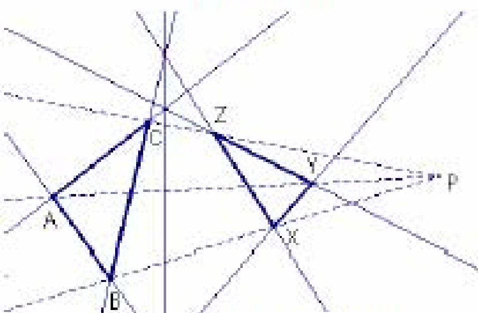


Figure 7. Desargues

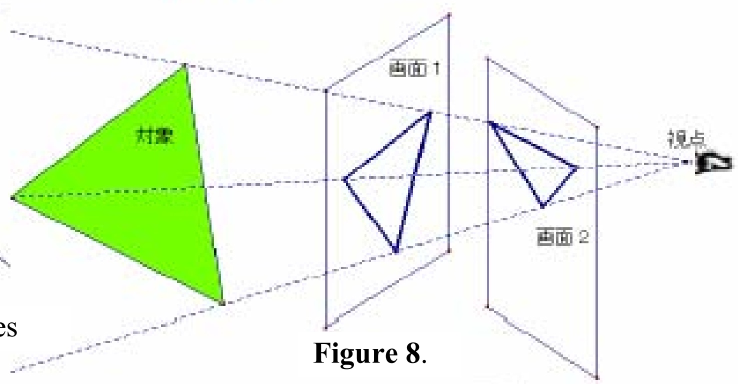


Figure 8.

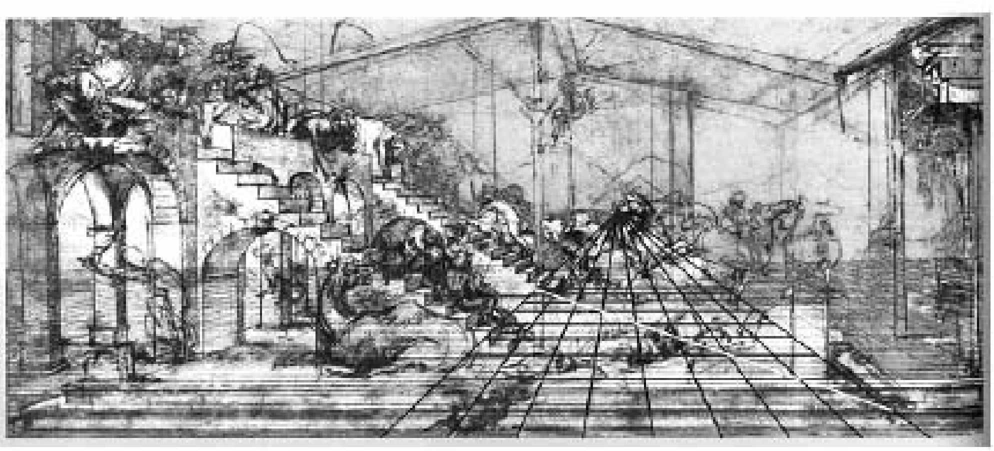


Figure 9.
da Vinci

The ways of how to construct the depth on the board using a set of parallel lines and focusing lines develop the mental object for mathematization because there are mathematical invariant three or more lines intersect at one points. Pascal reconsidered the phenomena upset ways and considered it from ‘ordonnance’ (beam, the set of parallel lines and lines thorough one point). If we use the pencil, it is not easy to write the lines which enable to meet at one point. Because we have experience the difficulty, we feel strange when three lines meet at one point.

The embodiment (Lakoff, Nunez, 2000), explained an even higher mathematical concepts originated from some metaphors derived from bodily motion, is the basic ideas when we think about the role of technology. One cannot understand mathematics appropriately without reliable metaphors. The theory gives a central role of the appropriate metaphors of bodily motion for understanding abstract mathematical ideas and overcoming discontinuity of learning process. If we do not know the difficulty to draw three lines intersect at one point, we can not recognize it as mental object.

In Figure 7, the Theorem of Desargues is explained by Figure 8, which uses the same metaphor as the pictures in Figures 5 and 6. Projective geometry generalized the eye-beam metaphor. The eye-beam also existed in the pictures of Christ in the middle ages but it was not a human eye-beam. It came from heaven and eyes of God. Painters imagined the existence of God and trying to draw the benediction with eye-beam from God or heaven on the drawing boards. Thus, normally, Christ should be larger than apostles because they follow and receive God’s eye-beam.

These new tools treated the eye-beam like the eye-beam of a human painter and humanized reality. Through the use of these geometrical tools, it was possible to see reality as a human construction and enable people again to use eye-beam metaphors as well as Euclid did at his Optics. Today, the latest technology, DGS, enables us to construct figure 12 and figure 13. Because we can drag the Eye Point and Screen, we can easily realize the appropriateness of the construction based on the perspective drawing theory. We can embody the perspective drawing theory to the real world which we feel in reality through the dragging the object. Through the embodiment by the DGS, mental object is now the really object could be dragged.

We have to note that this treatment by new technology is not the same recognition as da Vinci had because he did not have DGS. On the other hands, DGS supported to see those processes from the view point of hermeneutics that the understanding of mathematics in a social context includes the developer’s own, author’s or another’s perspective (Janke, 1994). Mathematics is most reliable subject to represent or reinvent by other people and thus understanding mathematics include expecting other people’s mind. Historical tools and current technology enable us to imagine or simulate developer’s perspective.

Because we know the meaning of Figures 5, 7 and 8, we can imagine the activity in Figures 9 and 10, and imagine the mind of da Vinci and his view of the world. If we understand the Figure 10 as Figure 11, we understand well that there is only one perspective drawing theory even if, in art, it is technically explained with many kinds of composition such as using one-point, two-point or three-point perspective drawings. If we do not know the perspective drawing tools used in Figures 5 and 6, and if we do not imagine that the structure is the same as looking outside through a window, we cannot imagine the process by which mathematics developed. Today’s perspective drawing theory in art that counts the number of vanishing points is only one technique for the composition of a given picture. It is not only graphics theory but also does not include the theory of perspective drawing that da Vinci and Dürer developed in their time.

Counting the number of vanishing points of the picture of da Vinci in art classroom in school teaches technique for the composition of picture but lacks interpretation of it as a human endeavor: For interpreting, it is necessary to imagine the author’s/painter’s wishes at that age in trying to explore with tools. This personal interpretation was not constructed without the tools of Figures 5 and 6.

The example illustrates Judicious using of old and new technology, and both technology supported to construct the important mathematical object on the context. This is the reason why we do not need to distinguish between old and new technology in the context of education such as mathematization. Depending on the time and aim, teacher can chose it in the most meaningful way.

Figure 11. By DGS

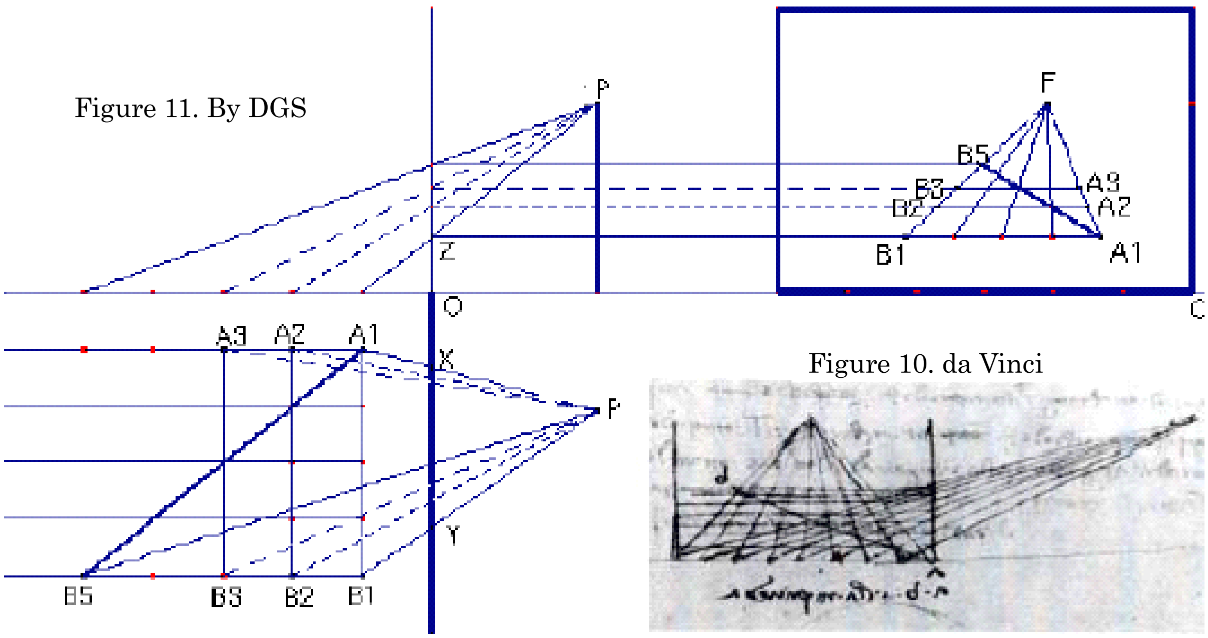


Figure 10. da Vinci

Figure 12

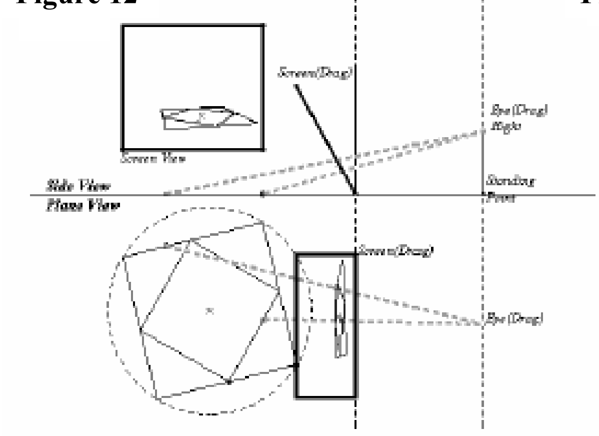
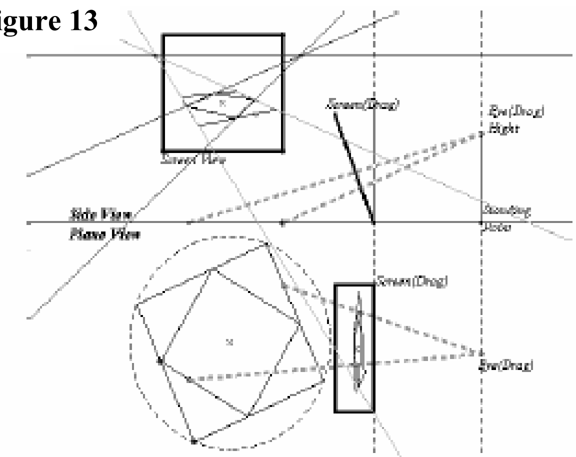


Figure 13



4. The intuition supported by mental object and visualization supported by technology.

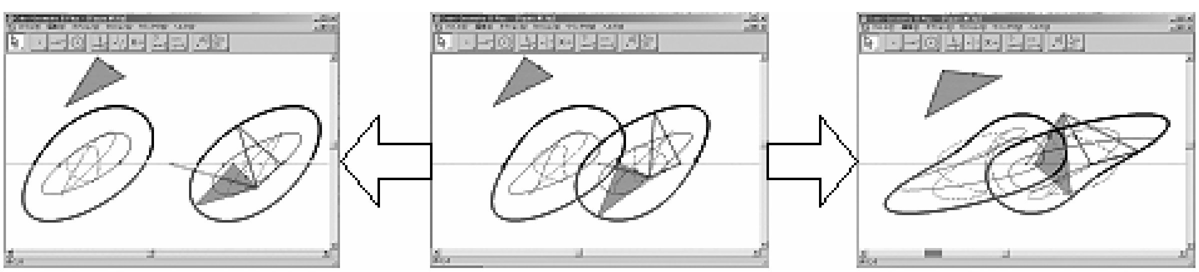
Every tool has certain intuition. It is this reason why we should choose the appropriate technology depending on the aims of education and necessary to develop the ability for judicious using.

The problems of locus on mechanics are good examples to illustrate what mechanical technology is. Technicians need the appropriate ability and knowledge in order to develop mechanics since majority of us do not have such mathematical intuition. Figure 14 was sited from Japanese Secondary School Textbooks in the 1943. In figure 14a, if point C move the slider AB, where point D is fixed and EC=ED=EF, then how does point F move? The guessing and answer could be illustrated cleary by the window mechanism (figure 14b). In case of figure 14c, how about the case of point C (middle) and point D (1/3). On further grade, they are analyzed by algebraic representations (figure 14d).

What is astonishing about this example is that figures 14a – 14d do look different but they are the same in mechanical meaning. To understand these different mechanics as one mechanics you must solve these problems geometrically according to the order of the textbooks. If you could imagine those mechanics are the same, you must have the mental object that is necessary to develop these kinds of mechanical tools. The similar descriptions existed in the textbook by van Schooten in 1646 (Figure 15). His textbook also began from guessing to geometric and finally algebraic reasoning. The intuition which we could acquire from those textbook is supported by the mental object of geometric reasoning. It bridges mechanics and geometry.

On the other hands, even if we recognize mechanics with geometrical mental object, it is very difficult to imagine the following locus (see bellow) because geometry can treat s limited number of curves as for mental object. This is why it is necessary to use technology for visualizing following curves (see bellow).

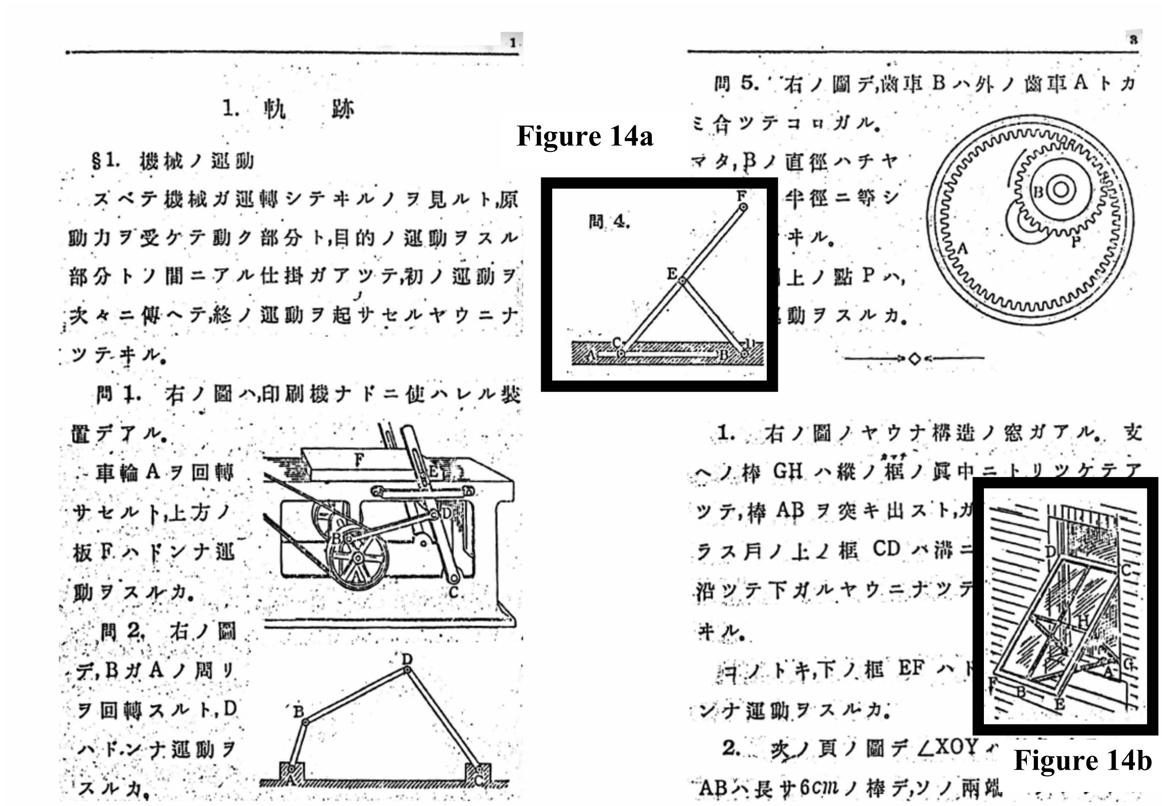
For technicians who develop mehanics, mechanical and geometric reasoning is important. From the view point of mathemaitzation, the following curves became the mathematical object which will be expressed by equations of functions. If technicians need to control the motion by computer, they must use the equations, too. Through mathematization, they will develop further intuitions to treat the following motions and develop better imagination as mental object without visualization by computer.



5. Conclusion

Logical inconsistencies for introducing technology come from the difference of society, curriculum and technology itself. Because of these inconsistencies, we should develop the judicious position to use technology. Teachers and students should be judicious user. Japanese approach which focused on teachers is one example. In any example, there is limitation. To enhance students' mathematical exploration, the importance of judicious using of technology is mentioned, in order to enhance students' mathematical explorations and developing the mental object that support their mathematization. The basic theories of mathematics education are used to explain this position. To illustrate the importance of both the traditional and computer technology, I used perspective drawing and mechanical motions as the main examples.

We use technology to improve mathematics education. This is the standpoint in this lecture even though there is some necessity to use mathematics education for technology in some occasions.



1

1. 軌　跡

§1. 機械ノ運動

スベテ機械ガ運轉シテヰルノヲ見ルト,原動力ヲ受ケテ動ク部分ト,目的ノ運動ヲスル部分トノ間ニアル仕掛ガアッテ,初ノ運動ヲ次々ニ傳ヘテ,終ノ運動ヲ起サセルヤウニナッテヰル。

問1. 右ノ圖ハ,印刷機ナドニ使ハレル裝置デアル。車輪Aヲ回轉サセルト,上方ノ板Fハドンナ運動ヲスルカ。

問2. 右ノ圖デ,BガAノ周リヲ回轉スルト,Dハドンナ運動ヲスルカ。

Figure 14a

問4.

3

問5. 右ノ圖デ,歯車Bハ外ノ歯車Aトカミ合ッテコロガル。マタ,Bノ直徑ハチヤ … 半徑ニ等シ … ヰル。… 上ノ點Pハ, … 動ヲスルカ。

1. 右ノ圖ノヤウナ構造ノ窓ガアル。支ヘノ棒GHハ縦ノ框ノ眞中ニトリツケテアッテ,棒ABヲ突キ出スト,ガ … ラス戸ノ上ノ框CDハ溝ニ沿ッテ下ガルヤウニナッテヰル。

コノトキ,下ノ框EFハド … ンナ運動ヲスルカ。

2. 次ノ頁ノ圖デ∠XOYノ … ABハ長サ6cmノ棒デ,ソノ兩端

Figure 14b

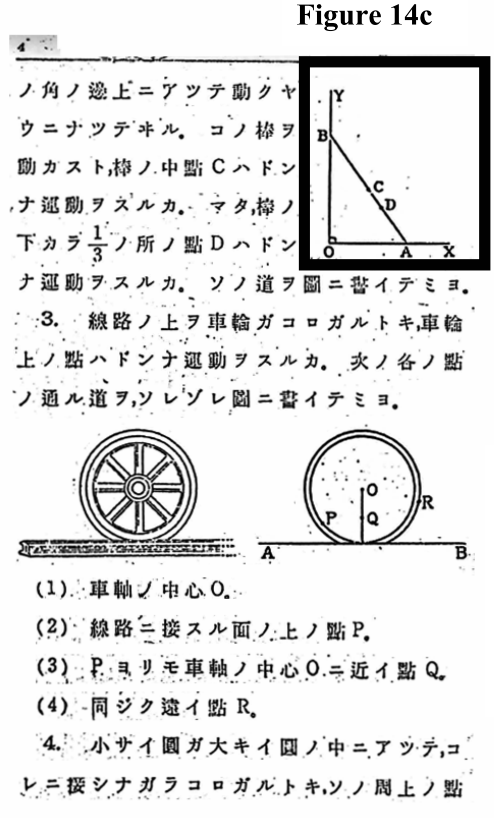


Figure 14c

4

ノ角ノ邊上ニアッテ動クヤウニナッテヰル。コノ棒ヲ動カスト,棒ノ中點Cハドンナ運動ヲスルカ。マタ,棒ノ下カラ$\frac{1}{3}$ノ所ノ點Dハドンナ運動ヲスルカ。ソノ道ヲ圖ニ書イテミヨ。

3. 線路ノ上ヲ車輪ガコロガルトキ,車輪上ノ點ハドンナ運動ヲスルカ。次ノ各ノ點ノ通ル道ヲ,ソレゾレ圖ニ書イテミヨ。

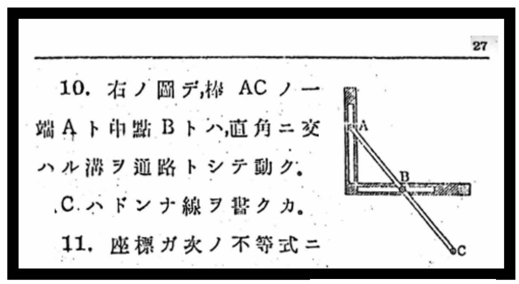
(1) 車軸ノ中心O。

(2) 線路ニ接スル面ノ上ノ點P。

(3) Pヨリモ車軸ノ中心Oニ近イ點Q。

(4) 同ジク遠イ點R。

4. 小サイ圓ガ大キイ圓ノ中ニアッテ,コレニ接シナガラコロガルトキ,ソノ周上ノ點



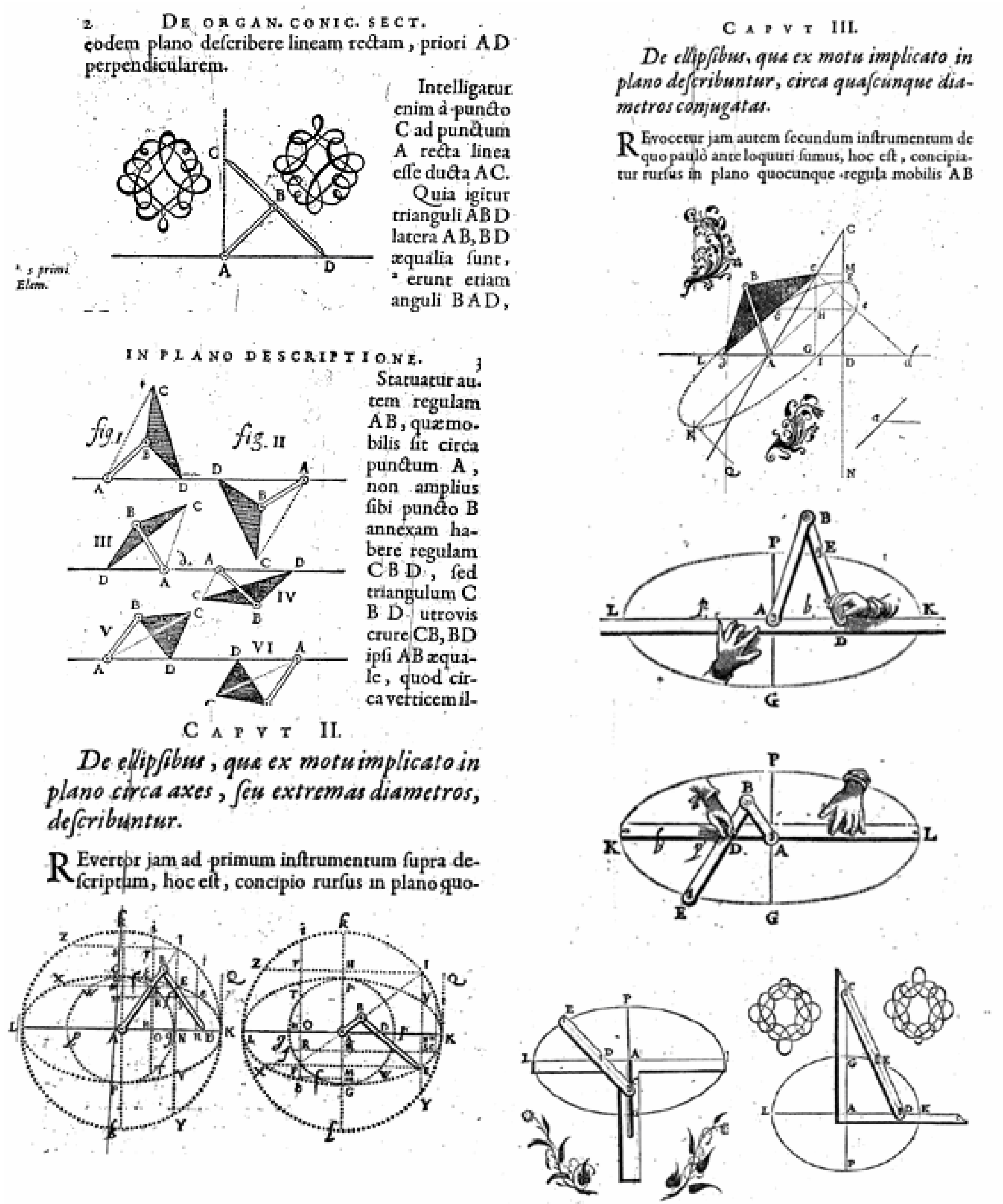
27

10. 右ノ圖デ,棒ACノ一端Aト中點Bトハ,直角ニ交ハル溝ヲ通路トシテ動ク。Cハドンナ線ヲ書クカ。

11. 座標ガ次ノ不等式ニ

Figure 14d

Figure 14
1943, Japanese Secondary School Mathematics Textbooks
a-c: from guessing to geometrically reasoning
d: after grade, in analytic geometry



2 DE ORGAN. CONIC. SECT.

eodem plano describere lineam rectam, priori AD perpendicularem.

Intelligatur enim à puncto C ad punctum A recta linea esse ducta AC. Quia igitur trianguli ABD latera AB, BD æqualia sunt, erunt etiam anguli BAD,

IN PLANO DESCRIPTIONE. 3

Statuatur autem regulam AB, quæ mobilis sit circa punctum A, non amplius sibi puncto B annexam habere regulam CBD, sed triangulum CBD utrovis crure CB, BD ipsi AB æquale, quod circa verticem il-

CAPUT II.

De ellipsibus, quæ ex motu implicato in plano circa axes, seu extremas diametros, describuntur.

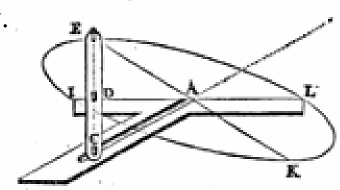
Revertor jam ad primum instrumentum supra descriptum, hoc est, concipio rursus in plano quo-

CAPUT III.

De ellipsibus, quæ ex motu implicato in plano describuntur, circa quascunque diametros conjugatas.

Revocetur jam autem secundum instrumentum de quo paulò ante loquuti sumus, hoc est, concipiatur rursus in plano quocunque regula mobilis AB

Figure 15 Schooten 1646,
from Guessing to Geometrically and Analytically.



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Free Software

Produced by Masami Isoda, CRICED, University of Tsukuba
GRAPES (Graphing Tools)
GCL (Dynamic Geometry Software enable to construct on the browser by Flash)
d-Book (Software to develop e-Textbook)
http://math-info.criced.tsukuba.ac.jp/software/

Exploring the Place of Hand-Held Technology in Secondary Mathematics Education

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While sophisticated technology for mathematics is available and used in many educational settings, there are still many secondary school mathematics classrooms in many countries in which student access to such facilities is either very limited or non-existent, either at home or at school. This paper focuses on secondary mathematics education for students and teachers who are without reliable and regular access to computers or to the Internet. The place of hand-held technologies, including scientific calculators, graphics calculators and integrated devices will be considered. The computational support such devices offer to students is described and evaluated. Opportunities for new approaches to teaching and learning mathematics are described. The significance of hand-held technologies for aspects of the mathematics curriculum, its evolution and its assessment will be outlined and some issues associated with effective integration of technology into the secondary school curriculum are identified.

1. Introduction

In recent years, mathematical use of computers has increased enormously in some settings, while in others it has not much changed at all. So there are still many secondary school mathematics classrooms in many countries (including affluent and industrialized countries) in which secondary school student access to technology for mathematics is very limited or non-existent, both at home and at school. This paper focuses on secondary mathematics education for those students and teachers who are without reliable and regular access to computers or to the Internet.

It is argued that there are good reasons for using hand-held technologies such as calculators to meet the needs of students, mostly deriving from the accessibility and affordability of the technology to a wide group of students. In addition, and importantly, hand-held technologies have been developed with the particular needs of secondary school mathematics education in mind, in contrast to more sophisticated technologies, which have been developed for quite different purposes and audiences.

While those less experienced with using technology in schools frequently think the main purpose is concerned with undertaking arithmetical calculations, in fact much more important issues of teaching and learning are at stake. Technology by itself is not enough: the capability of mathematics teachers and the nature of the school mathematics curriculum both need to be taken into account if secondary school mathematics is to be improved through the effective use of technology. When school mathematics curricula are dominated by external examination requirements, which is the case in many countries, hand-held technologies also take on a new significance.

The arguments in the paper draw on earlier ATCM and other papers presented by the author in the region, apply to a range of settings, and draw in part on experiences in developed countries, such as Australia and the United States. In developing countries, in which resources for education are more modest both at school and at home, the arguments for hand-held technologies are even more compelling, as they may represent the only realistic means to make progress connecting the mathematics curriculum to a modern world, already laden with technology.

2. Technology for education

Technology of many kinds is now widely available to most people throughout the industrialized world and in many parts of the developing world, especially in commerce and industry. A wander around Taipei makes this clear. It has now become a familiar part of the everyday world of citizens, parents and teachers. In addition, many technologies of potential interest to secondary school mathematics are manufactured in East Asia. Despite the widespread presence of technology, it seems that technology is not yet widely used in secondary mathematics teaching and learning in East Asian countries, such as Taiwan, China, Japan and Korea.

When considering 'technology' for education, it seems that many people interpret the term to refer to computer software and hardware of various kinds, and recently also to include the Internet. Although the ATCM has included aspects of other technologies, including hand-held technologies, over its entire history, the emphasis has been on computers, especially with the needs, interest and expertise of university teachers and researchers in mind.

It is much rarer for discourses regarding technology to refer to hand-held technologies, such as calculators and similar devices, although these are arguably of more importance to some parts of the school curriculum than computers. (Indeed, they are also arguably described as computers themselves, but for the present purpose, a distinction will be drawn.) It is common practice in schools and elsewhere for IT departments, policies and budgets to make no reference to calculators and similar hand-held technology devices in education, but to assume instead that the only technology of interest involves computers.

Indeed, some of the enthusiastic promotion and discussion of technology in mathematics education by both official sources and by commercial companies seems to take place under the assumption of an ideal education world. In the extreme, such an ideal world would be characterized by: (i) all students have unlimited access to modern high-speed computers; (ii) all software is free, or budgets for software are essentially unlimited; (iii) students and teachers have unlimited access to high-speed Internet lines; (iv) facilities in students' homes match those in their schools; (v) teachers are well-educated enthusiasts in mathematics and pedagogy, with unlimited free time; (vi) curriculum constraints, including externally imposed and administered examinations, do not exist.

Although such assumptions are mostly unrealistic, they do in fact provide a useful starting point to think about and study technology in mathematics education. Proceeding on the basis of such assumptions, teams of professionals can and should develop good uses of technology, free of the shackles of the present reality. Such professionals include mathematicians, computer scientists, software developers, mathematics teachers, education researchers and others.

Few school contexts today match these idealistic assumptions, however. The virtual world that has no constraints is not the same as the present world inhabited by most students in most classrooms in most schools, in most countries (including the more affluent countries). The present paper is concerned with the real educational world in which many students, teachers and curriculum developers find themselves, in these early years of the twenty-first century. In the real world inhabited by most people today, hand-held technologies continue to be of more significance than computers, and hence are the focus of this paper.

3. A hierarchy of hand-held technologies

In this section, a four-level hierarchy of sophistication of hand-held technologies is described, in increasing order of sophistication (and thus also of price).

Arithmetic calculators

First appearing more than thirty years ago, arithmetic calculators are in common use in commercial contexts everywhere. These include shops and street markets throughout Asia, where the main function is sometimes to communicate prices, especially to tourists and others who do not speak the local language well. Basic calculators essentially provide a means of completing everyday numerical calculations, using decimals, and are very inexpensive. They are generally restricted in capabilities to the four operations of addition, subtraction, multiplication and division; many models also deal (sometimes strangely) with percentages as well. More sophisticated versions have been developed for educational use. One embellishment is to use mathematically conventional priority order for arithmetic calculations, so that 3 + 4 x 5 gives the correct result of 23 instead of 35. Another is to include operations with fractions as well as decimals.

Arithmetic calculators have been available to elementary (primary) schools for many years now, although the extent to which they have been adopted has varied between teachers and between countries. Despite the concerns of some teachers and parents, extensive research has established that these are educationally useful, and not harmful [1], [2], and few researchers are interested any longer in looking for negative effects associated with their use. However, they provide insufficient capabilities for secondary school students, whose mathematical needs extend considerably beyond mere computation.

Scientific calculators

Scientific calculators offer students slightly more capabilities than numerical calculations. Most scientific calculators provide the same facilities as an arithmetic calculator, as well as some more sophisticated ones, such as powers and roots. Table functions are also provided: values of functions that previously had to be obtained from mathematical tables are available directly from the keyboard. These include logarithmic, exponential, trigonometric and inverse trigonometric functions. Statistical calculations are available, so that means and standard deviations are calculated for data entered, and for many calculators, bivariate statistical calculations (such as correlation coefficients and linear regression coefficients) are also included. Recently, sophisticated versions have included higher level calculations of interest to secondary schools, such as those involving complex numbers, probability distributions and combinatorics. In essence, scientific calculators provide students with the capacity to undertake numerical calculations relevant to the mathematics of the secondary school.

Scientific calculators have been routinely used by secondary school students in most western countries for almost thirty years now. They are generally regarded as inexpensive items of equipment, essential for computation in mathematics and science, and are usually permitted for high-stakes examination use. They reduce the need for extensive by-hand calculation and consulting of tables of values of functions, characteristic of secondary school calculations of the previous generation.

Graphics calculators

Graphics calculators are distinguishable by their relatively large graphics screen, which accommodates several lines of display or a visual image of some kind. As well as including the capabilities of scientific calculators, graphics calculators include their own software for a range of mathematical purposes, including the representation of functions in tables and graphs, statistical displays and two-dimensional drawings. The range of mathematical capabilities varies between models, but these days can include numerical calculus, complex numbers, matrices, spreadsheets, probability simulation, sequences and series, numerical equation solving, statistical analysis and hypothesis testing, financial analysis and geometry. Some more advanced (and thus more expensive) graphics calculators also include low-level versions of Computer Algebra Systems (CAS). An important difference between scientific and graphics calculators is the possibility of students using the latter for mathematical explorations, rather than just calculations, either spontaneously or under the direction of the teacher.

In most industrialized countries, graphics calculators have been well-received in schools over the past twenty years, and are now routinely used by many students in the senior secondary school years as well as the early years of post-secondary study. As an illustration of the reception of this technology by teachers, the Australian Association of Mathematics Teachers' graphics calculator communiqué [5] described several ways in which this technology was being used in many Australian schools to good effect. In many countries, graphics calculators are permitted for use in formal external examinations, including those for selective entrance to universities. Empirical research results (eg, recently summarized in [6], [7]) have generally supported the use of graphics calculators for student learning, especially conceptual learning, and have generally suggested that students do not lose important procedural skills at the same time.

Integrated devices

In recent years, powerful new devices have been manufactured to create a new category in the hierarchy of hand-held devices for school mathematics. Good examples (among others) include the *ClassPad 300*, manufactured by Casio, and the CAS version of *TI-Nspire*, manufactured by Texas Instruments. In some respects, these devices are similar to graphics calculators, with inbuilt CAS, and include the capabilities of graphics calculators within their software suite. They are distinguished from graphics calculators in at least three important ways, however. In the first place, they contain more significant mathematical software, dealing with a wider range of mathematical concepts.(In the case of the *ClassPad 300* for example, these include a powerful computer algebra system with exact solution of differential equations, three-dimensional graphing and vectors.) In the second place, they provide significant interactivity between the various software applications. Thirdly, they have substantial storage capacities and similar operating systems in some respects to computers, so that they can almost be regarded as small computers, dedicated to teaching and learning mathematics. In these respects, such devices enable both more sophisticated mathematical ideas to be handled (in addition to less sophisticated ideas) and offer extensive opportunities for student manipulation and exploration, with teacher guidance in various forms.

These are relatively recent devices, and are included here in part to make it clear that the hierarchy does not end with graphics calculators. Although these devices have been available for only a short time, they have already attracted considerable attention, and some are permitted for use in formal examinations in some locations (such as Melbourne, Australia).

4. Educational advantages of hand-held technologies

Although some computer technologies are very powerful, there are some very good reasons for using hand-held technologies for secondary mathematics education. Five important advantages include the following:

1. They are easily portable, and can be comfortably carried in a school bag along with other materials students need. A consequence of this portability is that they can be used both at home and at school, and can be easily taken from one school classroom to another.
2. They are less expensive than computers, especially when all the software needs are taken into account (as calculators contain their own software). This cost issue has important implications for accessibility, regardless of whether costs are met by individual parents or by schools.
3. They are potentially more accessible to more students than are other forms of technology, as a consequence of the first two advantages: curriculum developers can design curricula on the assumption that students can access technology, only if it is accessible on a wide scale.
4. They can be used in formal examinations, which are of considerable importance in many educational settings. This advantage is mostly a consequence of the preceding reasons, since it is realistic to design curricula and associated examinations only for technologies that are potentially available to all relevant students; to do otherwise is likely to be regarded as unfair.
5. Most of them have been designed, and continue to be modified, for the express purpose of school mathematics education. Unlike other technologies, designed for other purposes, today's calculators are developed solely for the purposes of education, and so can be expected to be sensitive to the needs and interests of those involved, such as students and teachers.

A possibly surprising consequence of this last advantage is that, unlike other more sophisticated forms of technology, hand-held technologies are less likely to be used by mathematics and science professionals in universities than by secondary students and their teachers. At least in the developed world, the present generation of professionals in the mathematical sciences are comfortable users of computers and computer software, but have often had little experience with the comparatively recent technologies of interest to this paper.

5. A computational role

It is important to recognize that there are different roles for technology in secondary mathematics education. For example, the *Technology Principle* of the National Council of Teachers of Mathematics, widely quoted, asserts that "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning." [3]. In elaborating this principle, the NCTM in the USA observed:

"Calculators and computers are reshaping the mathematical landscape, and school mathematics should reflect those changes. Students can learn more mathematics more deeply with the appropriate and responsible use of technology. They can make and test conjectures. They can work at higher levels of generalization or abstraction. In the mathematics classrooms envisioned in *Principles and Standards*, every student has access to technology to facilitate his or her mathematics learning." [3]

Some roles for technology concern computation, the provision of different educational experiences and influence on the school mathematics curriculum. These three roles are elaborated in an earlier paper [10]. In this section, we consider the important computational role played by hand-held technologies in secondary mathematics education

In brief, hand-held technologies can now meet all the computational needs of secondary education, providing a means to obtain reliable answers to numerical questions. This role is significant, as it potentially allows more time to be devoted to developing mathematical concepts, where previously a lot of time was required just to do computations. The centrality of calculation in mathematics was emphasized by Wong's observation that mathematics is a "subject of calculables." [8]

For students in elementary (primary) school, an arithmetic calculator allows everyday calculations with any measurements that are meaningful to them to be carried out, an important consideration if realistic applications of mathematics are to be included in the curriculum. The scientific calculator extends this capacity to large and small numbers, including those expressed in scientific notation (one of the many reasons that an arithmetic calculator is inadequate for secondary school use.) This is an important consideration for any mathematical modelling undertaken by students, whether in a mathematics class, a science class, or elsewhere. When confronted with calculation needs that could not be handled mentally, or for which reasonable approximations were insufficient, previous generations of students have been reliant on less efficient means of calculation, such as by-hand methods, or the use of logarithms and tables. A scientific calculator provides values for functions that were previously published in tables (such as trigonometric and logarithmic functions, as well as squares and square roots), and thus offers the opportunity to avoid long, tedious and error-prone calculations. Historically speaking, in many mathematics curricula, calculations have frequently became procedural ends in themselves, distracting from the important mathematical features of the work, and rarely offering much insight to students. This problem has often been exaggerated by the use of examinations emphasizing efficient use of procedural computational techniques.

As well as handling arithmetic calculations, scientific calculators also provide a means for efficiently dealing with easy, but lengthy computations, such as those associated with combinatorics (such as determining ${}^{52}C_5$, the number of poker hands possible from a standard deck of cards) or with elementary statistics (such as finding the mean and variance of a sample of 20 measurements). It is interesting that such calculations were not routinely available on early scientific calculators, but were added to later models, designed for education, almost certainly to accommodate the computational needs of secondary school students rather than 'scientists', for whom presumably the original scientific calculators were designed. It is noteworthy that recent scientific calculators also provide some exact answers as well as numerical approximations, consistent with the continuing support of computational needs. Figure 1 shows two examples from a recent scientific calculator model.

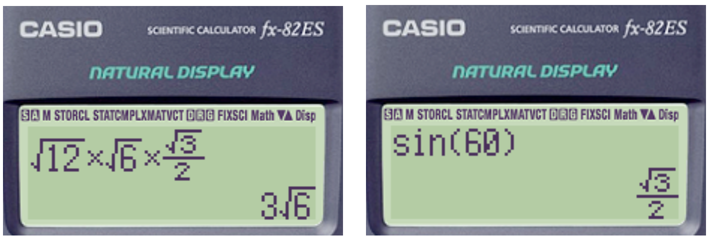

Figure 1: Exact computation on an entry-level scientific calculator

As well as providing reliable answers to computational questions, such capabilities might support student thinking and even curiosity about the mathematical ideas involved.

The scientific calculator now has a history of about thirty years in secondary schools, and has continued to undergo developments, partly fuelled by competition between rival manufacturers, and partly as a consequence of advice from mathematics teachers themselves. Over that time, they have become much easier for students to use, with more informative screens and a broader range of capabilities, which have together improved their capacity to fulfill the computational role for students.

Modern graphics calculators usually have at least the same suite of capabilities as scientific calculators (so that it is not necessary for students to have access to both kinds of devices.) A difference between the two is their relative ease and their range of computations. These vary a little between models, but Figure 2 shows some examples of evaluating expressions, calculating a logarithm, calculating with complex numbers and inverting a matrix.

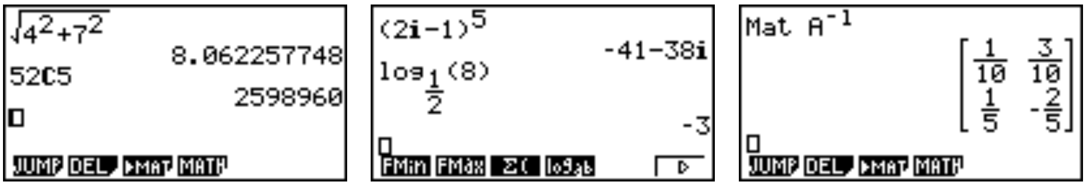

Figure 2: Some computations with a Casio fx-9860G graphics calculator

The examples illustrate routine numerical calculations that, if required to be done by hand, occupy a lot of student time. Although evaluating the logarithm involves conceptual thinking about the nature of logarithms, neither the inversion of a matrix nor the expansion of a complex power involve anything other than routine procedures, which some would argue are better left to a machine.

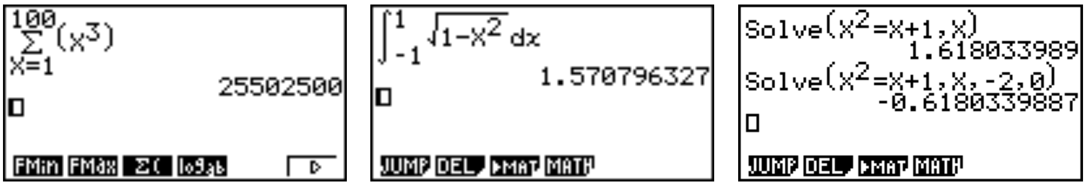

Figure 3: Further computations with a Casio fx-9860G graphics calculator

The further examples of computations shown in Figure 3 illustrate how significant numerical work can be completed on a graphics calculator, raising issues regarding the appropriate balance of mathematical concepts and skills in the curriculum.

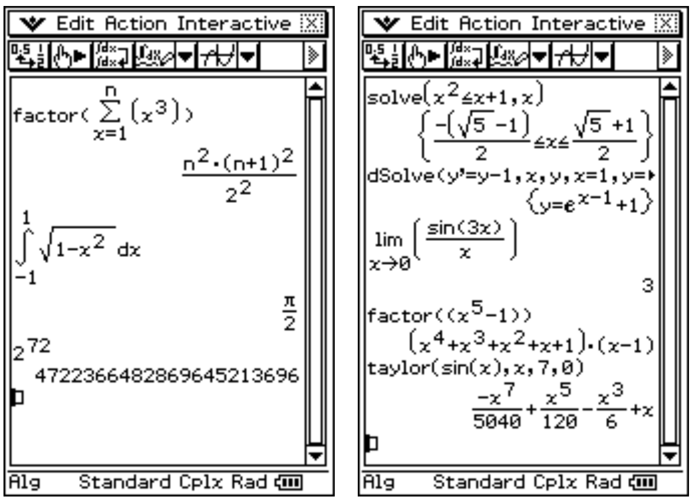

Figure 4: Some computations on a ClassPad 300

Finally, Figure 4 shows some examples of computations available on Casio's *ClassPad 300*, a good example of an integrated hand-held device in current use. The eight examples chosen for this purpose illustrate the powerful capabilities for exact computation and symbolic computation routinely available on this device. In each case, a computation has been entered on a single line, with the result displayed on the following line. Several of the examples chosen make direct use of the computer algebra system built into the *ClassPad 300* software; these show integration by parts, the solution of a quadratic inequality, factoring of simple and complicated expressions, and a Taylor series expansion. While secondary school students have previously undertaken computations of these kinds by hand, because the necessary results could not be obtained in any other way, the work involved has generally been routine and procedural in nature and has not added greatly to the quality of their mathematical thinking. It is important to recognise that sophisticated and routine activities of these kinds can now be accomplished by a few keystrokes on a hand-held device.

The fact that exact computations are available is also of significance, both for numerical work and for symbolic work. Some of the results shown in Figure 4 can be obtained numerically (although not exactly) on graphics calculators, while others can be obtained only on a CAS-capable device.

The examples in Figure 4 have been chosen from many possibilities, to illustrate and support the claim that any of the routine computational needs of secondary school mathematics can be readily obtained on an integrated device like a *ClassPad 300*.

6. An experiential role

While computation is important in mathematics, it is not the main contribution of hand-held technology to teaching and learning mathematics. The experiential role, describing the possibility of students encountering different experience, is arguably of greater significance. In this section, some examples of the ways in which hand-held technologies can offer students new experiences for learning and teachers new ways of teaching are briefly described. A major element is the possibility of provoking students to use technology to explore mathematical ideas for themselves, and thus to support cognitive development and not only procedural skill.

Scientific calculators are less powerful than graphics calculators in this respect, which perhaps accounts for the relatively little impact they have had on thinking about school curricula. The lack of a graphics screen, allowing for various representations of mathematical objects, is a significant limitation. Despite this drawback, scientific calculators can be used in intellectually productive ways, many of which are explored in [12], which contains many detailed ideas, ranging across several areas of mathematics, including algebra, functions, trigonometry, geometry, statistics, calculus and business mathematics. Some of the examples derive from the ability of the calculator to show different representations of numbers, such as fractions and decimals, powers and logarithms. Others derive from alternative approaches on a sophisticated scientific calculator to mathematical topics, such as numerical solution of equations or evaluation of integrals.

For graphics calculators, experiential opportunities are much more plentiful, as the availability of a graphics screen offers students ways of interacting with mathematical ideas that were not available to them prior to the advent of technology. There are very many examples in [14], but space here to include only a few.

Perhaps the most common ways in which graphics calculators offer students new experiential opportunities are those related to the representation of functions. A graphics calculator allows for both graphical and tabular representation of a function, in addition to symbolic representations, as Figure 5 illustrates.

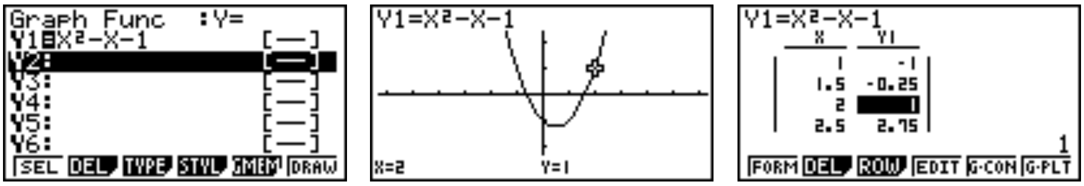

Figure 5: Three representations of a function on a Casio fx-9860G graphics calculator

Students can manipulate and explore these representations in a number of ways in order to understand better the mathematics involved. For example, they can modify the symbolic function and see almost immediately the consequences for the graph, compare the graphs of several functions at once, or a family of functions, to understand the effects of the coefficients, and can learn about functional forms (such as linear, quadratic, cubic) interactively. They can zoom in or out on a graph to study its shape and properties in detail. They can examine at close quarters the numerical values of the function, connecting th graph to the solution of equations or roots or both. They can study the intersections of graphs and connect these to the solutions of equations. They can examine the shape of a graph in detail to encounter ideas of rates of change informally. The experiences offered by these three representations provide both teachers and students with new ways of interacting with the mathematical ideas involved.

A powerful way of using these sorts of capabilities involves the derivative of a function. The idea of a derivative at a point (rather than the slope of a tangent to a function at the point) is well illustrated on the graphics screen shown in Figure 6, using an automatic derivative tracing facility built in to the Casio fx-9860G calculator. Students can get develop their intuitions about the relationship between the derivative and the shape of a graph by seeing how the derivative changes sign and magnitude at various points as the graph is traced.

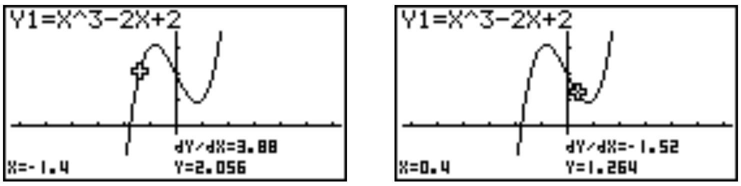

Figure 6: Using a derivative trace to explore the idea of a derivative at a point

The calculator can also be used to represent a derivative function automatically, by evaluating the derivative at each point of a function and graphing the result, as shown in Figure 7. Since both the function and its derivative are represented on the same screen, important connections between these are readily examined by students. In this case, the characteristic parabolic shape of the derivative function is important, as are the observations that the derivative function changes sign near the turning point of the cubic function.

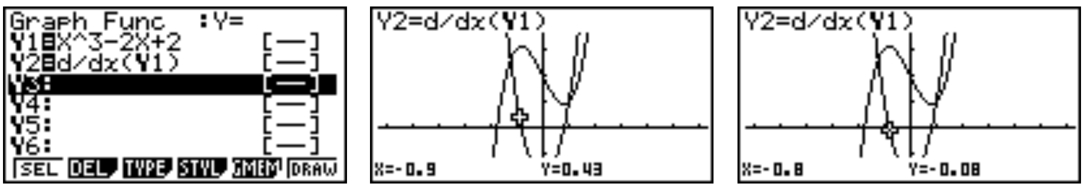

Figure 7: Simultaneous graphical representation of a function and its derivative function

Graphics calculators offer students much more opportunity to explore data and engage in statistical thinking than do scientific calculators, because data are stored in the calculator once entered, and then can be manipulated in a variety of ways. For example, data can be edited to correct errors or omissions, can be transformed (eg with a logarithmic transformation, in order to linearise an exponential relationship), can be represented in graphical displays (such as scatter plots, histograms or box plots), can be compared with ideal mathematical models and can be used to undertake standard statistical tests. Figure 8 shows representative screens for some of these sorts of operations.

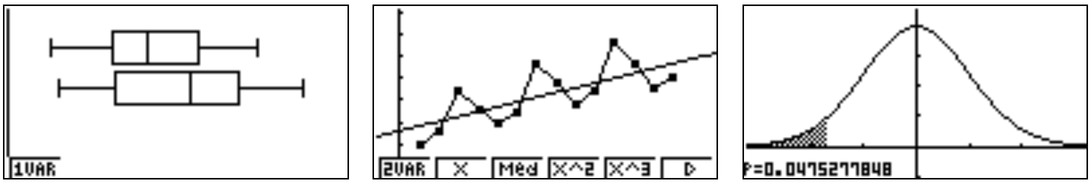

Figure 8: Examples of data analysis activities on a graphics calculator, taken from [14]

Taken together, these sorts of capabilities suggest that a graphics calculator can be regarded as device for exploratory data analysis in a range of flexible ways, supporting both descriptive statistics and inferential statistics, and allowing students an opportunity to develop important data analytic skills and understandings, using their own collected data or those of someone else.

Figure 9 shows examples of using a calculator spreadsheet to show the kinds of interactions with which students can engage in exploring the behaviour of the Fibonacci Sequence. In this case, the ratio of successive terms (in column C) converges quite quickly to φ = 1.61803…, as can be seen from the graph of column C. Students can manipulate spreadsheet elements (such as the starting value of the sequence) to see the (unexpected) effects. Such experimental activity is not practically possible without access to technology.

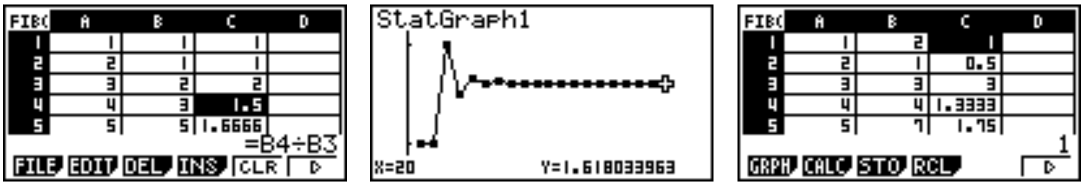

Figure 9: Exploring a series with a graphics calculator

As a geometric example of offering a different experience to students, Figure 10 shows some Casio fx-9860G graphics calculator screens concerned with plane geometry, in particular the idea of a locus. In each of the three screens, D is a point on a circle centred at A with radius AB. C is a point external to the circle and E is the mid-point of CD. When animated, the screen shows the locus of E as D traverses the circle, suggesting visually that the locus is itself a circle. Students can experiment with this situation by moving C (as suggested) or by changing the size or location of the circle; in all cases, they can examine the effects to seek invariants in the locus, with a view to understanding what is happening at a deeper conceptual level.

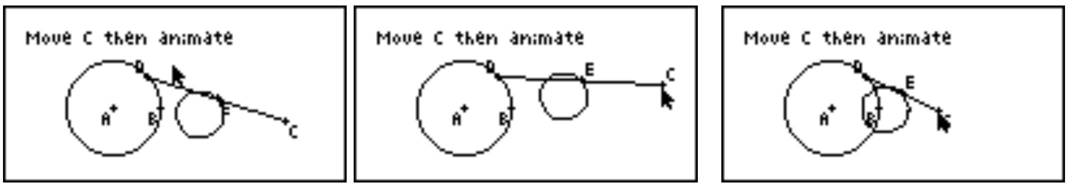

Figure 10: Experiencing geometry with a graphics calculator.

Another kind of opportunity for experimentation is made available through a calculator, using simulation, a powerful tool for both understanding probabilistic situations and for modelling contexts. The sample screens shown in Figure 11 contain histograms of simulated tosses of two dice. Students can readily perform the simulations on a calculator and then graph the results.

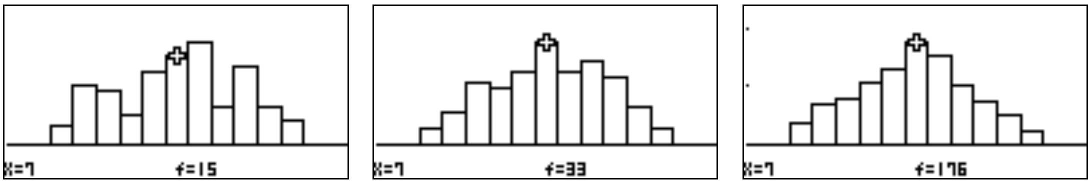

Figure 11: Simulations of the total of two dice thrown 50, 200 and 999 times respectively

While some practical work of this kind is useful, efficiency suggests that technology is needed to provide a good environment for learning about the regularities involved. In this case, a larger number of simulations results in a more symmetrical distribution, and a closer match to the theoretical result obtained on the basis of a probabilistic analysis. In a classroom, the potential for students to compare simulations offers opportunities for learning about the nature of randomness.

Opportunities for new learning experiences are even more plentiful on a more sophisticated device such as a *ClassPad 300*. To give an illustration, Figure 12 shows the reflection of the plane about line AB, with the triangle CDE and its image C′D′E′, both shown.

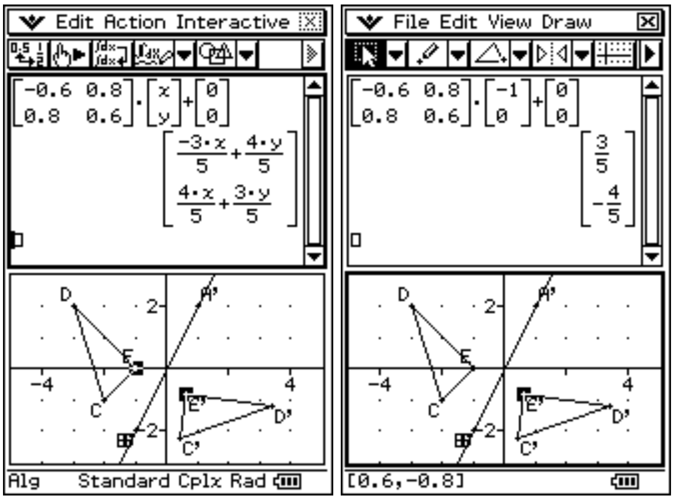

Figure 12: Experiencing reflections on a ClassPad 300

A student can manipulate the elements directly and see the results: moving line AB and changing the triangle CDE will both result in interesting effects. Powerfully, students can also drag a point and its reflection from the lower (geometry) window to the upper window to produce automatically the relevant transformation matrix, shown above. Individual points may be inserted into the general relationship to see the relationship in action; for example, the second screen above shows that the image of E(-1,0) is E′(3/5,-4/5). Such an environment offers new experiences for students (to learn about reflections and linear transformations in this case) that are not available otherwise.

In summary, this section has presented some examples, from a potentially much larger set, to elaborate the notion that hand-held technology can significantly alter the experience of learning mathematics for students, by providing them with access to tasks and opportunities not otherwise available to them. For the same reason, the opportunities for teachers to provide a different experience for students are also significantly changed.

7. Curriculum influences

The availability of hand-held technologies raises important curriculum questions for teaching, learning and assessment of secondary school mathematics, arising from the execution of the previous two roles concerning computation and experience. In this sense, the technology has become an important source of curriculum influence. There are at least three dimensions of this influence, concerned with the place of computation, the choice of mathematical content to be included in the curriculum, and the sequence in which material is presented to students.

Computation

Since most routine computational procedures can now be conducted efficiently on a calculator, curriculum developers need to decide the extent to which it is necessary, or desirable, for students to develop expertise at executing these same procedures by hand, probably less efficiently and less reliably than their calculators. Few would suggest that all procedures ought to be mastered by all students, partly because there is insufficient time likely to be available for this. Alternatively, few would suggest that hand-held technology be relied upon too heavily for computational purposes, since the development of some level of personal computational expertise is widely regarded as an important outcome of mathematics education. The issue instead is one of finding a suitable balance between these opposing views, which is not an easy matter.

Part of the problem of balance, of course, is that there is usually only a fixed amount of time available for mathematics in school; in many countries, even this fixed amount of time is reducing, as other pressures on the school curriculum.. It might be argued that only some students (not all) are expected to develop high levels of personal computational efficiency, or that by-hand expertise is left until later in the curriculum (for all, or for only some) students.

The situation can be illustrated by considering the solution of systems of linear equations, universally included in secondary school mathematics curricula. Consider the following system:

$$\{x - 2y - z = 2;\ 3x + y + 4z = 6;\ 2x - 3z = 7\}$$

Algebraic procedures involving Gaussian elimination are available to solve this system, and students have routinely been taught how to use them in secondary school. Such procedures are powerful, relatively efficient, generalizable and should always produce the correct solution; in addition, refined versions of the procedures using matrix representation and arithmetic are available, rendering the tasks even more efficient, because attention is paid only to the coefficients. On the other hand, the by-hand procedures themselves are complicated, heavily laden with elementary arithmetic and hence fundamentally error-prone. Furthermore, carrying out such procedures offers very little insight into the solution, in most cases. Developing student expertise at this sort of task requires a lot of time in class and time for practice (possibly at home) and considerable motivation and commitment by students. Using curriculum time for such tasks means that the time is not available for other tasks, since the total time available to students is always constrained.

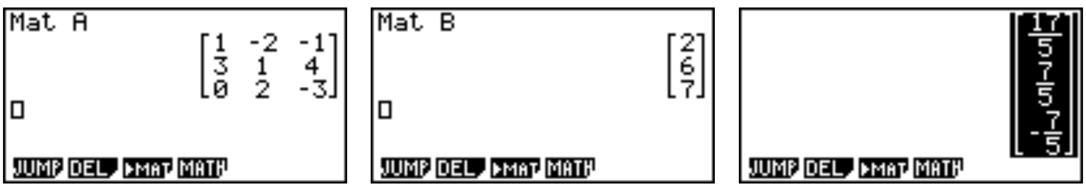

Figure 13: Matrix solution of the system of equations

On a graphics calculator, the matrices of coefficients are readily entered, and readily manipulated to produce a solution, as shown in Figure 13 on the Casio fx-9860G. To do this, students need to know both how a linear system can be represented by a matrix and how to use their calculator to do this. They also need to know how the matrix formulation can be used to represent the solution, and how to obtain this on their calculator, using the inverse of the matrix of coefficients.

The computations involved can be streamlined even further, however, on this graphics calculator, as shown in Figure 14, in which the augmented matrix of coefficients is shown and the student needs merely to execute a *solve* command to see the result, available as both a decimal and a fraction in this case, after scrolling the solution vector. While the same procedures are employed internally in this case and for the matrix formulation, this is not the case for students; this latter version of the calculator solution does not require any knowledge of matrices or their manipulations. Students merely need to recognize the problem as one that involves the solution of a system of three simultaneous linear equations and know how to enter the coefficients faithfully.

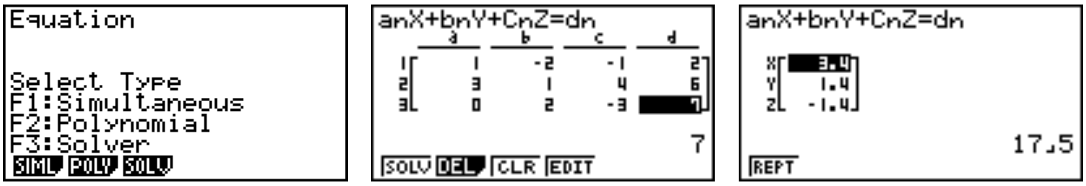

Figure 14: Direct calculator solution of the system of equations

In each of these cases, students need to both know about the idea of a system of linear equations and how to both represent and solve such a system numerically on the calculator. As for developing by-hand algebraic procedures, developing this knowledge also requires the use of curriculum time, although less time is likely to be needed with the technology.

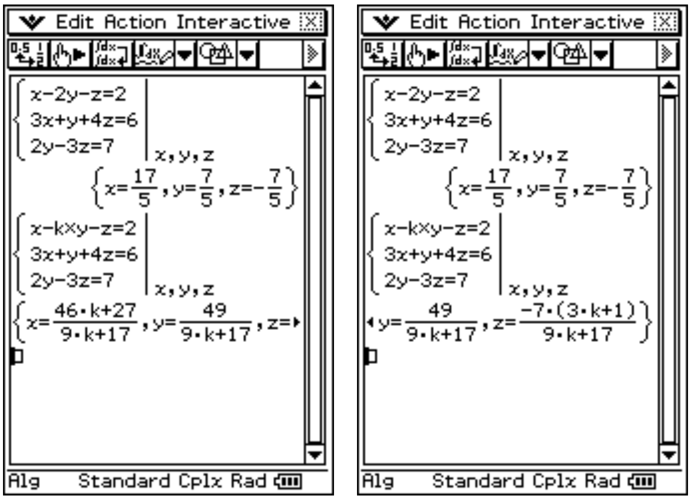

Figure 15: Solving a linear system of equations on the ClassPad 300

It might be argued that the regular use of a calculator to solve a system of linear equations might render students powerless when confronted with a more significant system, such as one that could not be solved numerically. In fact, similar procedures are available on more sophisticated devices, using a computer algebra system. To illustrate this, Figure 15 shows the solution of the linear system above, but with the first equation replaced with $x - ky - z = 2$, with the parameter k replacing the 2 as the coefficient of y in the first equation. (The result on the *ClassPad 300* occupies more then a single line, so a second screen is shown to display the result after horizontal scrolling.)

This example has been chosen to illustrate the influence that hand-held technologies might exert on thinking about the secondary school mathematics curriculum, at least as far as computation is concerned. The fundamental question that needs to be addressed concerns the extent to which students ought to be taught to imitate by hand the vast range of procedures that can be quickly, efficiently and reliably undertaken by a machine. While the answer to such a question is necessarily one of finding the appropriate balance, it seems inevitable that the balance will be shifted somewhat as a result of the influence of the technology.

Content

A second aspect of potential influence concerns the content of the curriculum, which represents an answer to the fundamental question: which aspects of mathematics are important enough to be included in the school curriculum, and which can be safely left until later, or excluded altogether?

The answers to such a question have always varied between countries and over time within a country. The answers depend to some extent of course on the amount of time that can be devoted to school mathematics in the wider school curriculum and the aspirations and interests of the students concerned. Thus, some countries have a range of courses to suit students of different kinds, with varying emphases on various aspects of mathematics, such as algebra, calculus, trigonometry, geometry, statistics, probability, and so on. In addition, there is varying emphasis given to mathematical processes, such as proof, problem solving, mathematical modelling, inductive and deductive reasoning.

Hand-held technologies influence the answers to this sort of question, as they provide access to different aspects of mathematics in a variety of ways. To continue the example of equations explored briefly in the previous section, access to hand-held technologies provides an opportunity to reduce the emphasis on some algebraic manipulations associated with solving equations, and at the same time extend the repertoire of equations and solution methods to which students are exposed. More attention might be paid in secondary school mathematics to numerical methods of solution of a wide range of elementary equations and less attention to exact algebraic methods of solutions of a very small range of elementary equations (mostly linear and quadratic). [11] Exact methods may receive more emphasis later in the curriculum and for only some students.

In a similar way, the calculus curriculum has always required that students develop substantial competence with essentially algebraic procedures, such as differentiation and integration, because progress in using calculus to solve problems requires such expertise. Although students in the past have developed the necessary expertise, for too many students this has taken so much time and energy that the (arguably more important) conceptual development has been less well developed. A good example involves integration, where students have been taught a suite of methods to handle various situations, such as integration by substitution, by parts and by partial factions. While there is some conceptual value in such procedures, their main place in the curriculum is as a means to an end, rather than because of their intrinsic mathematical interest. A hand-held device can evaluate integrals (numerically, symbolically or both) without focusing on the methods of integration employed, offering opportunities to reconsider the balance between the idea of integration and its many uses and the routines associated with evaluating integrals.

In other cases, the nature of school mathematics can be reconsidered because of the availability of hand-held technology. In the statistics curriculum, attention can be directed to statistical thinking and choices by students to represent data in various ways, rather than on the tedium of calculation or construction of graphical representations. A graphics calculator essentially provides students with a suite of data analysis techniques, so that attention might shift towards how, when and why to make use of these, and more attention can be directed at previously neglected aspects such as designing the collection of data to answer questions of interest and interpretation of statistical results.

Sequence

Another form of curriculum influence involves the sequence of ideas in the curriculum, which is also potentially influenced by the availability of hand-held technology. When thinking about the curriculum, under the assumption that technological support is available, some concepts appear earlier than might have been previously expected.

An elementary example of this on a calculator is when a subtraction results in a negative number or a multiplication results in a number expressed in scientific notation (before students have studied these. Similarly, a graphics calculator may produce results which suggest that some mathematical ideas may be introduced into the curriculum earlier than might otherwise have been the case. Two examples are shown in Figure 17. Both the evaluation of the square root of a negative number and the solution of the cubic equation $x^3 - 2x^2 + 2x = 4$ result in complex numbers.

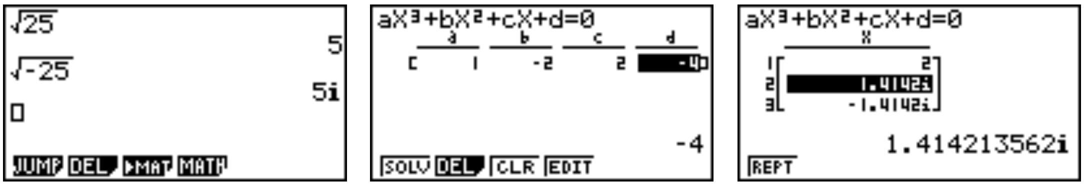

Figure 17: Unexpected appearance of complex numbers on a graphics calculator

More significantly, other aspects of the sequence of the mathematics curriculum may be affected by the availability of hand-held technology. Some of these are concerned with the capacity of a graphics calculator to display graphs of functions and allow them to be interrogated by the person using the calculator. The shapes of various families of functions (such as linear, quadratic, cubic and exponential) will be accessible to students at an earlier stage than previously, and without the necessity of tedious and extensive plotting of points. This might be expected to lead to a consideration of the nature of different kinds of functions earlier than previously.

In a similar vein, having ready access to a graph of a function changes some of the rationale for traditional approaches to the calculus. Before the technology was accessible, early approaches to the calculus focused attention on sketching curves in order to understand their shape, including their asymptotic shape, and also to identify key aspects such as local extrema. Some understanding of differential calculus and some expertise at finding and using derivatives was needed in order to consider such mathematical ideas. Similarly, the concept of the area under a curve was not introduced before the study of integration. However, as the screens in Figure 18 suggest, the mathematical ideas can be extracted from calculator graphs, which are accessible to students long before the study of calculus is introduced.

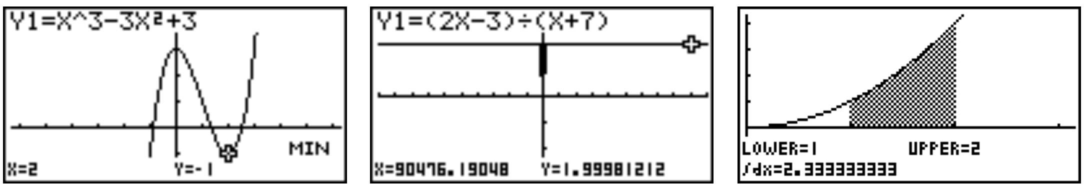

Figure 18: Early graphical introduction to local extrema, asymptotic behaviour and integration

Changes of this kind may serve to help us reconsider the role of the calculus in school: not merely as a means to answering questions about functions, but a means of doing so *exactly*, rather than with numerical approximations. Excellent numerical approximations are available to students well before the formal study of calculus, thus influencing the way in which we regard the calculus itself later in the sequence.

8. Issues for resolution

In this section of the paper, some key issues associated with the use of hand-held technologies in secondary school mathematics are identified and briefly explored.

Integration of technology

A major advantage of hand-held technology over other forms of technology for secondary school mathematics is the possibility of it being integrated into the curriculum, rather than being regarded as an addition of some kind. It is important that there be coherence in the place of technology for each of teaching, learning and assessment; hand-held technology offers the best prospects for this.

In many contexts, such as those in Australia [4], a key aspect of the integration of technology concerns its role in external examinations, especially high stakes examinations at the end of secondary school used for selection into universities. When technology is permitted for use in examinations, it is much more likely to be a part of the teaching and learning practices of schools, for obvious reasons. In the same way, when technology is not permitted in examinations, schools and teachers are understandably reluctant to use it in their teaching and learning activities. The small size, portability, relatively low cost and (perhaps ironically) limited mathematical power together render hand-held technologies much more likely to be permitted into formal assessment programs than, say, computers or the Internet. All of these characteristics enhance the likelihood of coherence and integration.

If technology is integrated into the curriculum, both curriculum developers and textbook manufacturers can develop materials that make sound use of it. Without widespread integration, it seems unlikely that the necessary curriculum changes (some of which were suggested in the previous section) can be seriously undertaken.

An additional aspect of integration concerns teachers, most of whom are unlikely to have extensive experience themselves with hand-held technologies. As noted earlier, graphics calculators are much more likely to be found in secondary schools than universities, so that even recent graduates may not have a lot of experience with using them as learning tools. Integration of technology requires that teachers be supported adequately, as elaborated later in this section.

Curriculum balance

Hand-held technologies of the kinds considered in this paper are frequently misunderstood as having *only* a computational role, especially and ironically, it seems, by people who do not make much use of them. Such a view is understandable, particularly when ‘calculators’ are naturally interpreted as devices whose main purpose is to ‘calculate’. However, this is a rather limited view of the many ways in which learning can be supported, some of which are described earlier with others described elsewhere (eg [14], [4], [5]).

A fundamental issue at stake in this respect concerns the significance of formal algorithms and procedures in mathematics. While none would doubt the importance of these, educators are increasingly questioning the balance of conceptual and procedural thinking in curricula. Extensive memorization and development of by-hand algebraic and arithmetical skills would seem to be both less necessary and less defensible in the opening years of the 21st century than they were a few decades ago, before technologies of the kinds discussed here were first developed.

In a similar way, formal assessment mechanisms, such as external examinations, seem now more likely to encourage conceptual development and careful mathematical thinking than smooth repetition of memorized procedures, so that the changing roles of a calculator to foster learning need to be considered in such a context.

Motivation

Reference has already been made to the role of external examinations as a source of influence, widely recognized in many countries (both East and West) as a key agent in directing the activities of both teachers and their students. When hand-held technologies are integrated into examinations, the motivation to use them thoughtfully and efficiently is considerable.

In addition, many teachers report that hand-held technologies themselves can be intrinsically motivating, as they offer students a responsive environment in which to experiment with mathematical ideas and explore connections between them. Many of today's students are accustomed to environments that are awash with technologies, and consequently are more inclined and less anxious than many of their teachers to experiment with them. As noted earlier, calculator display screens can themselves provoke students to explore new aspects of mathematics, when surprising results are given (such as the new kinds of numbers shown in Figure 17). The powerful range of software built into modern devices offers a platform upon which many interesting learning ideas and activities can be developed to motivate learning. (eg, see [12], [13], [14])

The work of the teacher

Effective use of hand-held technology requires teachers who are themselves competent and confident users of the technologies concerned. This is of critical importance: nothing important changes in mathematics classrooms without the teacher changing in some way. [9] Supporting the professional development of teachers for this task takes both time and effort. While publications such as [14] and websites such as [13] are important components of the professional development involved, experience in Australia suggests that hands-on time in workshops, together with the support of colleagues in a school are also key elements [4], [5].

By their very nature, hand-held technologies are personal devices that lend themselves to individual work or shared work among two or three students. Their use thus has some implications for pedagogy, with more emphasis on individual and small group work in a classroom than on whole-of-class instruction. Many teachers need help to develop expertise in such an environment, which differs from many traditional ways of teaching mathematics.

Of course, there continues to be a place for whole class instruction. A means of projecting a calculator for the whole class to see is useful; both overhead projection panels and emulators displayed via a computer projector can be used productively in classrooms. Some classroom time needs to be devoted to making sure that students can use their technology well, including thoughtfully deciding when not to use it at all. Similarly, productive class discussions can be provoked by a calculator display visible to all members of a secondary school classroom at once.

An interesting and recent development offers an opportunity to creatively mix both individual and collective work, through the connection of individual student graphics calculators to a computer projector, or through networking of calculators, so that the work of one student can be the subject of discussion by other students in a class, or engaged with by the teacher directly.

Changing the curriculum

Finally, it has often been recognized that it is very difficult to bring about deep change in mathematical curricula. There are many understandable reasons for this: a natural conservatism of teachers, especially older teachers; a reluctance to remove from the curriculum anything that the current generation of university and high school teachers themselves learned as students; the difficulties and anxieties associated with adjusting well-developed and well-understood practice; the reluctance to risk existing practices for new alternatives that have not yet stood the test of time; the extreme difficulty many teachers have to find any time to acquire new skills and take on new challenges, in the context of doing a job that is already very demanding; the ever-present threat in many countries of external examinations, which discourage adventurous and innovative teaching practices. In such circumstances, and with the history of curriculum development in mind, we should be cautious about expecting too much curriculum change too quickly.

Alternatively, if curricula do not undergo a process of gradual change, they risk a process of having to make very large changes periodically, which is much more difficult. Hand-held technologies offer an opportunity for teachers themselves to explore some new opportunities for teaching and learning in their classrooms, rather than relying entirely on external influences, such as curriculum and assessment authorities. It is clear that many teachers have found this opportunity valuable [eg 4], laying important groundwork for others to change and for larger-scale changes to be contemplated. From the perspective of curriculum developers, there seems to be more likelihood that technology can bring about curriculum change if it takes the form of hand-held technology than other variations, as these seem to have more prospects of being widely accessible. It seems most unlikely that large-scale curriculum change is possible, at least as far as technology is concerned, unless universal access to the technology can be contemplated. This probably explains why it is easier to change curricula on a local level (such as a single classroom, a single teacher, or a single school) than on a global level (such as a school district, a province or a country).

9. Conclusions

In this paper, we have surveyed the place of hand-held technologies in secondary mathematics education, relying on an analysis of the devices themselves, their mathematical capabilities and some educational consequences of using them. If secondary mathematics education is to be responsive to the changing technological world of the 21st century, there appear to be good prospects for using these kinds of devices to do so. While this is already the case in developed and affluent countries such as Australia and the USA, the arguments seem also to be relevant to less affluent and less developed communities, including many countries in Asia, Africa and South America.

By its nature, curriculum change is slow–much slower than technological change; however, hand-held technologies offer considerable promise to support sound curriculum change, provided teachers are given enough help and attention is paid to the curriculum constraints that militate against change, such as the expectations of universities and high stakes external examination systems. They continue to offer more hope for real change than do some more sophisticated technologies.

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Experiencing the Multiple Dimensions of Mathematics with Dynamic 3D Geometry Environments: Illustration with Cabri3D

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Abstract: In this paper, we start from the distinction between two processes : iconic and non iconic visualization. Both are involved in solving problems in geometry. The non iconic visualization consists in breaking down an object into parts of same or lower dimension. This cognitive process is critical for solving problems in geometry as very often the reasoning consists in establishing relationships between elements of the figure. However this process is not spontaneous and must be learned. 3D geometry is the source of new problems regarding iconic and non iconic visualization. On the one hand, iconic visualization is not always reliable as it is in 2D geometry, on the other hand non iconic visualization is more complex since it deals with a larger number of kinds of objects, from dimension 0 to dimension 3. The paper examines how 3D dynamic geometry environments with direct manipulation and the tools they offer may enlarge the iconic visualization and assist the non iconic visualization. The example of Cabri3D is used to illustrate the analysis.

1. Multiple dimensions of mathematics

At first glance, the notion of dimension refers to the dimension of geometrical objects such as point (dimension 0), line (dimension1), plane (dimension 2). But we also want to refer in this presentation to external representations or registers (in terms of Duval) used in mathematics for representing mathematical objects and relations or describing them, such as

- graphical representations like diagrams in paper and pencil or on a computer screen
- natural language.

This dual meaning of the notion of dimension is presented in this first section: the geometrical dimension of the objects and the registers in which mathematical objects are expressed or represented.

1.1. Deconstruction of a figure into components

Duval (2005) distinguishes between two ways of “seeing” a figure in 2D geometry or 3D geometry:

- an iconic visualization bearing on the shape: a child recognizes a round shape in a disc or in a circle and is able to distinguish it from a squared shape; the shape of a ball is also easily distinguished from the shape of a cube. The criterion for recognizing the shape bears on the contour of the global object. Shapes must be stable for being recognized.
- a non iconic visualization in which the figure is broken up into components or is transformed into another figure.

The iconic visualization of a cubic box does not consider the faces nor the edges or the vertices of this box. A strong evidence of this is the well-known difficulty children or even older students encounter when counting the number of faces or of edges of a cube even when the cube is available for manipulation. As long as the children do not have structured the cube into for instance the top and bottom faces and lateral faces, they must have recourse to marks on the material cube in order to memorize the faces already counted. This is why enumerating tasks are used in teaching as powerful tasks for fostering the construction of a mental structure for simple solid objects. Similar experiments with other solid objects (pyramids, prisms) carried out with students at the end of high school provide the same evidence (Mariotti 1992, 1996). Constructing a structure for solid objects require identifying the parts of this object of same or lower dimension and their relationships.

For example, a cube can be broken down
- into a set of polyhedrons, like three congruent pyramids with a common vertex and a squared base or two prisms on a triangular base (Fig.1)
- into a set of faces: it can be considered as a prism constructed on a squared face
- into a set of edges : four systems of three edges orthogonal to each other and sharing a common endpoint (points 1, 2, 3 and 4 on Fig.2)
- into a combination mixing edges and faces: it can be structured as made of two parallel squared faces connected by four edges perpendicular to the faces (Fig.3)

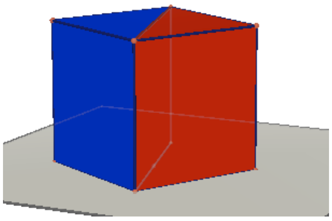


Figure 1 - Two prisms

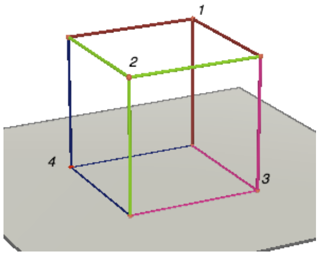


Figure 2 - 4 systems of 3 edges

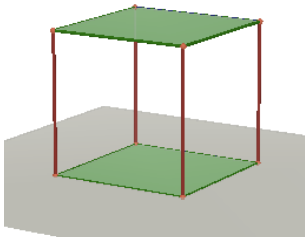


Figure 3 - 2 faces connected by four edges

According to the problem to be solved, one way of breaking down the cube is more appropriate than the other ones. This cognitive process of splitting up an object into subparts of same or lower dimension is the core of the non-iconic visualization and is required in any problem solving in geometry (Duval 2005). This process can be supported by adding some elements on the diagram and/or hiding other elements. The visualization of a cube as made of two prisms is more apparent when the common rectangle of both prisms is drawn.

Although geometry requires both types of visualization, the non iconic visualization is essential for identifying and reasoning about geometrical properties. The non iconic visualization must be learned. This is not an easy task as the iconic visualization which is immediate may sometimes hinder the non iconic visualization. The recognition of a prism is much easier for students when the base is horizontal. The prisms of the deconstruction of the cube (mentioned above) usually are not seen by students on a diagram if their base is not in a horizontal or vertical plane (Fig.4). The iconic visualization when the base is not horizontal or not vertical is more focusing on the corners of the prism than on its parallel edges.

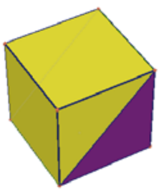


Figure 4 - Prisms in a non prototypical position

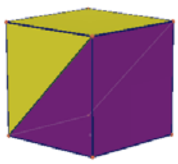


Figure 5 - Prisms in a prototypical position

The possibility of manipulating the cube to move the prisms in a prototypical position helps students see the half cube as a prism (Fig.5). This manipulation allows students to eliminate the conflict between the iconic and the non iconic visualization.
Too often the teaching of mathematics ignores that students have not yet constructed a non iconic visualization and does not help students to be able to develop it. Fishbein (1993) developed the notion of figural concept to give account for this dual role of figural and conceptual in geometry.

1.2. Graphical and textual registers

Each representation of a mathematical object brings some aspects to the fore, whereas it hides other aspects of the same object and thus affects the way the object is conceived. The meaning constructed by the individual is not only affected by the features of the representations available but also by the possible ways to use them. Mathematical activity requires manipulations of and operations on these representations. Various systems of representation in mathematics have been built over time, and these systems affect how we do mathematics. Netz (1999) argues that Greek mathematics was both supported and limited by the available media. Kaput (2001) claims that fundamental representational infrastructures, such as writing systems and algebra, play a major role in determining what and how people think and what they are capable of doing.
Learning mathematics and learning to have a mathematical activity require being able to choose the adequate register for the problem to be solved and possibly to move to another register. What we mean, is that the flexibility of moving between registers is not only supporting the construction of the meaning of a mathematical concept but is essential when «doing» mathematics. Duval (2000, pp.1.63-1.65) claims that understanding a concept requires coordinating at least two registers and being able to move spontaneously and rapidly from one register to another one. In geometry, two registers are indispensable: the graphical register of diagrams and the textual register. As a geometrical figure cannot be entirely determined only from its diagram, a textual description specifying the objects and relationships determining the figure is needed (Parzysz 1988).

It has often been stated that the difficulty of proving in geometry for students lies in the subtle role of diagrams in the elaboration of the proof. On the one hand, diagrams provide ideas about how to justify a statement and it would be impossible to write down the proof of a complex problem without a diagram (Laborde 2005). On the other hand, a proof cannot include elements coming from visual evidence. This subtle role of diagram is far from being used spontaneously by students. The role of the teacher is essential in making students more familiar with this game.
We believe that new 3D geometry environments offer useful tools that can be used by the teacher for the development of both a non-iconic visualization and flexibility between diagram and text. This claim will be illustrated in what follows by means of Cabri3D.

2. 3D dynamic geometry environments with direct manipulation

2.1. Amplifying the reliability of iconic visualization

One of the problems of 3D geometry is that 3D objects can be represented only in 2D even on computer screen unless these objects are represented by material solid objects (such as mock-ups). In 2D geometry, the iconic visualization could hinder the recourse to non iconic visualization but the evidence given by iconic visualization is generally reliable. It is no longer the case in 3D: it is not possible to be sure that two lines intersect from a diagram, or that four points or more are

coplanar. It was often observed by teachers that middle school or high school students believe that two lines intersect in 3D because they intersect on the diagram.
The possibility of changing the point of view in 3D dynamic environments with direct manipulation allows the user to obtain immediate visual evidence of such phenomena, as in the following Cabri 3D figures of two non intersecting lines (Fig.6) or of four non planar points (Fig.7).

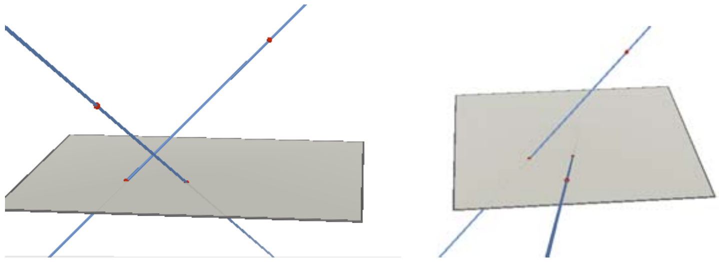


Figure 6 - Two apparently intersecting lines seem no longer intersect after changing the point of view

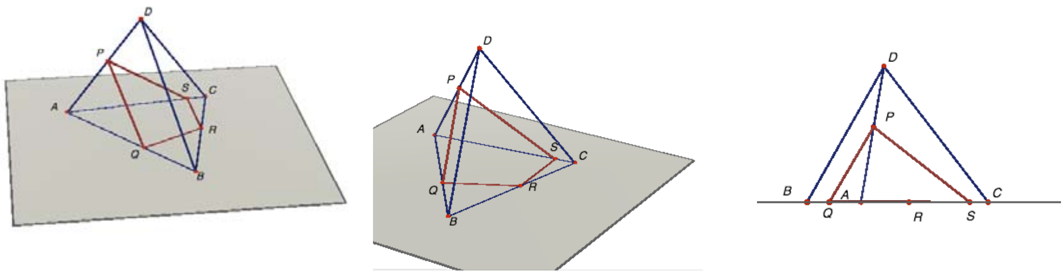


Figure 7 - Four apparently coplanar points seem no longer coplanar after changing the point of view

In 3D dynamic geometry environments, changing the point of view so that three points seem to be on a line provides iconic evidence whether four points are coplanar or not: if the fourth point seems to be outside of the line, the four points are not coplanar.

2.2. Assisting the development of non iconic visualization

Constructing an object by means of Cabri 3D can be done directly in some cases, like for example for usual polyhedra. However in most cases, it can only be done in constructing parts of this object by taking into account their mutual relationships. Construction tasks require a cognitive process of deconstruction of the complex object to construct. The novelty of 3D geometry environments is that this deconstruction is not only possible with 0 or 1 dimensional parts like on paper and pencil but also with 2 or 3 dimensional parts.
Construction tasks of 3D objects in those environments may call thus for an analysis of 3D objects focusing on components of dimension 2 or 3 and so contributes to a better knowledge of space. Before the availability of 3D geometry computer environments, construction by means of 2 or 3 dimensional parts were only possible with mock ups or games.
The very simple example of the cube will be used to illustrate this claim. The teacher asks the students to get rid of the tool « Cube[1]» and to construct a cube from a given square in the base plane. The most spontaneous strategy from students is to construct the edges of the cube and not the faces. They do it by working in the planes of the lateral faces obtained as perpendicular planes containing an edge of the starting square. They construct squares in each plane by using circle. This dimensional deconstruction structures the cube as a net of edges and consists in coming back to

[1] This possibility of costumizing the toolbar is available in Cabri 3D v.2.1.1

construction of squares in a plane. This is the most usual strategy for students as they are mainly familiar with 2D geometry. The students could use a more economical way of constructing each square by using a 3D tool transferring measurement from a point, i.e. a sphere. They could obtain vertices of the cube as intersection of spheres and perpendicular lines to the base plane (Fig.8). This is sometimes used by students, although this is not their spontaneous strategy as using a sphere to transfer measurement is not possible in paper and pencil.

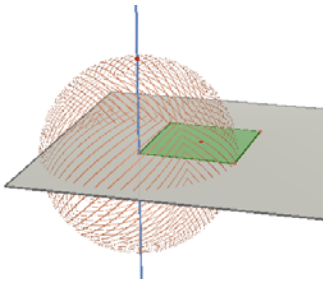


Figure 8 - Construction of a vertex of a cube from the base square

In Cabri3D like environments, there are still other strategies based on a deconstruction of the cube into 2D elements. A lateral square can be considered as the image of the base in a rotation with axis the edge of the base plane (Fig.9). The other lateral faces can be obtained as images of the previous one in rotations around the vertical axis of the cube (Fig10). Those strategies are not used by students. We hypothesize that the construction strategies learned in 2D geometry become obstacles to new strategies specific of 3D.

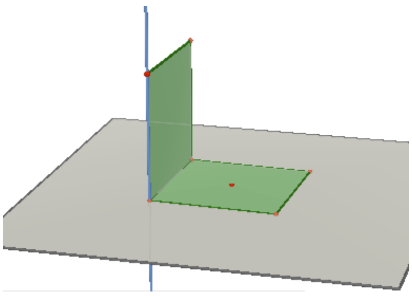


Figure 9 - A lateral face as rotated from the initial square

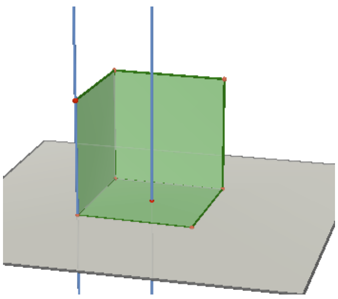


Figure 10 - The second lateral face as rotated from the previous one

We consider these strategies which are made possible by Cabri 3D as interesting from a learning point of view, for two reasons:

- they enlarge the scope of the non iconic vision
- and they use construction tools based on objects and properties of 3D geometry and as such contribute to the learning of 3D geometry.

In a paper and pencil geometry, these objects and properties are not operational for construction tasks, they are only operational in proofs. The strength of Cabri3D like environments is that those objects and properties become operational construction tools. They can be used in action in construction tasks before being used at the level of proof. In construction tasks, students can observe that these strategies provide the expected result. The visual feedback strengthens the power of these properties.

This claim of the contribution of construction strategies based on 2D and 3D objects and properties is convergent with the instrumentation theory. The use of a tool affects the way a subject solve problems depending on the possible actions made possible by the tool. In a first use of a tool, the subject (for instance a student) must learn how to use the tool to solve a certain kind of problems. He must develop an efficient instrumentation of the tool. Doing so the subject constructs knowledge about the tool but also about the knowledge domain of the problem.

Very relevant and beautiful examples of original and efficient instrumentation of Cabri3D are provided by Chuan (2006)[2]. They come from the lecture given by Chuan at ATCM 2006 entitled “Some unmotivated Cabri3D constructions”. “Unmotivated” was explained by Chuan as “non algebra, non routine, not found in Euclid, discovered accidentally, tailor made, so short, so beautiful, so fun”. For most all these reasons, we consider that facing students with these construction tasks is supporting students learning of a deeper non iconic visualization and thus a better knowledge of space geometry. These constructions are non routine and not found in Euclid because the tools they required were not available. Chuan insists on the efficiency of the constructions (short). This is a critical feature of problems that are able to promote learning of new knowledge according to the theory of didactic situations (Brousseau 1998). A new solving strategy is likely to be constructed by an individual when his/her routine or available strategies are tedious or inoperative for the problem. The beauty of the solution emerges from the conjunction of its efficiency and its unusual character. Let us comment one of the examples given by Chuan: the triangular cupola starting from an equilateral triangle (Fig.11).

Figure 11 – A triangular cupola

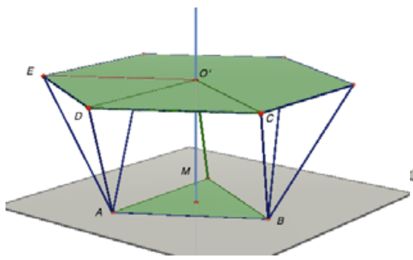


Figure 12 - Axis of the cupola

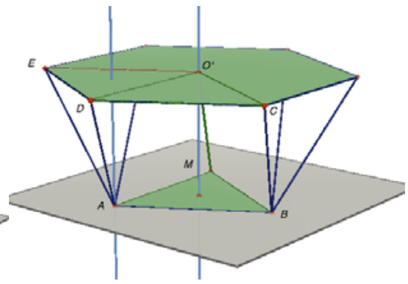


Figure 13 – Tetrahedra OEDA and O’MAB

Constructing the cupola requires analyzing it. The top of the cupola is a regular hexagon whose center O’ is on the perpendicular line passing through the center of the given triangle MAB (Fig.12). As soon as the position of O’ is determined, the hexagon can be constructed since its side is congruent to AB and the cupola can be constructed as the convex hull of AB and of the hexagonal face (tool Convex Polyhedron). How to obtain the distance of O’ to the base plane? One must notice that tetrahedron OEAD is regular and that its altitude is equal to the altitude of tetrahedron O’MAB since they are both regular tetrahedra with congruent edges (Fig.13). Point O’ can be constructed as the fourth vertex of the regular tetrahedron constructed on the given triangle MAB.

In this example, a regular tetrahedron is used as measurement transfer tool. This solution is based on a deconstruction of the cupola into 3D and 2D components of the figure. Of course a more traditional deconstruction into 2D and 1D components could be carried out. Such a deconstruction

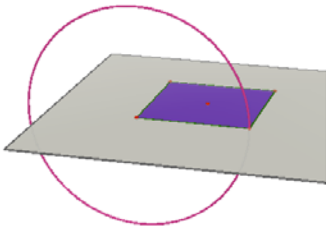
[2] at the address sylvester.math.nthu.edu.tw/ d2/talk-atcm2006-unmotivated/

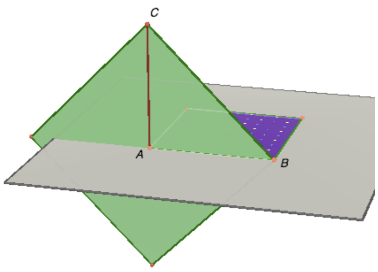
could be used in a strategy based on the determination of the plane ABCD and then on the construction of square ABCD in this plane. This would lead to a longer construction.

2.3 The role of the available tools in the environment

The available tools of the environment affect very much the possible construction strategies. We have just seen how the availability of regular polyhedra from a regular polygon makes possible the transfer of measures. Some other tools may play an efficient role in construction tasks avoiding long construction processes.
This is the case of the tool providing the convex polyhedron hull of four points or more (in particular of two polygons, of a segment and a polygon…). Constructing a convex polyhedron requires thus identifying the minimal number of elements of lower dimension determining the polyhedron.

The sphere is also a tool offering a transfer of measure from a point in any direction. The transfer of a measure in a plane from a point is also possible by using a circle around an axis (Fig. 14). Transferring the length of a segment on a perpendicular line from one of its endpoint can be done in the same way by using a square (Fig.15). The transfer of angles of some regular polygons is also possible by using the tools regular polygons around a line or a segment (Fig.16). All these tools are based on geometric properties of regular polygons and of circles.





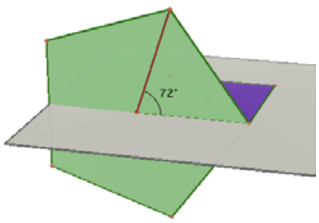


Figure 14 – Measurement transfer by means of a circle **Figure 15** – a square **Figure 16**- Transfer of an angle

The non iconic visualization is called twice

- in identifying that on the 3D figure to be constructed, two segments (elements of dimension 1) are congruent
- in adding another object of dimension 2 or 3 in which two segments with the same relative position are also congruent.

A process of going down and up in the dimensions of the objects is very much involved in the use of these tools. Therefore we consider that construction tasks in 3D geometry environments offering this kind of tools is very demanding in terms of geometric knowledge and conversely can be used by teachers to promote the development of non iconic visualization in 3D.

Transformations offered by Cabri 3D are also tools that can be used to construct 3D complex objects. Identifying that one part of the object is the image of another part in a transformation is also a matter of non iconic visualization.
The interface representing continuously the image in the construction process (Fig.17) of a rotation around an axis when the angle is increased continuously from 0 until its target value provides iconic visualization simulating a real motion in space and hence supports the non iconic visualization. We assume that this possibility of seeing a continuous movement between an object

and its image can be used by the teacher to provide imagery of movements in space that very often students do not have and to relate these movements to geometrical transformations. Students can be asked to simulate a triangle rotating around one of its sides, or a face of a polyhedron to rotate around one of its edges, or a polyhedron around an axis.

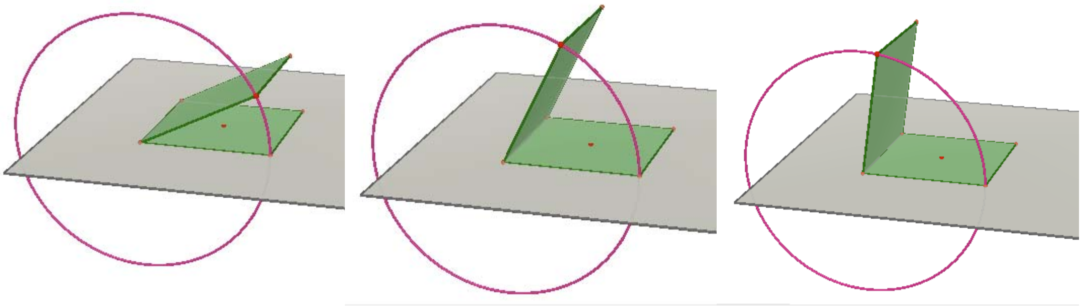
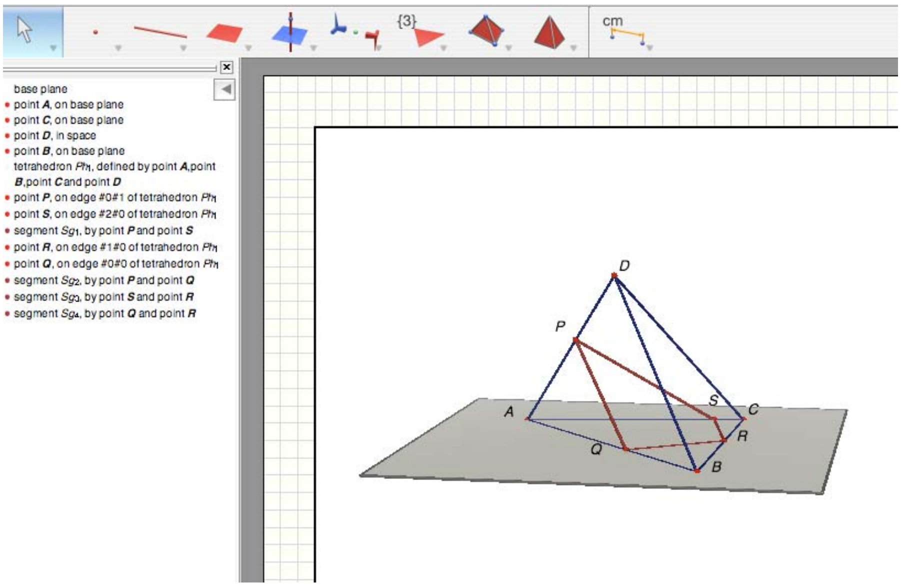


Figure 17 – A rotating square around an edge of another square

2.4 Complementary roles of graphical and textual registers

Let us come back to the example of four points apparently coplanar. As said above, changing the point of view allows the students to augment their iconic visualization and to invalidate this fact. However the reason why four points are coplanar or non coplanar can only be found by using theoretical knowledge. The textual description of a figure offered by Cabri3D provides the objects and their relationships (geometrical properties) that define this figure.



The surprise of the students discovering that four points they expected to be coplanar are not can be used by the teacher to motivate students to prove why. Students often have difficulties in not using evidence given by the figure in their proof: it is clear that three points are in the same plane and the fourth one is not. The existence of the textual description can be utilized by the teacher: the proof of the fact that these four points are not coplanar must be based only on information items given by the textual description.

The teacher can ask: Why do we know from the description that R, Q and S are in the same plane? Why do we know from the description that P is not in this plane? The link between the textual

description offered by the environment can help students to find in the description all information items about an object. When clicking on the object in the diagram, all the occurrences of the object in the textual description are highlighted and conversely when clicking on an object of the description, all the other occurrences of the same object are highlighted and its representation in the diagram is flashing. While the description provides an objective criterion for insuring that the proof is only based on properties used to build the figure, the interactive link between text and diagram allows the student to reason by using non iconic visualization coming from the diagram. It offers a way of overcoming the paradoxical situation which students face: they must elaborate a proof with the help of the diagram but are not allowed to refer to the diagram in the text of the proof.
Such a proof requires to make explicit some theorems and axioms of 3D geometry. This can be used to make university students, in particular preservice teachers, aware of the axiomatic system of geometry. Cabri II on the TI 92 was already used in this way to introduce university students to the axiomatic system of geometry and to do formal proofs (Perry Carrasco et al. 2006) or Italian high school students to construct a system of axioms (Mariotti 2000).

Teachers often consider that 3D geometry is hard matter to learn. In this paper, we pointed out two cognitive processes contributing to the difficulty of 3D geometry: iconic visualization and non iconic visualization. These processes are an essential part of any geometrical activity. We attempted to show that tools available in new 3D dynamic geometry environments may not only assist these cognitive processes but enlarge their range. Since it makes accessible in particular operations on 2D and 3D objects, it may extend non iconic visualization to those objects. Of course the teacher is still needed. One of his/her roles is to design challenging tasks requiring an extended non iconic visualization. Such tasks can even be fun when, for example, they consist in reproducing dynamic 3D objects given on the screen of the computer as in Chuan's examples.

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Modelling with real data and technology

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Abstract: There are several ways in which the teaching and learning of mathematical modelling may be motivated. In this paper, we describe how real data, together with technology, can provide a rich environment for mathematical modelling activities. Examples on modelling tasks for students at different levels are presented and described in detail. These examples not only illustrate the use of real data and technology in mathematical modelling activities, they also underline the importance of keeping such tasks in relevant contexts to provide added motivation for students.

1. Introduction

It is widely accepted that mathematical modelling is one useful way of injecting more real life activities into the mathematics classroom. However, although mathematical modelling has had a reasonably long history, mathematics and mathematics educators alike, cannot seem to agree on a precise definition for this area of work [6].

Some researchers view mathematical modelling as essentially the movement of a physical situation to a mathematical representation [15, 16], while others feel that all applications of mathematics are mathematical models [7]. Galbraith, however, believes that there is a difference between mathematical modelling and applications of mathematics [11]. In addition, Galbraith proposed that the teaching of mathematical modeling may take either an "open" or a "structured' approach. In contrast, Yanagimoto defines mathematical modelling as not just a process of solving a real life problem using mathematics but "applying mathematics which is useful to society" [17]. Still, there are those who believe that mathematical modelling consists of understanding, simplifying and solving a real life problem in mathematical terms [5, 8].

Notwithstanding the differing views, one aspect of mathematical modeling that seems to be commonly regarded as the most important is its connection with real world problems. To further emphasize and focus on this connection, perhaps mathematical modelling activities should involve the use of real data. Real data provides a rich and often relevant platform for developing, designing, learning and applying mathematical models.

The introduction of technological tools such as graphing calculators, computer algebra systems and dynamic geometry software has influenced the approaches to teaching mathematical modelling. Apart from enabling the user to perform computational experiments with the model, technology can help in making it more possible to work with real data [1], which are usually not as "clean" or "sanitized" as textbook or lecture examples.

In this paper, we discuss some examples of modelling tasks motivated by real data and aided or supported by technology. Real data are either collected or obtained through the literature or from some source in the open domain. In addition, the problems in these examples are both real and relevant to students and set in the context with which they are familiar. Hopefully, this will help raise the level of motivation in tackling these tasks.

2. What is mathematical modelling

While there may be different interpretations of mathematical modelling, for the purpose of the current discussion, we shall adopt the definition given in [2]. That is, we define mathematical modelling as the process of representing or describing physical systems or problems in the real world using mathematics so as to gain a more precise understanding of the problem. This process may be depicted as a flow of events as illustrated in Figure 2.1.

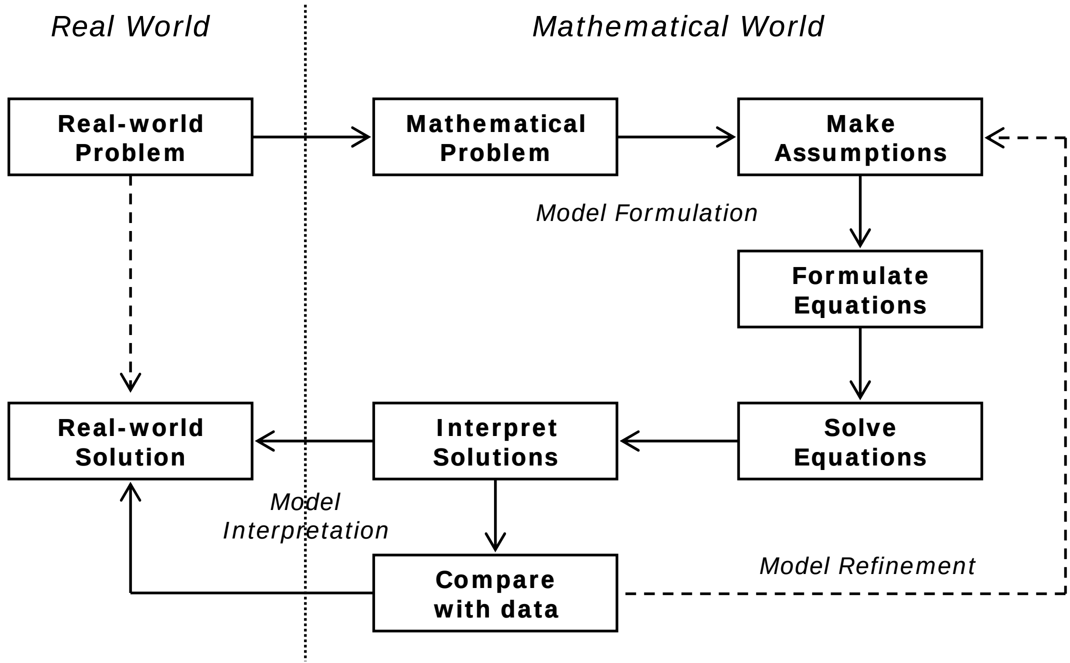


Figure 2.1 Mathematical modelling process

The process begins with a problem in the real world. In the first step, one hopes to describe or represent this problem in some mathematical terms. This may involve stating some variables, and forming some relationships amongst these variables. Usually, some assumptions have to be made in this model formulation phase. With these assumptions and an understanding of the mechanics of the problem, the next step is to formulate some equations to represent the dynamics of the system. The equations (or set of equations) thus form the mathematical model and the next step is to make an attempt to solve them.

The solution of the model equations may be where technology can come in handy. Quite often, the process involves estimation of some unknown parameter. Sometimes, some computational or numerical techniques may be involved, or perhaps some other approximations have to be made. At other times, collected or known data need to be analyzed before a solution method can be proposed or employed to solve the equations. Whatever the case, it is likely that when real data is involved, the use of technological tools will help in solving the problem.

To link the mathematical solution back to the real world problem, one needs to interpret the results or solutions of the model. It would be ideal if one could make some comparison of the results with some real data to validate the model. This represents the end of one cycle of the process because it is often possible to refine the model, improve the methods, or gather more data and repeat the steps in the process.

Although there may be many different versions of the process, most of them are similar and convey the same message essentially. However, the implementation and teaching approaches can vary a great deal, depending on what the teacher wishes to emphasize on. In the next section, we examine three examples, each of which illustrates the use of real data either collected by the students, or obtained from some real source. In every case, some form of technology or technological tool is used to solve or investigate the problem.

3. Examples

In this section, we consider some examples of how the process of mathematical modelling may be introduced in the classroom. In each of the examples, a problem with real data is considered, and a solution approach using a mathematical model and some form of technology is introduced. The examples presented are chosen to illustrate the range of possibilities of introducing modelling tasks to students at different levels of cognitive development.

Example 1: Largest Box Problem

Consider the problem of making an open-top box using a square piece of cardboard (of side S cm) by cutting a square of side x cm from each corner of the cardboard. The resulting piece is folded to form the box, as illustrated in Figure 3.1.

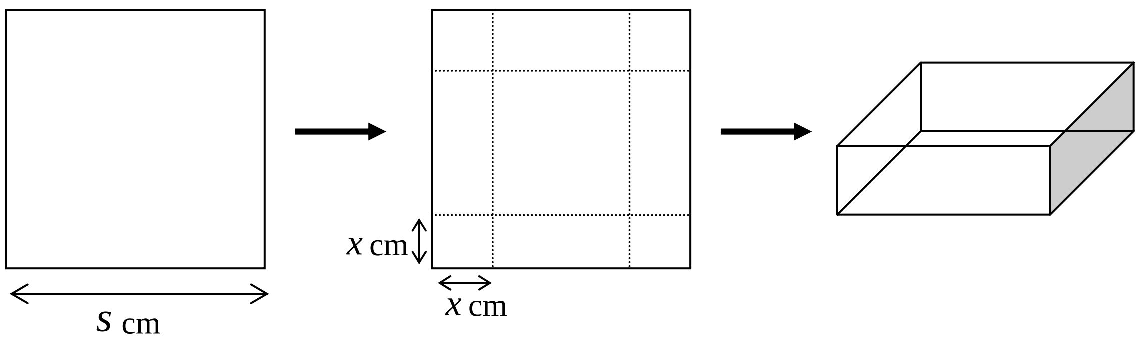
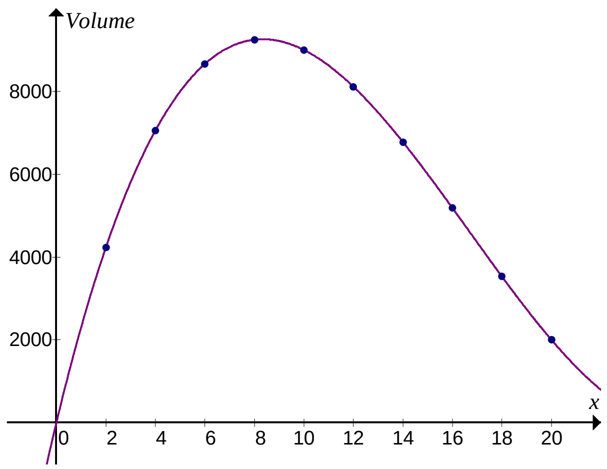


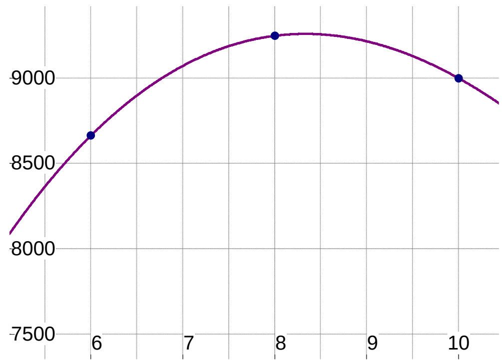
Figure 3.1 Largest Box Problem

The question is: what should x be in order to make the biggest box (in terms of volume)? This could be viewed as an optimization problem for an industry concerned with packing.

This problem was given to some Primary 6 students (11-12 year olds) in a Singapore school. It is important to note that at this level, students have very limited knowledge of algebra and, of course, yet to come across calculus. However, most would have been exposed to simple graphing tools. The students were given sheets of cardboard measuring 50cm by 50cm. In groups of 4 or 5, they made boxes with different values of x and computed the corresponding volumes. By collecting all the measurements, a graph of volume against x similar to that shown in Figure 3.2 may be plotted using a graphing tool such as Graphmatica or a graphics calculator. One could then repeatedly zoom in to estimate the maximum value of the volume and the corresponding value of x.



(a) curve fitting data points



(b) zooming in near maximum

Figure 3.2 Graph of Volume of box against x

Example 2: A logistic model for a disease outbreak

In 2003, some Asian nations including Taiwan, Hong Kong and Singapore were inflicted with an infectious disease known as Severe Acute Respiratory Syndrome, or SARS. In Singapore, there were 206 cases of infection. Among these, 31 lost their lives. The number of SARS cases in Singapore over the period of 70 days has been reported by Heng and Lim [12] and is available in Appendix A.

A modelling task that may be posed to Singapore's Junior College (17-18 year-olds) students is to construct a model (or apply a known one) to represent the SARS outbreak in Singapore. One plausible model based on the so-called "S-I" model for epidemics is described below.

The "S-I" epidemic model consists of two compartments, the susceptible population and the infected population (Figure 3.3). This is called the "S-I" model as it involves susceptible individuals ("S") becoming infected ("I").

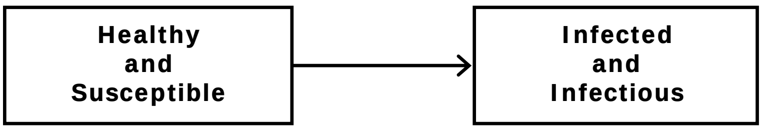


Figure 3.3 A simple epidemic model

Suppose $x(t)$ and $y(t)$ are the number of infected and susceptible individuals at time t (in days) respectively. We further assume that during the course of the epidemic, the total population of the community remains constant. Thus, $x(t)+y(t)=N$, where N is the size of the population. The spread of a highly communicable disease such as SARS may be modelled by the logistic equation given by

$$\frac{dx}{dt}=k\,x\left(1-\frac{x}{N}\right) \tag{3.1}$$

where k is a positive constant representing the transmission rate [3]. Equation (3.1) may be solved using the standard method of separation of variables and integration after performing partial fraction decomposition. Suppose the initial condition is $x(0) = x_0$, then the solution to the equation may be written as

$$x = \frac{N}{1 + \left(N/x_0 - 1\right)e^{-k\,t}}. \tag{3.2}$$

The transmission rate k may be estimated from data. For instance, we may use the data for the SARS outbreak in Singapore in 2003 (Appendix A) to find an estimate for k. To do so, we define an "average error",

$$E = \frac{\sqrt{\sum_{i=1}^{n} (\hat{x}_i - x_i)^2}}{n}, \tag{3.3}$$

where $\hat{x}_i$ and x_i are data values and model values respectively. A good estimate of k is obtained when E is minimised. One way to do this is to use the "Solver tool" in *Microsoft Excel*. The Solver tool essentially allows the user to minimise (or maximise) the value of a selected cell by varying the values of other cells specified by the user. In the present case, the Solver tool returns a value of $k = 0.1686$ (to four decimal places) with a minimum value of $E = 1.9145$. Figure 3.4 shows the graph of the model, with this value of k, plotted against the real data.

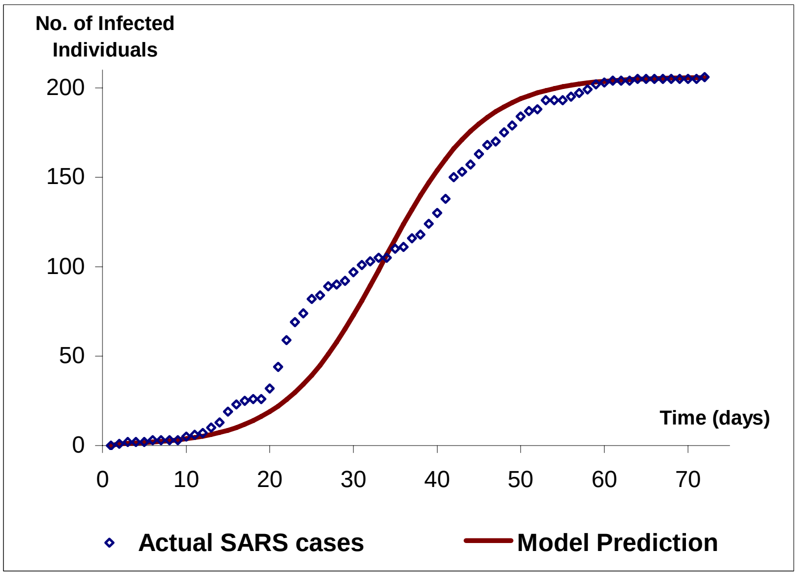


Figure 3.4 Graph of SARS outbreak model and real data

Although the model compares fairly well with the real data, it can be improved and refined, and this is discussed in detail in [3]. The challenge here may be to try and improve or refine the model to provide further insights into the dynamics of the outbreak.

Example 3: Modelling the spread of Dengue

Despite the *Aedes* mosquito control programmes, public health education and law enforcements, there has been a resurgence of dengue in Singapore in recent years. Dengue is transmitted when infected female *Aedes* mosquitoes, notably the *Aedes aegypti* and *Aedes albopictus*, bite human beings. After an incubation period of 5 to 8 days, dengue fever manifests itself with symptoms such as severe headaches, bone or joint and muscular pains, fever and rash. A complete recovery requires 4 to 7 days, and fatality is rare.

In Singapore, the annual incidence rate of the disease has risen progressively from 33 per 100,000 in 1993 to 166.2 per 100,000 in 1998. There are many possible reasons for the resurgence and increase in the number of dengue cases. It is suspected that Singapore's hot and humid weather plays a significant role in the breeding patterns of mosquitoes. As the daily average temperature in this city state remains fairly constant (at around 30°C) throughout the year, it is deemed that changes in the precipitation level (that is, rainfall) are probably the main environmental factor that is linked to the larvae densities of the *Aedes* mosquitoes. Thus, data on the number of dengue cases, larval densities and rainfall in Singapore may prove useful in providing insights to the dynamics of the disease. Such data for the year 1996-1997 are available in [4] (Appendix B).

The challenge, which may be posed to undergraduate students, is to construct a plausible model for the spread of dengue in Singapore based on the available data. One approach is suggested below.

In modelling the spread of dengue, one has to consider the host (that is, the human being) and the vector (that is, the mosquito) and the interactions between them. A possible compartment-model, in which each population type is compartmentalized, is depicted in Figure 3.5.

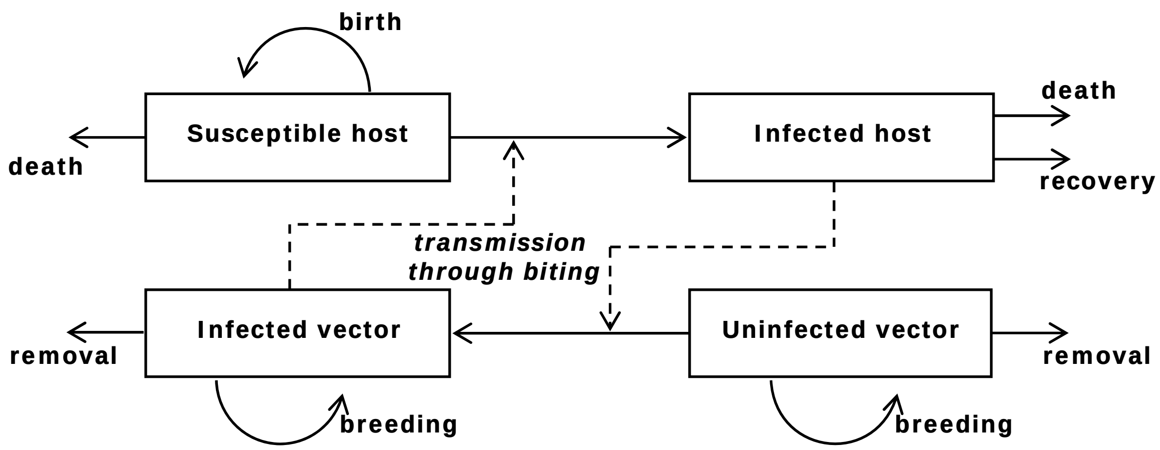


Figure 3.5 A basic model for spread of dengue

Let $S(t)$ and $I(t)$ be the number of susceptible and infected host individuals respectively, and let $N(t)$ and $M(t)$ be the number of uninfected and infected mosquitoes respectively. The model presented in Figure 3.5 may be represented by the following set of differential equations

$$\frac{dS}{dt} = (a - b)S - \upsilon SM \tag{3.4}$$

$$\frac{dI}{dt} = \upsilon SM - (\alpha + \gamma)I \tag{3.5}$$

$$\frac{dN}{dt} = (d - c)N - \omega NI \tag{3.6}$$

$$\frac{dM}{dt} = (d - c)M + \omega NI \tag{3.7}$$

where a and b denote the birth and death (from old age or other illnesses) rates of the host respectively, c denotes the removal rate of the vector by natural death or insecticides, and d denotes the breeding rate of the vector. The transmission efficiency of the host is given by υ while that of the vector is given by ω. The death rate due to the disease, and recovery rate from the disease are represented by α and γ respectively.

While this set of equations may seem complete, the underlying difficulty is the lack of information on the parameter values. Some of these (such as c, d, υ and ω) are non-measurable biological quantities and it is difficult, if not impossible, to obtain accurate values of these quantities. The remaining parameters a, b, α and γ may be estimated from known data.

Although it is possible to solve the set of equations using estimated parameter values, one missing component of this model is the impact or effect of environmental factors on the breeding rate of the vector. Mosquitoes breed rapidly in hot weather and where there are ample breeding sites. It turns out that in Singapore, breeding sites tend to increase during the rainy season, and the hot weather that follows promotes mosquito breeding. It is therefore not unreasonable to assume that the amount of precipitation (or rain, in this case) is linked to larvae densities.

The challenge is to find a way to model the relationship and incorporate it into the set of equations. One of the simplest ways to do so to assume that the population of infected vector is a fraction of the larvae density, and that this fraction depends linearly on the amount of rainfall. Thus, we may write

$$M(t) = P_i L(t) \tag{3.8}$$

$$P_i = \frac{P_{\max}}{R_{\max}} R(t) \tag{3.9}$$

where $L(t)$ and $R(t)$ are the larvae density and rainfall level at time t respectively, P_i is proportion of infected vector, and $P_{\max}$ and $R_{\max}$ are the maximum values of P_i and $R(t)$ respectively. $P_{\max}$ will need to be estimated experimentally.

Based on this assumption, Equations (3.8) and (3.9) can be used to replace Equations (3.6) and (3.7) in the model. Using the data given in Appendix B, the system of equations may be solved numerically using any suitable numerical method such as the Runge-Kutta method of Order 4.

Results from a typical run of the Fortran program written to implement the method is shown graphically in Figure 3.6. In this case, suitable parameter values, as reported in [4], that may be used are $a = 0.012$, $b = 0.007$, $\upsilon = 5 \times 10^{-7}$, $\gamma = 0.99$ and $\alpha = 3 \times 10^{-4}$. Figure 3.6 compares actual dengue cases with the model predictions using $P_{\max} = 0.7$ and $P_{\max} = 0.9$.

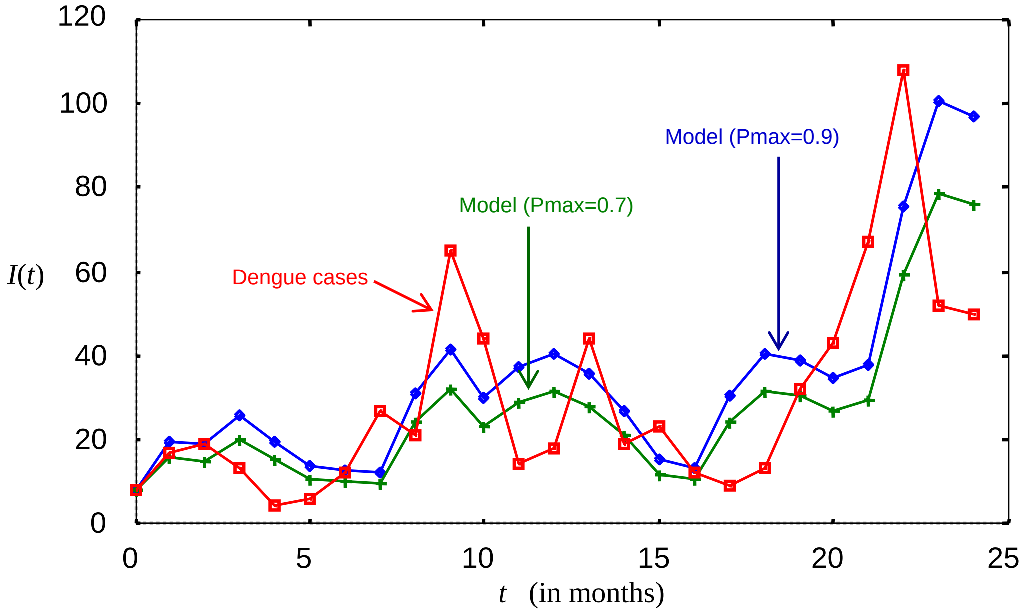


Figure 3.6 Actual dengue cases and model predictions

As can be seen from the graphical output above, the model compares fairly well with recorded data. The two spikes (around Months 9 and 22) in the dengue cases are reasonably well predicted. These spikes represent the dengue outbreaks during the period in question. While it is true that it can certainly be further improved, the present model serves well as a first approximation upon which more accurate models may be based.

4. Discussion

As can be seen, the common thread in all the examples discussed in this paper is the use of real data. In the first example, real data were collected, while in the next two examples, they were obtained from published sources.

In each case, the real data used had provided something concrete for students to work on. In the "biggest box problem", students could construct the boxes and then take measurements. Of course, the problem may be solved easily using calculus. However, the main point in this example is to illustrate how a problem or task may be tackled by modelling with real data. In addition, the use of appropriate technology can help fill some gaps in mathematical skills or knowledge, although one has to bear in mind not to trivialize the mathematical tasks. In this particular instance, technology has served to enrich the learning possibilities as students who do not have the "assumed knowledge" will still be able to construct the model physically and mathematically to tackle the problem.

The next two examples discussed problems related to disease outbreaks and control. In both cases, the context is real and relevant, and of immense public concern to people living in Singapore.

Although the problems are far from being solved, students who embarked on these modelling exercises would have gained a deeper appreciation of real life application of mathematics. The use of real data in modelling a SARS outbreak and dengue transmission has given students a rich mathematical experience in a highly relevant context. Moreover, in using of the real data, the application of technology has helped in reaching the solution of the models in both cases, making the mathematical modelling tasks more accessible.

While it may be possible to look for examples from textbooks or other sources in modelling epidemics or disease outbreaks, the use of data in real life provided learners with realistic opportunities to connect mathematics to important social issues or problems. Even if real data are found or cited in textbook examples, they may not provide as rich an experience as actual real life examples placed in a context that students can identify with. Real problems with real world concerns serve to heighten student interest and motivation [10].

5. Conclusion

Mathematical modelling can be thought of as a form of a scientific inquiry process for mathematics. This paper discusses some practical aspects of this process that may serve to enhance the learning experience in mathematical modelling.

The use of real data in real life problems helps the learner link the mathematical world and the real world. This appears to be an area of weakness in students and has been reported by various researchers [9, 14]. Examples such as those described in this paper will help students build connections between the mathematics they know and the world they live in.

Apart from using real life examples with real data, the timely and appropriate use of technology can help empower students in their modelling tasks. All examples presented in this paper have made use of some form of technology. Notwithstanding the usefulness of technology, it is important to note that the learning of mathematical modelling should not be limited to specific technological tools [13]. In other words, the focus should still be on using mathematics to model or solve a real world problem, and the tools are meant to support the process.

To truly experience the process of mathematical modelling, it would help if one deals with a real problem, handle real data or even test the validity of one’s models through various experiments, including computational or numerical experiments. Performing these tasks by hand without the aid of technology can be tedious, time-consuming and sometimes counter-productive or even impossible. A modelling task set in a real life context and supported by technological tools can provide a very enriching and engaging mathematical modelling experience.

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Appendix A: Data for the SARS outbreak in Singapore

Day	Number	Day	Number	Day	Number
0	1	24	84	48	184
1	2	25	89	49	187
2	2	26	90	50	188
3	2	27	92	51	193
4	3	28	97	52	193
5	3	29	101	53	193
6	3	30	103	54	195
7	3	31	105	55	197
8	5	32	105	56	199
9	6	33	110	57	202
10	7	34	111	58	203
11	10	35	116	59	204
12	13	36	118	60	204
13	19	37	124	61	204
14	23	38	130	62	205
15	25	39	138	63	205
16	26	40	150	64	205
17	26	41	153	65	205
18	32	42	157	66	205
19	44	43	163	67	205
20	59	44	168	68	205
21	69	45	170	69	205
22	74	46	175	70	206
23	82	47	179		

Appendix B: Dengue Cases, Rainfall and Larvae density

Month	Dengue Cases	Rain (cm)	Larval density	Month	Dengue Cases	Rain (cm)	Larval density
0	8	63	5896	13	44	159.1	5973
1	17	227.46	5070	14	19	65.54	4257
2	19	210.81	4801	15	23	200.4	2959
3	13	261.48	5962	16	12	88.78	4351
4	4	157.6	3813	17	9	111.03	8655
5	6	120.43	4947	18	13	187.04	7100
6	12	129.5	4436	19	32	123.7	7289
7	27	65.48	4878	20	43	82.46	6034
8	21	228.57	9264	21	67	152.82	7964
9	65	117.45	6849	22	108	295.96	12313
10	44	188.48	5281	23	52	88.46	10285
11	14	129.15	8512	24	50	149.77	8700
12	18	103.25	5925				

Managerial Issues of Teaching Mathematics

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Abstract: Mathematics belongs among the least popular subjects. The reasons are quite known and well described for example in [1]. For that reason, we concentrate on methods which could improve its image among general audience. Instead of talking about internal problems of mathematics, our paper discusses the issue as a managerial problem. We use an analogy: *What has to be done if Teaching Mathematics would be an enterprise with a similarly bad reputation?* Based on the parallel, we propose long-term solutions and later exemplify them. As shown, the most of them requires substantial changes in mathematicians' minds and in their approach to their teaching methodologies.

1. Introduction

Let us discuss the case of a company X:

- *X*'s public reputation of is very bad.
- Just a few employees of X care about improving its picture.

One does not need to be a businessman or economist to guess that *X* faces a gloomy future.
Teaching mathematics satisfies both above criteria:

- The public reputation of *Teaching Mathematics* is very bad.
- Not so many individuals try to improve the popular picture.

Based on the analogy, we can conclude: if there would be an enterprise named *Teaching Mathematics*, it would have probably already crashed and *Mathematics* disappeared from many schools. We have to ask ourselves:

- Is there precedence to such a case?
- Can we prevent its repetition?

A precedent exists. The *Latin* language was the lingua franca of scientists during the Medieval Ages and was taught as a compulsory subject at high schools long after – in Central European countries for a big portion of 20th century. In a stepwise manner, its dominant position has been diminished. Now it occasionally appears in the form of an elective. Such a limited offer seems to fully satisfy its market's demands. Voices for its return to a more significant position practically do not exist.

Mathematics teachers and enthusiasts should therefore ask themselves: *Can we prevent Mathematics to become a follower of Latin?* Due to his background as a Professor at the Management College in Bratislava and the author of the course *People, Technology and Management* for the Laureate Program of the University of Liverpool, the author proposes his response based on a managerial point of view. At first glance, making interconnections between Mathematics and Management may look strange. For that reason, our first aim is to convince our reader that there are good reasons to do so. Above we stated: "*There is no public demand for the return of Latin*". Evidently, if parents would feel the need, they would form a strong pressure to the establishment to introduce it – and the authorities would obey. The present situation of Latin is the result of non-existence of such pressure – a typical "demand-offer" relationship well-known by salesmen.

In our argumentation we are looking for additional relations, too. Below we therefore consider *Teaching Mathematics* as a (virtual) company and try finding answers to the principal questions:

- What's wrong with Teaching Mathematics that makes its reputation so negative?
- What business principles can propose ways out?
- What does their application mean for our community?

The last question is demonstrated by a series of examples which are "purely mathematical" despite the fact that they have been inspired by our managerial considerations. In such a way they illustrate that a bridge between the disciplines exists and can be efficiently applied for benefits of the both.

2. What Danger Do We Face?

We start with using managerial terminology for clarifying the subject of our discussion. When people talk about "bad reputation of *Mathematics*", their criticism may address either science itself or its teaching in schools. In our opinion, only the latter is correct. Those who deal with Mathematics as professionals know its importance well. The critics are predominantly grouped from people who have got just a limited contact with it during their school education. Despite the fact that throughout their adult lives they have a little to do with it, their bad feelings and attitude persist.

From the managerial point of view, *Mathematics* is in the position of the parent company; *Teaching Mathematics* is its daughter. The verification of this statement is quite obvious. If there would not be *Mathematics*, there would be no *Teaching Mathematics*. (If there would be no car producer, car dealers could not exist – they distribute what was produced by others.)

As usual, the dependence is reciprocal. Car dealers cannot exist without car production, but car production becomes meaningless without cars' sale. Similarly, there would be no mathematics education without mathematics but mathematics would face the extinction without (some extension of) teaching it. Compare the situation with *Latin*. It is still taught at a limited scale – just sufficient for "keeping it alive". Similarly, *Teaching Mathematics* would unlikely fade away entirely. It would persist at a limited scale necessary for its survival.

For that reason, our questions do not address teaching "*Mathematics for its followers*". In this form, it will survive anyway. What we do face is a danger of extinction of teaching "*Mathematics for others*". Our Latin lesson says: "To prevent the disappearance, concentrate on creating demand. The greater is public demand, the lower becomes the probability of extermination." While forming demand, we have to prioritize *Mathematics for others*. Danger coming from this "market segment" is more imminent as this is the group of "potential troublemakers". If we win their endorsement, the danger of extinction vanishes.

Unfortunately, the group is specified by exclusion. Consequently, it is not a homogenous body. For that reason, a single strategy will hardly work – the approach described in our paper should be considered as an example as well as our call for alternative ones. Their combination could lead to a better image of Mathematics education in eyes of public.

3. Setting up Priorities

As explained above, the relationship between *Mathematics* and *Teaching Mathematics* resembles that between "parent" and "daughter" companies. The parent establishes its daughter(s) for special purposes: banks set up leasing companies or assets management organizations; car dealers establish brand-oriented dealers or spare part producers. The daughters allow their mothers concentrate on their core activities. *Teaching Mathematics* releases its mother from the need to solve daily tasks of spreading its earlier discoveries among population. As a result, the mother can concentrate on exploring and finding new knowledge.

As managers, we may then ask: *Does Teaching Mathematics distinguish itself enough from its mother - Mathematics?* Again, an analogy is handy. Everyone likely agrees that everyday functions and work organization of car dealers substantially differ from that of producers. Similarly, in our paper we discuss the activities of *Mathematics for others* in the maximum degree of isolation from its mother. We pose ourselves into the role of a management that designs its new daughter. Our aim is to establish it in a manner that will help to maximize benefits for both mother and daughter and to minimize unnecessary overlapping between them – we will form the "flesh and spirit" of the new body. Tasks of this sort are solved during the strategic planning – the function each company must regularly execute every few years. So, even if *Teaching Mathematics* exists for centuries, its aims and methods should be revised and redesigned at regular intervals. We can therefore regard our task as another stage of this standard process – the moment when we analyze whether or not there are reasons for building a new daughter or changing its methods of operation.

Wikipedia [2] characterizes Strategic Planning in the following manner: *Strategic Planning is the formal consideration of an organization's future course. All strategic planning deals with at least one of three key questions:*

"What do we do?"
"For whom do we do it?"
"How do we excel?"

4.1 What do we do?

The first question targets the current state of the organization. It should take into account not only the subject of the company's interest but also effectiveness of its procedures and quality of its output. Many analyses documented the inferiority of mathematical education for our market segment – for "all others". It is often expressed by the statement "I hate Mathematics" [1]. The reason for the loathing comes from its most frequently applied method: "*Repeat what mathematicians do and you will (hopefully) start understanding Math*". In principle, "learning by doing" is not a bad idea and it functions in many cases. At the same time, it presumes that the learners are capable of understanding the output of their efforts and can evaluate its correctness and quality. This is important for building the feedback that enhances learners' further progress.

The misconception is somewhere else. The execution of formal manipulations – the simplification of complex expressions or construction of geometric bodies – is not the primary task of mathematicians. They first analyze the problem in their minds. (If) they get a promising hypothesis, they try to verify it. Using formal algebraic manipulations and geometric constructions is therefore a supportive method, not their main aim. Traditional *Teaching Mathematics* went from this concept too far. Too much accent is done to their performance changes learning to drill. Drill prohibits understanding.

So, the lesson from the first question should be: *Formal algebraic manipulations and geometric constructions themselves are not the best way to understanding Mathematics. If we want to simulate the work of mathematicians in a more realistic manner, we should concentrate on formulation of problems, forming hypotheses and verification of solutions.*

4.2 For whom do we do it?

The second question addresses another key aim. The target groups of *Mathematics* and *Mathematics for all others* radically differ. The former one is formed of professional mathematicians and active users of Mathematics (engineers, designers, physicists, biologists, and other specialists) who need new or improved methods for solving their problems. The latter one consists of much more general audience. As our intention is to increase the demand, we have to demonstrate the usefulness of mathematics to them.

The "usefulness" can be expressed by various means: practicality, beauty, joy, etc. In fact, many mathematicians and teachers of mathematics claim that they are familiar with them and try to implement then into their classrooms. The question is whether their perception of "beauty" coincides with that of their pupils and students. For example, everyone will agree that the fractal in Figure 1 (taken from [3]) is beautiful but people may argue what makes it nice. Mathematicians will probably prefer the possibility to define it by a formula – and to deform and modify it by entering different parameters. To artists, similar characterization of beauty will likely sound strange. To minimize misunderstandings, mathematicians should allow the artists to manipulate with parameters and let them to feel their influence to the form and the size of the picture. Later, they could ask them to predict forthcoming modifications caused by changing parameter values. Notice that even without knowing the formulas both groups are coming much closer to "genuine mathematical activities" – forming hypothesis and verifying them than with a traditional classroom.

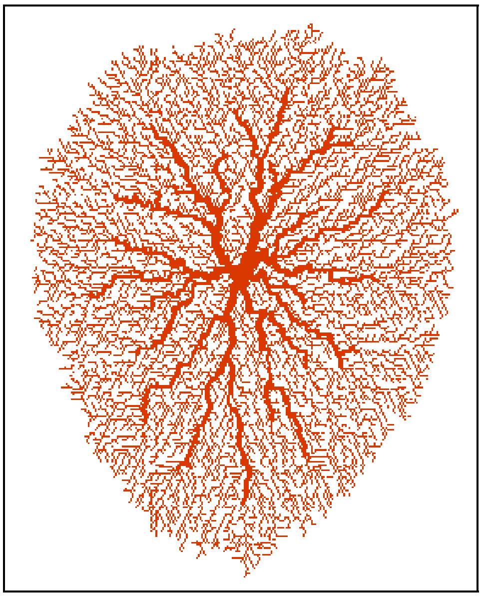

Figure 1. *Fractal*

Using fractals and other complex structures, we can also demonstrate the general character of mathematics. Ask several people what the object resembles – their answers will differ. It could be a tree viewed from above, a coral, a river system ending in a lake in the middle of the picture, windpipes in lungs, and others. All of them are defined by the same formula!

A lot about Mathematics can be told and explained without routine calculations and formula memorizing. It does not mean that we entirely reject teaching them. We just underline that their role seems to be exaggerated. Our second lesson sounds: *For the most of population, it is difficult to understand raison d'être of mathematical concepts, formulas, expressions, and manipulations unless they assign meanings to them. Individual interpretations do not need to be unified – each person may have his/her own.*

This idea explains why even illiterate persons can perform four basic arithmetic operations – they understand their meaning and importance. Troubles start when similar connotations are lost.

4.2 How do we excel?

The third question addresses the role of the teacher and its specifics. As this conference is oriented to Computers in Education, we should likely prefer methods intensively exploiting IT but there can be other parallel methods of education with different accents.

What then appears is the problem of compatibility of these different streams. Contemporary mathematics is extremely large. It consists of many fields, each of them with its specific problem-solving methods, terminology and results. Professionals in one field have often problems to understand those belonging to other ones. Designers of different courses have to decide which fields to select and why. Again, the situation resembles that of car dealers. No one sells all cars. They specialize to certain brands, categories and price intervals. This implies that *Teaching Mathematics* may develop into a series simultaneously acting "companies" each of them possibly "selling" different products. Some branches will be relatively independent e.g. graph theory or logic. The same holds for their problem-solving methods. Proofs in *Logic* are very formal; proofs in *Graph Theory* often use pictures and intuition. The methods applied in Logic are appropriate for *Mathematics for followers*, whilst Graph Theory fits well to *Mathematics for others*.

The "sister companies" – alternative branches of *Teaching Mathematics* – that will stress them have to be (to a certain) degree independent. At the same time, all cars are "compatible" in the meaning that driving skills learned using a particular car can be easily transformed to any other. So, a driver can change a car without necessity to be retrained. Designers can learn from this analogy, too. Their course content does not need to be identical, learning styles may substantially differ. However, there must remain something that helps characterizing all of them as "mathematically associated". The most affordable way of achieving such alternatives is a stepwise evolution from the traditional "Arithmetic – Algebra – Calculus" approach. A high proficiency in formula manipulations was a necessary pre-condition for solving equations and inequalities as well as a prerequisite for Calculus. Today, Computer Algebra Systems perform these operations much faster and with higher precision. Much less time needs to be spent on drilling them. The saved time can be devoted to activities that are more important to "non-followers" and more welcome by them. Thus, our last lesson says: *Release your students from drill. Give them opportunity to appreciate the core of mathematics in a very individual way whenever possible designed directly for them.*

Some of these alternative branches can target quite narrow "market segments". Similar moves have already been done. For example, Haapasalo and his colleagues wrote alternative textbooks [4], [5] and [6] for vocational schools primarily based on the concepts needed in their particular professions. In next chapters we show an approach based on separation of the roles between learners and information technology. While IT is much better in performing any sort of standard procedures, humans excel in considerations and putting things into the context. We try to intensify the separation of their functions and develop our student's intuition in recognizing mathematical aspects of reality.

To close our trip to *Management*, we can conclude that results are not entirely revolutionary. Most of them are known for long:

- Building relationships between mathematics and reality;
- Student-centered problem-solving approach;
- Satisfying individual interests and needs of learners.

What is new is the fact that – in order to satisfy its clients' needs – *Teaching Mathematics* must place them as its priorities. Its mathematical content must become just an "integrated part of the game". The current formal content of classes must be (intelli)gently covered by means that make it attractive. In our paper, we focus on how to make it in a user-friendly, “edible” and “digestible” way.

4. Making Mathematics More User-friendly, Edible and Digestible

4.1Mathematicians as Commons

As a part of a CASIO FX-9860G project supervised by Wei-Chi Yang (and later edited by L. Paditz in [7]), the author worked on a series of educational programs aimed to help understanding mathematical concepts, notions and procedures. Hyperbolic functions were among his assigned tasks. Suddenly, the author realized that what he learn during his university study were isolated facts like their definitions

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

and their derivation rules. He had no idea how the curves look and whether they appear in reality (and if so, where). His knowledge was based on memorizing, not comprehension. As his perception of mathematics has changed since his graduation, he wished to form teaching material in accordance to the above principles.

Luckily, at the present time, teachers and students can be better informed than it was typical for the period of my study. The Internet pages offer relevant and reliable information. Among many pages referring to hyperbolic functions, three [8, 9, 10] provided enough data for the author to accomplish his task.

Above formulas describing hyperbolic functions can hardly attract attention of a common learner. We recommend an opposite approach. One should propose simple problems which produce hyperbolic functions as their results. There is such a problem: *Hold the ends of a rope in your hands. The hanging rope forms a curve. What curve is it?*

Many people wrongly guess that the curve (named *catenary*) is a parabola. Even famous mathematicians (including Galileo Galilei, at about 1600) made the identical mistake. In 1669, Jungius found the right formula. The curve is cosine hyperbolic. It can be drawn using a graphing calculator – see Figure 2.

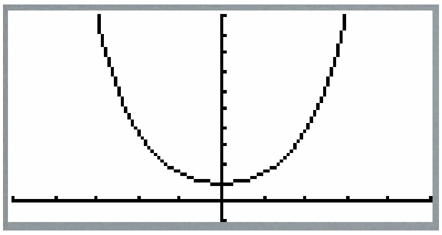

Figure 2. *Catenary*

We suggest teachers telling to their students about Galileo’s mistake. Mathematicians are often taken for as people never making errors. Showing a fault of the well-known personality can make him more human, more similar to “all of us”. We believe that frequent confessions of the movie stars and popular singers: “I was always bad in mathematics” originate in their desire to express: “I am one of you, I am not perfect”. Pointing to errors of top scientists may serve to a similar purpose. Mathematicians are people like any others. It is quite strange if for example biologists or geologists are appreciated as common humans, but mathematicians are not. While this picture in human minds persists, the negative (or better saying repelling and discouraging) picture of mathematicians will persist.

4.2 Visualization of Formulas

During the next step, the students can simulate the process of changing the shape of the curve by moving their hands closer (farther) to (from) each other. The formula of parameterized version of the cosine hyperbolic looks as follows

$$y = a * \cosh(x/a) = 0{,}5 * a * (\mathrm{e}^{\frac{x}{a}} + \mathrm{e}^{\frac{-x}{a}})$$

The learners equipped with advanced calculators with built-in hyperbolic functions do not need to know the formula. To perform simulations, they simply type them with varying coefficients. In the case shown in Figure 3, the parameter *a* changes from 1 to 4. Figure 4 shows the result. It simulates stretching hands with the rope – the minimum of consecutive curves goes higher and higher.

Figure 3. *Formulas with 4 different parameters*

Notice that we could create the identical picture using different ways. Our example uses the "pure" cosine hyperbolic:

$y_1 = cosh(x)$

$y_2 = 2*cosh(x/2)$

$y_3 = 3*cosh(x/3)$

$y_2 = 4*cosh(x/4)$

The other method may apply more complex formulas with exponentials:

$$y_1 = \frac{1}{2} * (\mathrm{e}^{x} + \mathrm{e}^{-x})$$

$$y_2 = \mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-\frac{x}{2}}$$

$$y_3 = \frac{3}{2} * (\mathrm{e}^{\frac{x}{3}} + \mathrm{e}^{-\frac{x}{3}})$$

$$y_4 = 2 * (\mathrm{e}^{\frac{x}{4}} + \mathrm{e}^{-\frac{x}{4}})$$

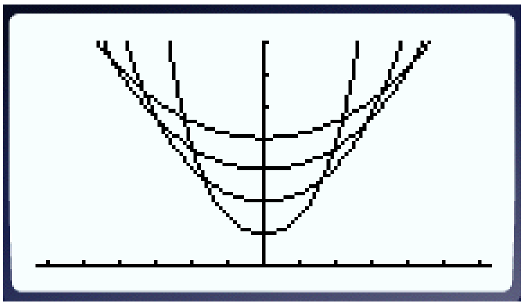

Figure 4. *Stretching hands with the rope*

There is no reason to introduce the latter set in *Mathematics for others* unless the teacher want to point to the fact that the same function(s) can be specified in different ways.

Is there any reason to give a preference to the notation with exponentials? It for sure existed in times of tables of logarithms and exponentials. The needed value did not need to be calculated – it could be found in the book. So, the formula could be manually evaluated much faster. In the Computer Age, this argument is no more valid. If its language allows typing the expression with cosine hyperbolic, the calculations are virtually equivalent. One the other hand, the formulas with *cosh* are more legible and express teacher's intention better.

4.3 Assigning Meaning to Abstract Objects

There are other applications of cosine hyperbolic. The website [9] contains an excellent animation showing regular polygons smoothly rolling on rails formed from identical sections of cosine hyperbolic – see Figure 5. The learners can discuss the application of the principle. In accordance to our experience, their views differ quite radically – from a railway with polygonal "wheels" appropriate for mountain cable railways to in-time delivery in which the number of polygon sides corresponds to the number of deliveries per constant period and the rail to the production process. Their conclusion is: *The more frequent deliveries, the smoother production.*

Even if some of these interpretations might not be entirely correct, there are not many reasons to forge learners to put them into accordance with the theory. In our opinion, building their own perception of the problem and starting understanding that it can be characterized by means of mathematics is more important that the making their interpretations perfect.

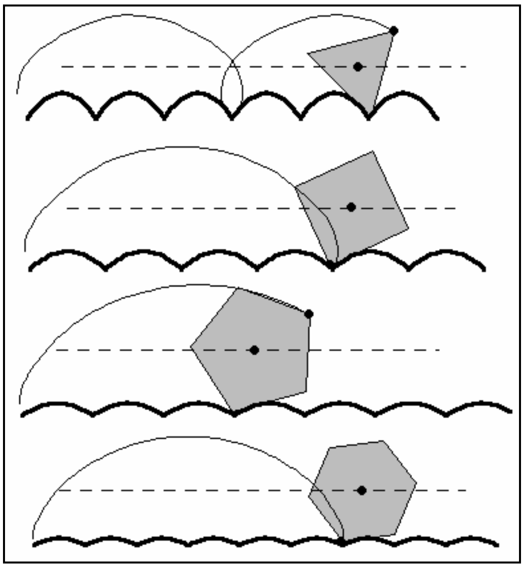

Figure 5. *Rails from sections of cosine hyperbolic*

4.4 Rising Students' Interest in Solving Mathematical Problems

The previous sections might create a false impression that we are against any elements of "traditional" education which includes calculations, manipulation with expressions and geometric constructions. Not at all. We only wish to put them into a real-life context and create a more pragmatic picture of mathematics and mathematicians. Calculations are the heart of mathematics but again they should not be "calculations for sake of calculations". They should be interconnected with reality whenever possible. Our next example shows a way.

In 1965 in Saint Louis (USA), a large arch has been built. The following formula [10] describes it:

```
y = 693.8597 – 68.7672*cosh(0.0100333x)
```

The teacher can ask the students first to find its photos in order to realize its magnitude and beauty. After showing them, the questions like: *How high is the arch? How far is each pillar from the other?* will become natural consequences of their findings. Using computer graphics, they can also draw its model – Figure 6.

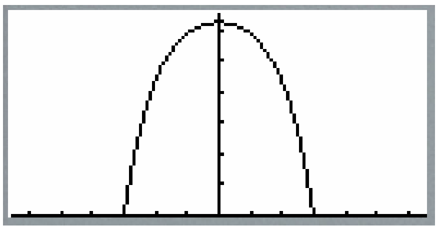

Figure 6. *Drawing Saint Louis Arch*

There are two ways of calculating the height:

1. The graph shows, that the highest point corresponds to x = 0. Taking this as a fact, we get the height as the result of calculating 693.8597 – 68.7672*cosh(0). With an advanced calculator, easily produces the result 625,09. Americans calculate in feet, so the result is to be converted into 190,5 meters.

2. As our eyes may cheat us (and x = 0 does not need to correspond to the maximum), calculating the exact maximum is another method. Presuming that the maximum is evidently somewhere in the interval [-10; 10], we can order its direct calculation. In such a case one can benefit from the FMAX function calculating the maximum of its argument with a given interval. The result is identical.

Figure 2 also shows that the pillars touch the ground in the points having their y-coordinate equal to zero. The students easily conclude that the bases are the roots of equation

```
693.8597 – 68.7672*cosh(0.0100333 x) = 0
```

This – otherwise extremely complex – calculation can be quickly solved using an advanced calculator. The roots are -299.226 and 299.226 (in meters –91 a 91) so the distance between pillars is 182 m.

Notice that even many university courses do not teach to solve similar problems due to their complexity. Manual calculations "must" end up with linear and quadratic equations. Problems with linear (or at most quadratic) solutions were preferred because the use of manual calculations prohibited finding their solution(s) in affordable time. With advanced calculators, all equations become "equally difficult". There is no need for favoring problems with simple (one-digit) coefficients and integer solutions. This opens doors for introducing more realistic problems into classrooms.

Another benefit of the proposed approach lies in its friendliness to the "user". Learners are not repelled from complex mathematics by a necessity to make long and boring calculations. Instead of them, they can focus on grasping principal concepts like the "maximum of a function", "value of a function in a point of our special interest", "roots", etc. Also, the formulation of problems is more "digestible". Traditional courses would ask the students:

- Find the maximum of the function `y = 693.8597 – 68.7672*cosh(0.0100333x)`;
- Calculate the roots of the equation `693.8597 – 68.7672*cosh(0.0100333 x) = 0`.

We prefer questions: *How high is the arch? How far is each pillar from the other?* Notice that the problems are identical – just the latter formulations look more human.

4.5 Imperfect Calculations for Our Imperfect World

There is an elevator inside a pillar of the Saint Louis Arch that transports visitors up to the viewpoint at the arch top. *What distance do the passengers travel?*

The direct method requires using calculus. However, one can calculate an approximate value using a spreadsheet. From our above calculations we know that the topmost point of the arch has coordinates (0; 625) and the right base (299; 0). One can draw a direct line between them. It represents the shortest possible lift between the considered points. Pythagoras Law can be used for calculating its length – 693 feet, i.e. 211,2 m.

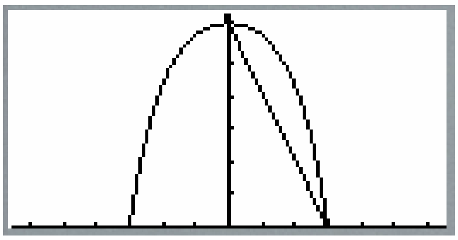

Figure 7. *The shortest possible lift*

Because the lift is located inside the arch, it is in fact longer. One can see in Figure 7. A closer value can be calculated by splitting the straight line into two (touching the arch also in x = 150). One can easily calculate (150; 531,35) as the coordinates of the point. (That's why we stressed above "calculations related to the points of our special interest".)

The arch curve is than approximated by two broken lines: the first from (0; 625.0925) to (150; 531.35), the second from (150; 531.35) a (299.226; 0). Again, using Pythagoras Law give us the result 728.79 feet, i.e. 222.14 m. Everyone sees a substantial difference. So, the new result is a significant improvement.

By increasing the number of sections, we can reach even better approximations. Let us divide it for example using six dividing points (for x = 0, 50, 100, 150, 200, 250, 299.226 feet). The calculation can be speed up using a spreadsheet calculator. Figure 8 shows a method:

SHEE	A	B	C	D
1	x	y	Part	Total
2	0	625.09		738.9
3	50	616.25	50.774	
4	100	587.47	57.691	
5	150	531.35	75.162	
6	200	433.47	109.91	
7	250	268.68	172.21	
8	299.22	7.8E-4	273.15	

Figure 8. *Using spreadsheet calculations*

- The column A contains the x-coordinates of the dividing points.
- The values in the column B are the values of the function defining the arch. First we write

  = 693.8597 – 68.7672 * cosh(0.0100333 * A2)

  and then copy it into the remaining cells of the column B.
- In the column C we calculate the distance between two consecutive points. Again, one formula must be typed but the remaining can simply be copied.
- Their sum is in the cell D2. It is 738,9 feet i.e. 225,22 m.

The difference compared to our previous result is about 3 meters. So we have achieved another – even better approximation.

The same process can be done for any number of dividing points. In practice, a dozen of them produce a sufficiently exact value. The new value differs from the one in Figure 4 by one foot (30 cm) only. Because the lift itself must be at least 2 meters high to accommodate passengers, further improvements can hardly give us more realistic answers. The current result is “reasonably good” for “everyday purposes”. Pure mathematicians will probably have difficulties to digest that but in *Teaching Mathematics* we should live with it because mathematics becomes closer to technology in this way. For example, when building constructions like skyscrapers, bridges or lifts, we can only achieve a certain level of precision. We can discuss the limits with our students: *Should it be centimeters or millimeters?* But everyone likely agrees that even if there is a method of calculating the results with a total precision, it is useless for practical purposes because factories and construction companies simply cannot guarantee it.

5. Hurdles and Obstacles

Based on above explanations, one could falsely conclude that all problems of teaching can be solved equally easily. Again, the answer is negative. The main problem lies in the complexity of the change and in the necessity to implement in the present environment. There are many potential opponents: *teachers* (as they know the traditional approaches better), *school administrators* (as they have to invest into re-training, textbooks, and methods of control), *parents* (as they will have smaller chance to help to their kinds and to communicate with them). For these reason we now point to some challenges that require further research. Its results have to make us being capable to explain to general public why the change should be introduced and how – and what we expect as its result.

In this chapter we concentrate on this sort of argumentation.

5.1 Contradictions between Naïve and Formal Solutions

Mathematical methods often cover more than “natural” solutions that can be accomplished using trial-end-error methods. Let us exemplify it:

A shoemaker has been asked to make 100 special-purpose shoes. During the first week he produces nine of them, on the next week eleven, on the third week thirteen. He sees that due to his growing experience and improved skills he will be capable of producing two shoes more during every next week compared to the previous one.

How long will it take for him to produce all pairs?

SHEE	A	B	C	D
1	Week	Shoes	Total	
2	1	9	9	
3	2	11	20	
4	3	13	33	
5	4	15	48	
6	5	17	65	
7	6	19	84	
8	7	21	105	

Figure 9. Calculating the weeks needed for completion of 100 pairs

A naïve solution can exploit a spreadsheet calculation – see Figure 9. In the first column, the weeks are calculated; in the second one, the number of pair produced during the particular week; in the third, the total of pairs. As we see in the cell C8, during the seventh week the desired amount will be achieved.

If we are satisfied with the solution, everything is OK. The problem appears when we would like to demonstrate the general approach based on the sum of the arithmetic progression:

$$S_n = (b_1 + b_n)\frac{n}{2}$$

$$b_n = b_1 + 2(n-1)$$

$$S_n = (2b_1 + 2n - 2)\frac{n}{2} = b_1 n + n^2 - 1 = n^2 + b_1 n - 1$$

$$100 = n^2 + 9n - 1$$

$$n^2 + 9n - 101 = 0$$

The solution of the last expression (of the quadratic equation) is the solution of our original problem using another method. Its roots are 6.5113 and –15.51. The first one corresponds to our approximate (naïve) solution. But what does the second one mean? *Can one hundred pairs of shoes be produced in six and a half weeks as well as in minus fifteen and a half weeks?*

No doubt, none of us would enjoy facing our class asking this question. On the other hand, *Teaching Mathematics* should have prepared “standard answers” for similar situations. In this case, we should claim (with a smile) that we are interested in positive solutions only because “time-travelling is not the subject of our interest”. We could even add that we cordially believe that flying with a time-travel-machine one might be capable to produce one hundred pairs of shoes within less than minus sixteen weeks.

Many students will probably be pleased about this sort of humor. If no more, it would relieve the rigid atmosphere – too frequently present in mathematics classes. At the same time, such jokes also introduce students into a sort of “mathematical beauty” based on “what-if-ever”. They open door to abstract concepts which are much more difficult to visualize but *may* exist (like irrational or complex numbers).

5.2 Providing Realistic Models

Formulations of problems should use a terminology comprehensible by common public. Whenever possible, they should address situations our students can meet with:

A student body discusses a possibility to organize a fund-raising dinner. The presumed price of a ticket is \$24. One member of the organizing committee has found an appropriate space for the event which can be rented for \$350. Another one learned about a company providing chairs for \$1.50 per night plus free tables. The committee needs to know the minimum number of people which has to come for covering their expenses.

Figure 10. *Revenue*

Such a problem can be first exploited for training student in reading drawings. Figure 10 shows how the revenue grows with the number of visitors; Figure 11 shows the same for the costs.

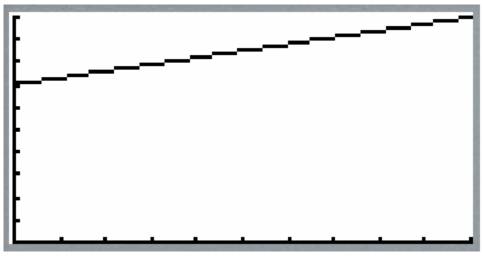

Figure 11. *Costs*

There is no reason for hurrying. The teacher should first make certain that all concepts expressed by the drawings are understood well: *Why do both drawing contain straight lines? Why does the first line grow faster? Why the second one does not start at zero?* Only then the final task can be posed: *How do the lines refer to our key task?*

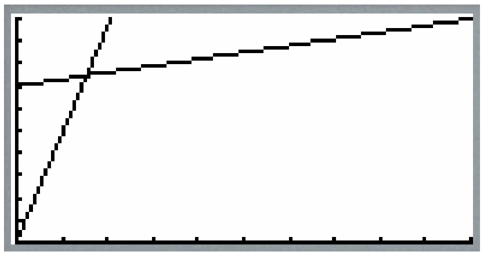

Figure 12. *Break-even point*

The answers should preferably come from students. Such an approach helps in building their self-confidence in viewing mathematics as “something that can be understood”. Mathematical software allows reading the position of the intersection as [15.55555; 373.33333]. Again, questions should follow: *What is meaning of each number? Do they have a meaning in their present format?* Students easily conclude, that the *x* value must be 16 – the nearest higher integer.

The problem can be easily prolonged. As students are rarely interested in pure business-oriented tasks like break-even points, their interest will be attracted by additional questions as: *The student body plans to buy a TV set to their club for approximately \$500. How many people must show up to achieve this level of profit?*

All problems should be preferably solved using several alternative methods. A half of the class can use the above graphic method, the other one the equations:

$$P(x) = R(x) - C(x)$$

$$P(x) = 24x - (1{,}5x + 350) = 22{,}5x - 350$$

$$500 = 22{,}5x - 350$$

$$x = 37{,}77$$

Again we do not use “nice” coefficients producing whole-number-results. We feel them as artificial simplifications appropriate in pre-computer times and used for speeding-up manual calculations. As they are very rare in real-life, the students facing practical problems may feel cheated when asked to interpret non-integer results: “*No one taught as this*.” For we ask: *Why?* Under the presumption of availability of information technology in classrooms makes formulating similar problems more realistic. The students will not only learn to implement them but also become capable of interpreting their results regardles of their „imperfect“ values.

5.3 Introducing Non-linear Models

Our surrounding world is full of non-linear models. Often, we even do not realize their non-linearity e.g. "*Buy two and get one free!*" As we showed with the cosine hyperbolic, introducing non-linear functions does not need to make our teaching material more difficult. Let us discuss a problem similar to the previous one but using a non-linear costs function.

Two girls want to make money to buy Christmas gifts for their relatives and friends. They see their opportunity in making and selling necklaces from glass beans. They realized that first they have to invest $50 to various tools. For each necklace they also need a set of beans. The supplier offers them for the basic price $2, but the price declines by 1 cent per set.

Again, we can ask about the break-even point and profit but we prefer pointing to a different problem. The cost function (Figure 13) is a quadratic function:

$$C(x) = 50 + x(2 - 0.1x) = 50 + 2x - 0.01x^2$$

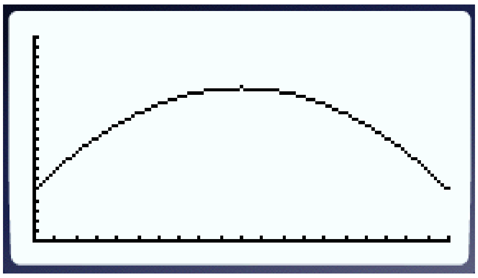

Figure 13. Quadratic costs function

As its function is a parabola – see Figure 13 – the cost initially grows, then reaches its maximum and starts declining. This is counterintuitive as we all know that with the growing production costs *must* grow. The reason of the irregularity lies in the permanent decrease of the unit price – after certain number of purchased items, the price per unit becomes *negative*. It indicates that similar discount can only be used for a limited number of products. This may lead to searching a better business model – to relative discounts. When the price of the consecutive item is a portion of the previous one (e.g. 90% of it), all prices remain positive. The model is more complex but can easily be calculated for example using a spreadsheet.

Such an approach may not be appreciated as perfect from the traditionalists' point of view but it is very important for building students' self-confidence. "*I can do it*" should be considered by *Teaching Mathematics* as more important than "*I know all theory behind it*". (How many of us successfully drive cars without knowing the theory of combustion engines?) The most complex models can be omitted – or shifted to *Mathematics for its followers*.

5.4 Syntax, Semantics and Mathematics

Some people feel frustrated by the fact that in mathematics "always works perfect". They may feel happy to learn that it is not always true. One example has been demonstrated by Galileo's error. Another one can be exemplified by problems looking as mathematical ones but wrong otherwise. Compare two problems:

The weight of a horse is 450 kg. What is the weight of 20 horses?

The speed of a running horse is 20 km/hour. What is the speed of 20 horses?

They differ only negligibly from the syntactical point of view. At the same time, there is a huge semantic gap between them. The speed must not be accumulated; the weight can. From the example we see that our considerations must be guided by the problem semantics, not by syntax. Unfortunately, most textbook problems (*A train from A to B …*) are "disinfected" to the manner that does not open questions about their semantics. What about introducing problems like "*A train from Tokyo to San Francisco …*"?

Mathematics is supposed to be about thinking. Rather than fill in all our students' time with calculations and drill, we should submit them material for thinking about mathematical problems, possibly with nonsense interpretations or with interpretations correct under certain conditions and strange in others.:

- *Our old TV set has collapsed so we could not watch any programs. Daddy went to the shop and bought 16 TV's. Mommy also went to the shop and bought 27. How many TV sets do we now have?*
- *A teacher and his class of 23 kids went to the mountains for hiking. After a while, all kids got tired and could not walk. The teacher put them to his backpack and walked carrying them on his back. Later on, eight pupils said that they have rested enough and can walk by themselves. How many children are still in the backpack?*
- *Peter went to the forest behind his house and saw 5 giraffes, 24 gazelles and 7 lions. How many animals did he see?*(The problem might be fair in Kenya but hardly in Slovakia).

Using them, we can teach our students to become capable of recognizing the core of problems, analyze whether it is a mathematical problem or not, finding the "right" method of its solution and become capable of interpreting its result. Even if some people might calculate 400 km/h as the speed of the herd of horses, they should be capable of stopping in a certain moment and start considering: "*Isn't that too much?*" They should learn to interpret the outcomes, assess them critically and ask him/herself: "*Presuming that the result is formally correct – is it also logical?*" The importance of similar questions grows with the application of information technology as we are unable to follow them step by step. IT outcomes are often taken for granted beyond discussion. The previous question combined with common sense can prevent many flaws in applying its deviated results.

6. Conclusion

The results of mathematical education in this present structure are quite sad. Even excellent students do not understand why it is taught. Their picture of a "mathematician" is either "a teacher of Mathematics" or "a researcher solving mathematical problems". They can hardly imagine a mathematician participating in economic, medical or construction problems because they have never been told that some mathematicians do that. Until the public is not convinced about its usefulness, the picture of mathematics and of the role of mathematicians in society will remain distorted. Consequently, we are loosing plenty job opportunities because top managers do not see reasons to invite mathematicians into their teams.

Mathematics is not here just for the pleasure of mathematicians. Its elements were discovered and developed due to practical problems surrounding us. Whilst Latin was a vivid language (at least in the community of highly educated individuals), it was also surviving as a school subject. Our lesson must be straight – *Mathematics Teaching* will continue until people value it as a vivid, vital and no-nonsense subject. Contemporary advanced software tools are much better in performing calculations, formula manipulation and construction of geometric bodies than most people (including highly qualified specialists) are. If Mathematics would only consist of them, its learning would become obsolete and it would follow the destiny of Latin. Thus, we have to:

- Frequently demonstrate that mathematical problems relate to practical life;
- Show alternative methods of solving the same problem and discuss their advantages and disadvantages – see for example [11];
- Stress that finding one practical solution is for users more important than knowing a variety of methods and all theory behind them;
- Train our learners to comprehend the problems, interpret their outcomes and discuss to what degree they are in a good correspondence with our daily experience.

In our paper, we have tried to indicate a few ways of doing so. At the same time, such attempts cannot be isolated. We all should be looking various alternative methods fitting to different "market segments". Attempts have already been made – see for example [12]. What we lack is the feeling of emergency among our community to wake it up to act instantly and thoroughly as a whole.

As a community, we will have to make many experiments to understand which methods work and which not. We should also do "marketing research" that reflect the attitudes of our "customers" and level of their comprehension. A nice and practical method is described in [13]. At the same time, our pupils should feel the same pleasure from our experiments as we do. Mathematics will hardly be an eternal fun – but the proportions might change. Even if it is not always fully understandable, the students should have a feeling that they understand at least a part of it. We humans never understand all processes around us but believe that when we concentrate on some, we can manage them. Give the same feeling to our pupils and students.

We can learn a lot from managerial disciplines how to achieve our aims. I personally hope that the entire community will participate in this learning.

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MuPAD for the Classroom—a Discusion on Using Computer Algebra Systems in Teaching Mathematics

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Abstract

It is well known that Computer Algebra Systems (CAS), for instance Mathematica, Maple, MuPAD or Derive, provide computational power to solve many mathematical problems faster and more efficiently than using the old paper-and-pencil methods. The availability of such software and its ability to deal with most undergraduate mathematics problems cannot be ignored. A mathematics educator cannot stay neutral in this issue. Indeed, the emergence of such software has divided mathematics educators into two almost separate groups — those who believe that we should use such software in teaching mathematics as much as possible, and those who think that we should completely avoid technology in mathematics classes.

In this paper we will analyze various points of view on using CAS in teaching mathematics. We will show how some of the major concerns can be overcome. Finally, we will discuss the *MuPAD for the classroom* project, where the main objective is to develop a comprehensive set of materials to help educators incorporate MuPAD efficiently into undergraduate mathematics teaching.

Most of the issues discussed in this paper are related to teaching high school mathematics. In a few places we will also consider the university point of view. However, it is important to note that in many aspects, the CAS situation at the university is quite different than in high school.

1 Introduction

This paper starts with some issues related to MuPAD and moves towards more general analysis of using Computer Algebra Systems (CAS) in teaching mathematics. Later we will return from those general issues back to MuPAD. Therefore, all MuPAD related issues will be discussed in a wider context of teaching mathematics with CAS. Let us start by introducing MuPAD.

MuPAD is one of a few CAS used in teaching mathematics, as well as in scientific computing. Originally MuPAD, full name of this CAS is MuPAD Pro, was developed at the Paderborn University by the MuPAD Research Group under supervision of Professor Benno Fuchssteiner. In 1997 SciFace Software company was created in order to handle further development and distribution of MuPAD. For the last ten years, one of the main objectives of MuPAD development has been to create a didactical tool that could be used in teaching mathematics — starting from high school mathematics courses all the way to advanced university level courses.

The author of this paper has been involved in numerous projects related to MuPAD — suggesting and developing of some of its features, writing books and papers on using MuPAD, organizing MuPAD conferences and workshops for mathematics teachers. His main MuPAD activity in Europe is a series of *mathPAD Conferences* — six annual conferences targeting mathematics teachers in Central and Eastern Europe. He organized MuPAD workshops in Poland, Germany, Singapore, Malaysia, Thailand, Macao, and Taiwan. He is also the editor of an online magazine for the MuPAD community (www.mathpad.org).

Having been involved in MuPAD activities for so many years, a natural question is to ask *how do mathematics teachers use MuPAD in their classroom*? The findings are quite surprising. In Germany, the country where MuPAD was developed, MuPAD is used in teaching mathematics in many minor ways. However none of them is monitored by the Ministry of Education or any other educational authority. There is no common policy on using technology in the classroom and in many of federal states, using CAS in mathematics teaching and examinations is still forbidden.

Therefore, another natural question is to ask *are there countries or places where using CAS for teaching undergraduate mathematics is taken more seriously*? It was surprising for the author to find that implementing CAS in teaching mathematics in many countries is far from what we would expect, and in many developed countries is still completely ignored.

Before going into further discussion let us remind ourselves what CAS can do for mathematics educators.

2 CAS and Mathematics Education

Major CAS, e.g. Maple, Mathematica and MuPAD, are command-line tools for solving mathematics problems. Therefore, in order to get a response on a mathematical question, we have to type in a command, and press the [Enter] key to produce a result or a graph. Let us start with a simple middle school example.

Example 1 *Converting a recurring decimal into a fraction*

Suppose

$$k = 0.076923076923076923076923076923076923076923076923... \qquad (1)$$

We can easily observe that the recurring cycle is 6 digits long. By multiplying the above equation by 10^6 we get

$$1000000k = 76923.076923076923076923076923076923076923076923076923... \quad (2)$$

Now, by subtracting (1) from (2) we produce the equation

$$999999k = 76923$$

therefore $k = \frac{76923}{999999}$

Here comes the use of CAS:

```
simplify(76923/999999)
```

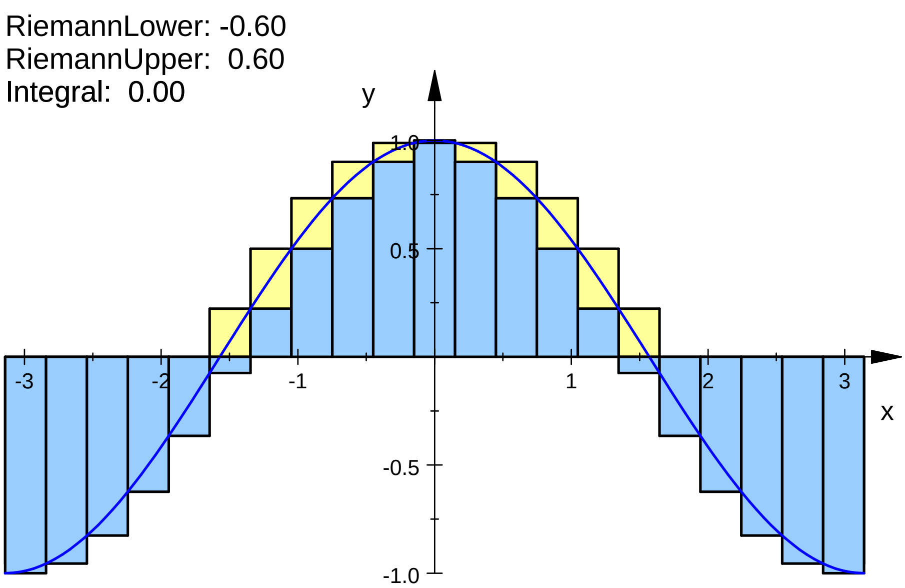
$\frac{1}{13}$

Note, we used MuPAD here in a situation where students, in this case middle school students, had to perform a most time-consuming operation of simplifying a complex fraction. There are many more situations in mathematics where CAS can free the students from spending time on tedious calculations.

Here is a simple example showing the MuPAD code to produce a graph of the function $y = \sin x$ and graph representing its Riemann lower and upper sums.

Example 2 *Riemann lower and upper sums for the function $y = \sin x$ using 21 equal size intervals.*

```
f := plot::Function2d(cos(x), x = -PI..PI):
g := plot::Integral(f, IntMethod = RiemannLower, 21,
                       ShowInfo = [IntMethod, "", Integral]):
h := plot::Integral(f, IntMethod = RiemannUpper, 21,
                       Color = RGB::LightYellow):
plot(h, g, f)
```



Example 3 *Consider the quadratic polynomial $f(x) = ax^2 + bx + c$. A graph of such a polynomial has an extreme values at $\left(\frac{-b}{2a}, \frac{4ac-b^2}{4a}\right)$. Determine the locus of the extreme points of quadratic polynomials where $a = 1$ and $c = -4$.*

This is a perfect example to demonstrate many applications of CAS. Let us start with visualizing the family of curves. In MuPAD we start by loading the easy plot package (more about eplot in the last section of this paper).

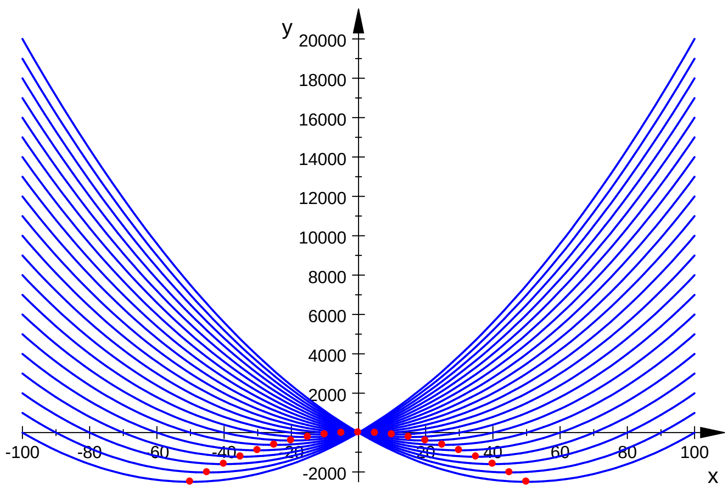
```
reset():
package("eplot"):
```

Then we define the family of polynomials and their vertices,

```
f := x -> x^2+b*x-4:
curves := {f(x) $ b=-100..100 step 10}:
points := {[-b/2, (-16-b^2)/4] $ b=-100..100 step 10}:
```

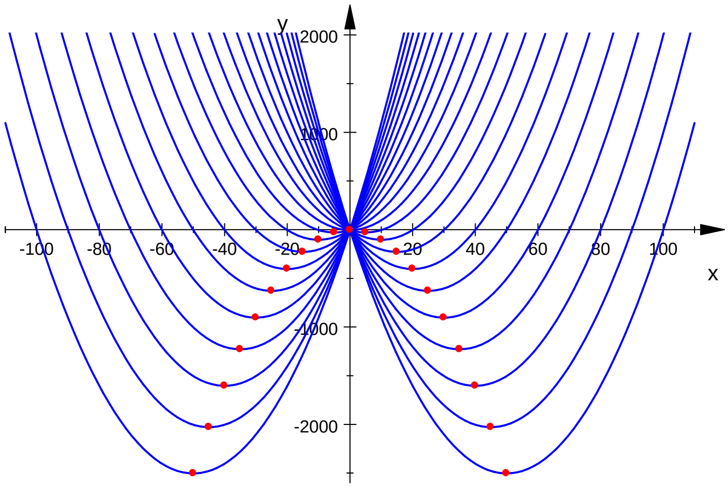
Now we plot all of them together,

```
plot(curves, points, x=-100..100)
```



Before we proceed with our investigations, we should limit the graph to the area where the vertices are shown by defining a more suitable viewing box for the plot.

```
plot(curves, points, x=-110..110, ViewingBox=[-110..110,-2600..2000])
```

At this point, a few simple operations are needed to produce the equation of the locus curve,

```
f'(x) // obtain derivative of the polynomial
```

$b + 2x$

```
u := op(solve(b+2*x=0,b)) // solve equation f'(x)=0 in respect to b
```

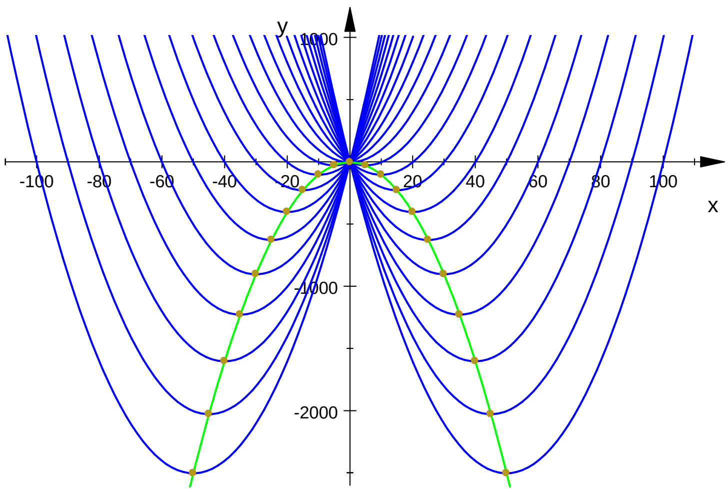
$-2x$

```
locus := subs(f(x), b=u) // substitute the solution for b into f(x) to
                          // obtain the locus curve
```

$-x^2 - 4$

Finally we can plot all the polynomials, vertices and the locus curve together:

```
plot(curves, points, locus, x=-110..110, ViewingBox=[-110..110,-2600..1000])
```



Although typing in commands requires at least some minimal knowledge of the software, the obtained results can be invaluable tools in visualizing many mathematical problems and theorems (example 2 and 3).

As we said before, some of the CAS use command-line instruction. Other CAS are menu-driven, e.g. Scientific Notebook, Scientific Workplace, and Derive. In all these programs, we do not need to type in even a simple command, and all the operations can be obtained by selecting them from menus and toolbars. In such cases, we do not need to memorize commands but we still need to have a reasonably good knowledge of what is included in each menu.

Finally, we have to notice that all the command-line CAS offer some menus with shortcuts to many commands. MuPAD, for example, has a sidebar with menus and toolbars for the most frequently used commands. It is important to note that even the most extensive menu system is not big enough to accommodate all commands and parameters available in a CAS. Therefore, the menu-driven CAS will have always limited functionality.

Example 2 tells us much more than only how the Riemann sums define the approximate value of an integral. It shows us that operations, in this case calculating the Riemann sums, that were usually beyond student abilities, in particular when it came to high school students, can be taken over by CAS and students will still be able to deal with and understand a mathematical concept. A similar situation occurs in example 1. Finally, example 3 shows that CAS can be used as an integrated environment for solving mathematical problems, experimenting with mathematical concepts and visualizing them.

Mathematics education researchers as well as mathematics instructors emphasize the numerous advantages of using CAS in teaching mathematics. There is a huge number of publications analyzing the use of CAS in the classroom and showing its role in transforming mathematics education. Some of the mentioned advantages have a concrete topical value, like creating mathematical graphs illustrating a given topic (see Bowers [4]), some other may lead to serious consequences for the whole teaching process. Let me mention a few of the most important arguments from the second group (compare Heugl [13]).

1. *CAS help in abstracting mathematical problems.* Most mathematical problems start with a concrete problem, then an abstract model is created. By formulating such problems in a CAS language students gain a better understanding of the problem and the algorithms needed to solve the problem (see [13]).

2. *CAS help to increase the value of the knowledge and the degree of interest of students.* Example 2 shows that a student can easily go beyond calculations and concentrate more on the knowledge itself than on tedious transformations needed to produce the Riemann sums.

3. *With CAS we can offer students a more application-oriented, and definitely more interesting mathematics that has a strong meaning.* This, in particular, is important for students who do not posses mathematical skills and do not see the point of learning mathematics.

4. *CAS build skills in translating mathematical problems from a native language into the language of mathematics.* Before students will be able to solve a real life problem

with CAS, they have to translate it into the mathematical model (see Kutzler [17]). We know that such models, developed by students, can be very inaccurate and sometimes even miss some special cases of the problem. While translating such problem into mathematical language and later into CAS commands students will be forced to think about roles of constants, variables and parameters in the problem, their ranges, etc.

5. *Using CAS will result in expanding students' mathematical language and bring new meanings in mathematical reasoning.* For example, in the past mathematical activities of students were frequently limited to transforming mathematical equations, with CAS students will move to the level where they treat equations as objects and apply CAS operations to them. In many situations, a process approach will be replaced by an object approach (see [14]).

6. *While solving problems with CAS there will be a significant shift from calculating to planning and interpreting* (see [8], [13], [7]). Students will be able to spend less time on performing calculations and more time on interpreting the obtained results and applying them in real-life situations.

7. *With CAS students will be able to develop new elements of their thinking technology.* New heuristic strategies will become part of their problem solving skills: experimenting with mathematical concepts, proceeding step by step through calculations, visualizing concepts using various types of graphs, developing models or real-life problems and simulating them with a computer (see [13] [6], [24], [19]).

8. *With CAS students will be able to develop a modular way of thinking and working.* In traditional mathematics, i.e. mathematics without CAS, we did not emphasize the role of modules (see [21], [20]). However, modules have existed in mathematics for a long time. For instance, as soon as we learn how to integrate, integration becomes a kind of black-box tool that we apply whenever we need it without even thinking about what the definition of the integral is. With CAS, we will be able to skip the whole process of integration and use a respective command. Therefore, with CAS students will be able to create more general modules, then modules of modules etc.

9. *CAS will become a medium for creating interactive and dynamic prototypes.* This is a completely new technique in mathematics education. For years, we created mathematical prototypes of given situations. However, we were rarely able to observe how these prototypes change with the change of their parameters. Besides the prototypes we had already been using: word formulas, symbolic prototypes like terms and equations, graphs, and tables; CAS offer us some new prototypes like recursive models or programs (see [13]).

The list of advantages of using CAS is certainly much longer (see the discussion section in Bowers [4]). I did not mention here some disputable arguments (see [22]), like increasing motivation of students, increasing their understanding of mathematics, making students more confident in problem solving, etc. The ongoing discussion on the benefits of using

CAS in mathematics teaching is enormous and almost every conference on technology for mathematics can be a source of papers advocating for the use of CAS and against it (see [1], [3], [4], [12], [14], [17], [18], [24], etc.)

3 Who teaches mathematics with CAS and how?

After this brief overview of the benefits that CAS may bring to our mathematics classes, let us come back to the question: *are there countries or places where using CAS for teaching undergraduate mathematics is taken more seriously*? An extensive survey of literature, conference papers and online discussion list shows that there are very few countries where CAS or computer technology in general, got more attention from educational bodies or government institutions. In most other places using technology for teaching is completely ignored, or even prohibited. However, even there, we can find enthusiasts experimenting with technology in their classes. Professor Fred Szabo in one of his letters wrote *technology has produced fabulous tools and is providing us with an environment that allows us to reach our intellectual potential on a grand scale and beyond our wildest dreams* ([23]). I have reasons to believe that many of us feel the same way.

For six years now, I have been working for a university that uses information technology as one of their learning outcomes. The university has a university-wide Maple license and I had my own unlimited MuPAD license. I have been discouraged many times in using any of these programs in my discrete mathematics classes. I guess this is a rather extreme case, but nonetheless some of my colleagues from other universities have had similar experiences.

More positive experiences have been noted by our colleagues from Austria, Victoria in Australia, France, USA and perhaps a few other countries. The Austrian initiatives are worth noting here. According to my observations, no other country has made as much effort to implement CAS in teaching undergraduate mathematics.

CAS in Austria is tightly connected with Derive and a few people pioneering the use of Derive in Austrian mathematics educations. Let me mention them in alphabetic order: Josef Böhm, Bruno Buchberger, Helmut Heugl and Bernard Kutlzer. Professor Buchberger is the author of a famous black-box/white-box principle that we will analyze later in this paper.

In the early nineties school authorities and a group of teachers created the Austrian Center for Didactics in Austria (ACDCA). ACDCA was created *with the intention to attend and support teachers in using revolutionary tools in a meaningful and responsible way unlike to the introduction of pocket calculator several years ago* (see Böhm [3]). In 1991 the Austrian Ministry of Education purchased Derive for all general secondary schools. It is important to note that the ACDCA for many years was, and still is, the main power involved in a very systematic way of introducing CAS in undergraduate mathematics teaching. The word *systematic* used here is very important. Since 1991 we have witnessed five large, country-wide projects (see [3], [25]):

- CAS I 1993-1994 Derive project
- CAS II 1997-1998 TI-92 Project (creation of teaching materials, research on influence of CAS on teaching), 44 schools, 70 classes, 65 teachers, 680 female students, 1570

male students

- CAS III 1999-2000 2nd TI-92 Project (research on influence of CAS in teaching, learning, curriculum and assessment), 94 classes with more than 2000 students
- CAS IV 2001-2002 CAS Project (research on the new culture problems, establishing a Service Center for Teachers), 140 classes with more than 2200 students
- CAS V 2003-2005 Variety of Media Project (together with GeoGebra, Mathe-Online, online learning, teaching in laptop classes)
- CAS VI (in preparation).

Josef Böhm in his paper entitled *What is Happening with CAS in Classrooms? Example Austria* presents a history of the ACDCA; he also analyses the outcomes of the projects initiated by this organization (see [3]). Very extensive information about the ACDCA can also be found on their web site (see [1]).

A number of similar, well-organized, albeit on a much smaller scale, attempts to introduce CAS in teaching mathematics can be observed in literature—Jos Bertemes describes some experiences from Luxemburg (see [2]), Peter Flynn presents the experiences of introducing CAS in Victoria in Australia (see [9]), David Bowers presents some valuable experiences in introducing CAS in a middle school in England and Wales (see [4]).

For a few years now, the MuPAD Education Group has been organizing regular training for German teachers in using MuPAD. The MUMM project (Mathematik-Untericht mit MuPAD) was first large scale project organized in 2003/2004 by the MuPAD Education group and Ministry of Education of Nordrhein-Westfallen. In this state MuPAD was evaluated by the ministry and recommended to all schools. In the MUMM project participated 280 secondary schools, and about 50 teachers training sessions took place. One of the outcomes of this action is a collection of about 100 MuPAD notebooks that demonstrate selected aspects of high school mathematics. After three years the collection contains about 500 notebooks and is growing rapidly. In Germany each federal state has so called *Kulturhoheit*, which simply means that each state can make their own school laws independent from the German Government. Therefore, in each state situation of CAS is different. A very positive example is state Baden-Württemberg—a precursor of using CAS in Germany. They take teaching mathematics with technology very seriously.

For the last six years, the author of this paper has been organizing regular MuPAD conferences at the Nicholas Copernicus University (about 60-70 participants each year). These conferences, sponsored by the university and SciFace Software, have been mostly targeting mathematics teachers, and their general purpose is to provide training on using MuPAD in the classroom as well as to prepare some printed and electronic materials for mathematics teachers.

In each case, we can observe one important thing — all the actions like those mentioned above are very important as they build an interest of teachers, students and education ministers regarding the use technology in teaching and, in particular, in teaching mathematics.

However, the successful implementation of CAS in teaching mathematics requires much more than official conferences, research projects, etc. We have to consider a few other parties that are or will be involved in this process. These are teachers, examination boards, and students. A deep analysis of this aspect in relation to Austrian environment we can find in Wazir's article (see [24]). In this paper, I will add also some experiences from Europe and the Middle East.

3.1 Teachers' point of view on teaching mathematics with CAS

The first and the most important observation is that we are still far away from a good methodology for using CAS in mathematics classes. While talking to Kai Ghers (see [11]) I discovered that for example German teachers still do not have a clear strategy on how to use MuPAD notebooks provided to them by the MuPAD Education team. According to him, *some of the teachers used notebooks to demonstrate certain topics by themselves, without letting the students work in these notebooks on their own. Later on, when students learned enough of MuPAD, the teacher took students to the computer lab and each student could work through the topics demonstrated in notebooks. ... Some other teachers were not using MuPAD in the school at all, but they provided the notebooks to the students so they could experiment with them at home. Finally, some other teachers were using only graphical features of MuPAD to demonstrate certain topics, e.g. solids of revolution, planes and lines in 3D, etc. without letting the students use MuPAD at all.*

I observed a completely different situation in Poland. Polish teachers, at least some of them, are keener to involve students in mathematical explorations with MuPAD and some of them are very open to use any technology in the class in order to build the students' interest in learning mathematics. This way they can motivate students to develop a kind of ownership of the learned mathematics. Similar aspects are also mentioned by other authors (see Bowers [4]).

In general, teachers hesitate to integrate CAS into their teaching practice (see Wazir [24]) and there are many reasons for this. Any technology, including CAS, requires learning it first, developing strategies how to use it, and developing new teaching materials. CAS is a new teaching aid and very few teachers learned how to use it during their studies. Therefore, teachers are often afraid that students, especially some of the computer-addicted students, will come to know the computer program better than they do. In fact, in most of the cases where CAS is used at school, teachers are one step behind their students. Another serious reason is the lack of mathematics textbooks using CAS as a teaching tool. Most the teachers prefer to stick to a textbook and follow it tightly in the classroom. They will not try anything new unless they are not forced to. We have to remember that in many countries and schools, teachers are heavily overloaded with the amount of classes and assessments. They simply may not have time to work out a strategy and methodology to use CAS in their classes. Finally, some teachers simply hate computers and/or prefer pure mathematics.

The textbook issue seems to be different at the university level. Here we can find a few textbooks that claim to use Maple or Mathematica to a particular mathematics course. However, many of these books contain in reality a very traditional course material sprinkled with a few CAS commands. Usually, in the first few chapters of such a book we can find more technology-based material, while the last chapters completely ignore the technology and the course goes on like it went 20 or more years ago.

Finally, it is important to mention that, according to my knowledge (July 2007), there is no country-wide high school curriculum in which CAS is officially accepted as a tool for teaching mathematics. There are some exceptions allowing the use of calculators, though often with some restrictions to calculator type. In such situations many teachers are afraid that their time for introducing CAS in mathematics classes will be wasted if CAS will be banned by the ministry of education or examination boards.

3.2 The students' view on using CAS in mathematics classes

In literature there are quite mixed points of view on students' appreciation of CAS. Some authors claim that introducing CAS in their mathematics classes developed more interest of students in mathematics, in particular mathematics with computer. Others say that their students did not like technology in mathematics classes at all. According to my personal experience, liking or disliking technology by students depends on their individual skills and attitudes. Intelligent students, who usually do not have problems with mathematics, like the technology component in the class and they quickly learn how to use it to expand their possibilities. There is no doubt that such students, at least some of them, with CAS can go far beyond of what they could do using traditional paper-and-pencil approach.

For students, who are struggling with mathematical concepts, the introduction of CAS is just another burden that will take their time. In such cases, we have to very carefully design the strategies and methodology of using CAS in teaching.

3.3 Examination boards and CAS in mathematics classes

It is well known that in most countries, the requirements of the examination boards have a major influence on what we teach and how we teach. This refers to both international examination boards like IB and local national-level school examination boards. The issue of what will be allowed on the exam, in reality, legalizes the tools used in the classroom. Therefore, why teach mathematics with CAS, if CAS are prohibited in all international exams and also prohibited in many countries in their countrywide exams? There is always a concern that students who learned to solve problems with CAS will fail on a paper-and-pencil exam. Although in literature there are opinions that students using CAS are still able to solve problems using traditional methods, some authors claim that their CAS students performed on the exam even better than non-CAS students, most educators have rather mixed feelings about this outcome. A serious step towards using CAS in exams was taken recently in Austria. In this country all final exams have two parts—one to be solved manually, and one to be solved using technology. Some critical comments in literature show that this solution is also not perfect.

A major question is how long we will have to wait for a more favorable look of examination boards on CAS? We have to accept that they have their rational arguments also — not enough computers in schools, lack or examination rooms equipped with computers and CAS, etc. However, even in some wealthy countries where such problems are marginal, the decision to introduce CAS in mathematics exams are not easily pushed through. There is one issue that should not be neglected — it was mentioned by a few authors that for many examination boards, the preparation of exam questions with a CAS component was a major difficulty and the preparation of such questions took much more time than standard exam questions.

I believe that if we at the university level start demanding from our candidates a better knowledge of technology for mathematics, then we will be able to change the point of view of the examination boards. This process may take many years.

4 Black-box/white-box principle

Until now we have discussed some social and psychological issues. Let us investigate one of the reasons why some teachers or university instructors do not wish to use CAS in teaching and why some other are in favor of CAS in the classroom. This discussion started when the first CAS were invented and it is still present in various conferences and seminars.

The major question asked frequently is — what will my students learn if such a tool does everything for them? Will they learn how to solve equations, differentiate functions or integrate if any CAS will do it for them in a result of a single command? At the same time some other educators ask — why do students still have to learn this *boring stuff* if computers can do it faster and better? We will leave the second question without an answer and concentrate for a while on the first one.

In 1990 Professor Buchberger formulated his famous black-box/white-box principle (see [6]). The major point of it is when and how we should use CAS in mathematics classes. According to Buchberger, while learning a given topic there is a moment when the topic is *trivialized*. For example, after learning how to integrate functions of one variable we are coming to the stage when we know the concept well, or reasonably well. This is the moment when we no longer need to think about how to integrate functions and we move on to topics where we can apply integration to solve another problems. We consider this to be the moment when integration is trivialized.

According to the black-box/white-box principle, we should not use CAS as a black-box in mathematics classes if the studied concept is new to students and is not trivialiazed, and we should use respective black-box commands as soon as the concept has been thoroughly studied and trivialized (like in the example 2). For example, while teaching how to solve systems of linear equations, we should go first through the phase where students learn the principles of solving such systems, theorems, algorithms and the theorems they're based on, and calculate by hand a number of examples. During this phase, we could use CAS in the form of a white-box to simplify some calculations or show how algorithms work (like in the example 3), but without using the command to solve a system of linear equations. After this phase, when we are sure that the concept was learned properly, we can introduce the command `linsolve` (MuPAD) and allow students to use it to solve systems of linear equations in further applications. In practice, the white-box and black-box phases occur at the same time in respect to different topics. The table on the next page presents an example of such a situation from a course of linear algebra:

Topic	Black-box phase for	White-box phase for
Determinants	arithmetical operations	determinants
Inverse matrices	arithmetical operations calculation of determinants	finding inverse matrices
Solving systems of linear eq.	arithmetical operations calculation of determinants finding inverse matrices	solving systems of lin. eq.
Eigenspace	arithmetical operations calculation of determinants finding inverse matrices solving systems of lin. eq.	eigenspaces
...	...	...

The black-box/white-box principle was discussed in a number of papers. In literature, we can also find some modified versions of it. The principle was consequently applied in all Derive materials produced during CAS projects in Austria. The Austrian experience shows how the black-box/white-box principle allow us to produce more structured, and definitely more interesting teaching materials encouraging students to experiments with mathematical concepts on a computer.

5 MuPAD for the classroom project

Having been involved in MuPAD-related matters for many years, I am confident that MuPAD can be a very useful software for teaching mathematics. Therefore, in the nearest future I am going to start, with a group of mathematics teachers and university professors, a research project called *MuPAD for the classroom*. At the moment, there is no institution formally supporting this project or attached to it, and there are no official participants of the project. The project is completely open and everybody interested in teaching mathematics classes with MuPAD can join the project as long as they want to share their works with the wider community. There is also no budget for the project. However, I expect that some educational institutions may support it. There will be certainly support from SciFace GmBH in the form of web space for the project and MuPAD licenses for the participants.

The objectives of the *MuPAD for the classroom* project will be research on implementing MuPAD in teaching undergraduate mathematics classes, the development of methodology of teaching with MuPAD, the development of strategies for using MuPAD in the classroom, and the development MuPAD notebooks and printed materials for many courses in mathematics. Such materials should be constructed using the black-box/white-box principle as well as the relevant Gagne's events of instruction (see discussion on Gagne's events of instruction in [10] or[15]).

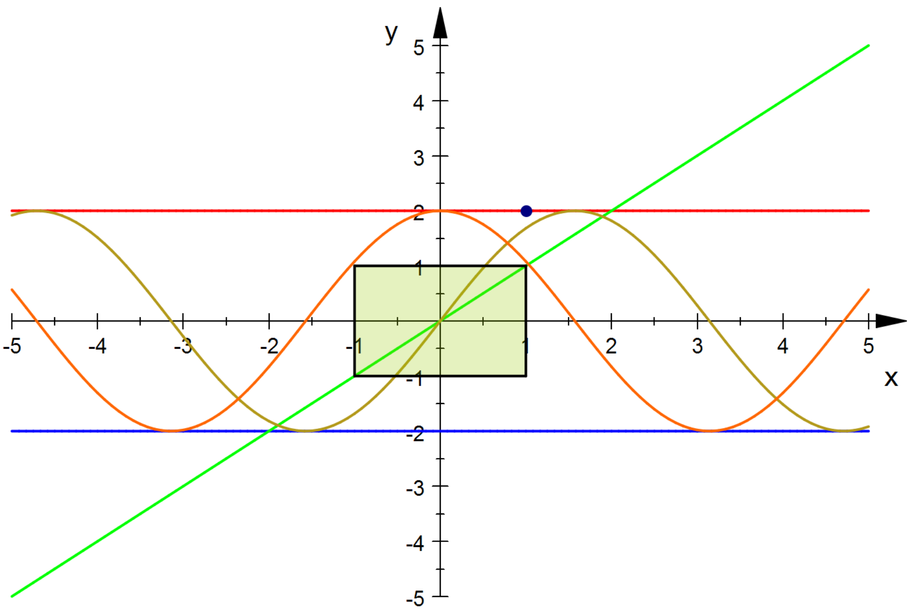
Recently, MuPAD has gone through many changes that are important from a didactic point of view. Here I wish to mention two of them: the intelligent plotting procedure, so-called eplot, and a new, very flexible interface.

The intelligent plotting procedure was developed recently and makes producing MuPAD graphs less formal than with the standard plotting tools. The procedure can plot almost everything what we drop into it. Here is an example showing how it works.

While using the standard plotting concept in MuPAD, we have to declare each object to be plotted with its parameters. Such a concept can be acceptable for an university student in some advanced courses (see the code in example 4). However, it may be too difficult for a high school student.

Example 4 *Standard plotting commands against the intelligent plot procedure*

```
f1 := plot::Function2d(-2):
f2 := plot::Function2d(2):
f3 := plot::Function2d(x):
f4 := plot::Function2d(2*sin(x)):
f5 := plot::Function2d(2*cos(x)):
p1 := plot::Point2d([1,2]):
poly1 := plot::Polygon2d(
   [[-1,-1],[-1,1],[1,1],[1,-1]], Filled=TRUE, Closed=TRUE
):
plot(f1, f2, f3, f4, f5, p1, poly1)
```



We can obtain the same result, or almost the same, using the smart plot concept:

```
plot(-2, 2, x, 2*sin(x), 2*cos(x), [1,2],
  {[[-1,-1],[-1,1], [1,1], [1,-1], [-1,-1]], Filled}
);
```

Another important thing that was introduced with MuPAD 4.0, is the new concept of a flexible interface. In fact, the interface in MuPAD, including its toolbar, can be modified for the needs of any specific mathematics course. The figure 1 shows two of the many possible versions of the toolbar.

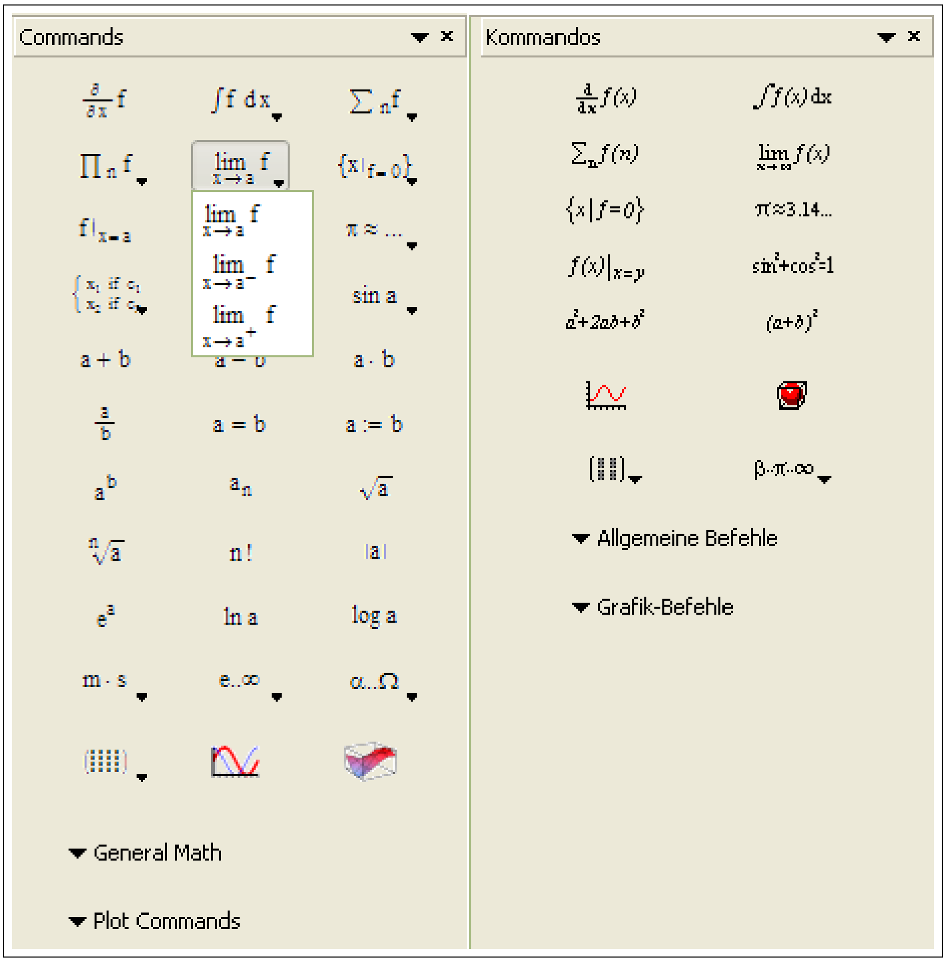


Figure 1: *Examples of two different toobars in MuPAD*

The new concept of MuPAD's interface is based on XML files, and it is a significant step towards a software interface that is user friendly and at the same time very flexible. The need for such interfaces has been discussed in many publications (see Kutzler [18] and Buchberger [6]).

There have also been some other developments in MuPAD addressing its usability in mathematics teaching.

6 Conclusions

There is no doubt that CAS can bring a number of improvements in teaching mathematics. However, before CAS will be widely used we have to change many things, and many developments have to be done. Here are some of them:

1. We have to develop methodologies for teaching mathematics with CAS and strategies for their introduction in our classrooms.

2. We have to produce teaching materials (textbooks, notebooks, online materials),

that will incorporate CAS into mathematics content according to good educational practices and principles.

3. We have to organize workshops, seminars and training sessions for current mathematics teachers as well as introduce CAS into teaching programs for mathematics teachers at the university and teachers' colleges.

4. We need appropriate changes in the official mathematics curriculum as well as changes in examination policies allowing CAS to be used on exams.

5. We need a lot of research on many aspects related to using CAS in mathematics teaching, in particular research on the influence of CAS on students' attitudes and the learning of mathematics, psychology of learning with CAS, etc.

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From String Art to Caustic Curves: Envelopes in Symbolic Geometry

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Abstract: In this paper, we use the symbolic geometry program *Geometry Expressions* to analyze three problems involving envelope curves. First we examine the envelopes of families of lines through points which are equally spaced on a pair of line segments. We use a combination of symbolic geometry and algebra to develop an expression for the area of the void in a popular string art figure consisting of 3 parabolas inscribed in a triangle. We use an envelope approach to reduce a popular calculus problem - that of finding the longest ladder which fits around an asymmetric corner – to an algebra problem which is readily solved using CAS. Finally we study the caustic curves generated by reflecting a point light source in a shiny cylinder. We analyze these both experimentally and theoretically, and focus on determining the parametric and Cartesian locations of the cusps. These examples illustrate how symbolic geometry technology can be used to make mathematics fun, accessible and challenging.

1. Introduction

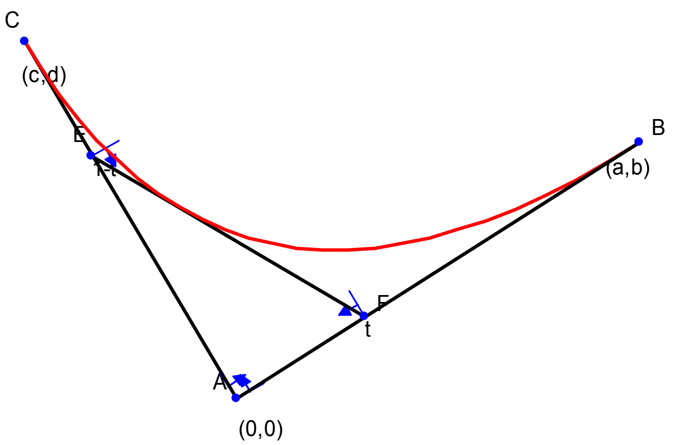
The envelope is that phantom curve your eye picks out when you look at a family of straight lines. Perhaps the most familiar example of the envelope is found in string art, where a family of lines is created by attaching strings to artfully located sets of pins (Fig 1.). Envelopes also occur naturally as light caustics, here the family of lines is a family of reflected rays, and the caustic appears as a fringe of light.
Geometry Expressions (www.geometryexpressions.com) is a symbolic geometry system. As such it is able to generate symbolic expressions from geometric figures. Amongst other things, Geometry Expressions is able to generate the envelope curve of a family of lines or circles, and to derive both parametric and implicit equations for the curve. In this paper, we will use envelopes in the symbolic geometry program Geometry Expressions to explore a variety of mathematical problems ranging from string art to light caustics. We will note how the use of a symbolic geometry system in concert with a CAS allows a mode of thinking which is part geometric, part algebraic, and which facilitates problem solving, exploration and discovery.

2. String Art

Mathematically, the envelope of a one parameter family of lines is the curve which is mutually tangent to that family. In Geometry Expressions a locus curve is specified by selecting a point and specifying the parameter which is to vary to create the locus. An envelope curve is created in a similar fashion except that instead of a point, a line, line segment or circle is selected.

Figure 1: We ask the question, what is the area of the void in the center of this string figure?

In Figure 2, we model a typical string art configuration where pins are evenly spaced along a pair of lines.
E is specified to lie proportion t along CA while F is proportion t along AB. The envelope of EF is constructed with respect to parameter t.



$$\Rightarrow b^2\cdot c^2-2\cdot a\cdot b\cdot c\cdot d+a^2\cdot d^2+Y^2\cdot\left(a^2+2\cdot a\cdot c+c^2\right)+X\cdot Y\cdot(-2\cdot a\cdot b-2\cdot b\cdot c-2\cdot a\cdot d-2\cdot c\cdot d)+X^2\cdot\left(b^2+2\right.$$

Figure 2: Geometry Expressions model of the envelope curve created by a string art model

The equation of the curve is second order and hence clearly a conic. Inspection of the coefficients shows that the second order term in Y and the term in XY disappear if a= -c. Hence, if points B and C are positioned equal distances on either side of the y axis, the parabola will be vertical (fig. 3)
Examining the parametric equation for the envelope (fig 3b), we see that X=0 occurs when the parameter t = ½. At this location the Y coordinate is $\frac{b+d}{4}$, or the midpoint of the median AG.

In the situation of figure 2, the triangle median AG is clearly vertical, as is the axis of symmetry of the envelope parabola. Hence, if we consider the general triangle (figure 1), we observe that the axis of symmetry of the parabola is parallel to the median of the triangle.

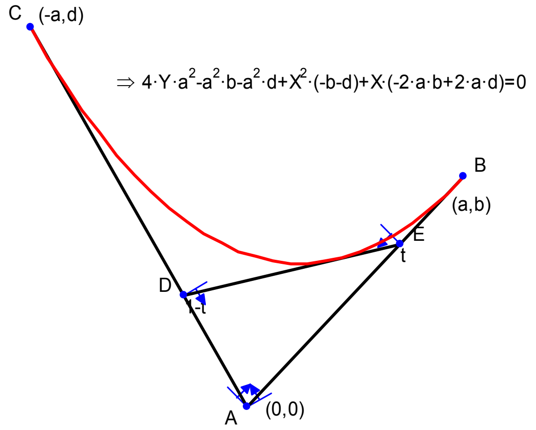


Figure 3: (a) Implicit equation of the envelope, where points B and C are equal distances along the x axis from A. (b) parametric equation of the envelope, along with the median AG of triangle ABC.

2.1 Areas

We address the question: what is the area of the region bounded by the three envelope curves in the string art of figure 1? We first ask the question: what is the area bounded by a single curve? We could do this using integration, but here we develop a geometrical argument.
It is convenient initially to transform our piece of string art so that the parabola is of the form $y=ax^2$ (fig 4).

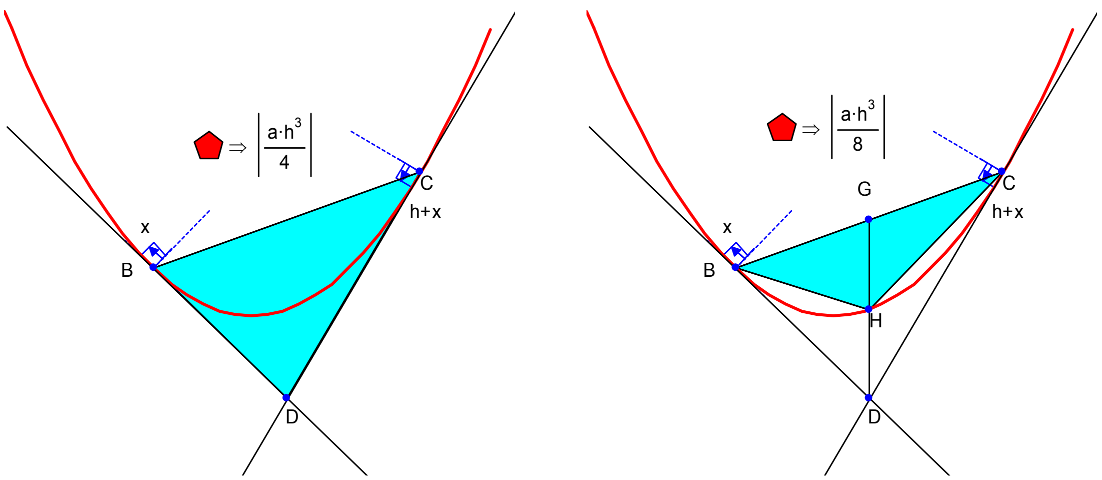


Figure 4: (a) Area of the triangle formed by a chord of the parabola and the tangents at its end depends only on the width of the chord, but not its location. (b) Area of BHC is half the area of BDC

Examining the triangle formed by the point on the parabola with x-coordinate x, the point with x-coordinate x+h, and the point at the intersection of the tangents to the parabola at those points, we see that its area is cubic in h, but independent of x.
If we now examine the triangle BHC, where H is the center of the median DG, we observe that its area is half that of BDC.

Now if we add a point I on the parabola at x coordinate $x+\frac{h}{4}$, the area BHI will be $\frac{ah^3}{64}$ and as the formula is independent of x, adding a point J at x location $x+\frac{3h}{4}$, the area HJC will also be $\frac{ah^3}{64}$.

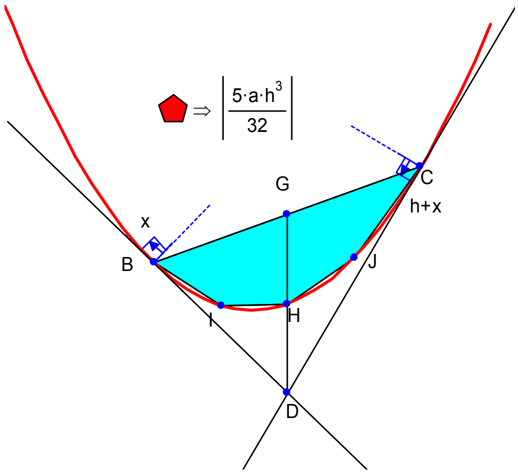


Figure 5: Area of BIHJC is 5/4 that of BHC

If the area of BHC is A, we see that the area of BIHJC is $A\left(1+\frac{1}{4}\right)$.

Continuing the process, we see that the area between the chord and the parabola is:

$$A\left(1+\frac{1}{4}+\frac{1}{16}+...\right)=\frac{4}{3}A$$

Now we look at three parabolas drawn on the sides of the same triangle. To calculate the points of intersection between the parabolas, we copy their equations into an algebra system and solve
We find 3 roots:

$$\{Y=0, X=0\}, \{Y=\frac{4\,c}{9}, X=\frac{4\,a}{9}+\frac{4\,b}{9}\}, \{$$
$$Y=\frac{4}{3}\,\mathrm{RootOf}(\_Z^2+1-\_Z, \mathit{label}=\_L4)\,c, X=\frac{4}{3}($$
$$-a\,\mathrm{RootOf}(\_Z^2+1-\_Z, \mathit{label}=\_L4)+5\,b\,\mathrm{RootOf}(\_Z^2+1-\_Z, \mathit{label}=\_L4)$$
$$+5\,a-4\,b)\,/\,(1+4\,\mathrm{RootOf}(\_Z^2+1-\_Z, \mathit{label}=\_L4))\}$$

The second is the interesting one. From its form, we see that it is 8/9 of the way along the median. We can therefore draw the straight edged triangle covering this area by placing points 8/9 of the way along the medians (fig 6):

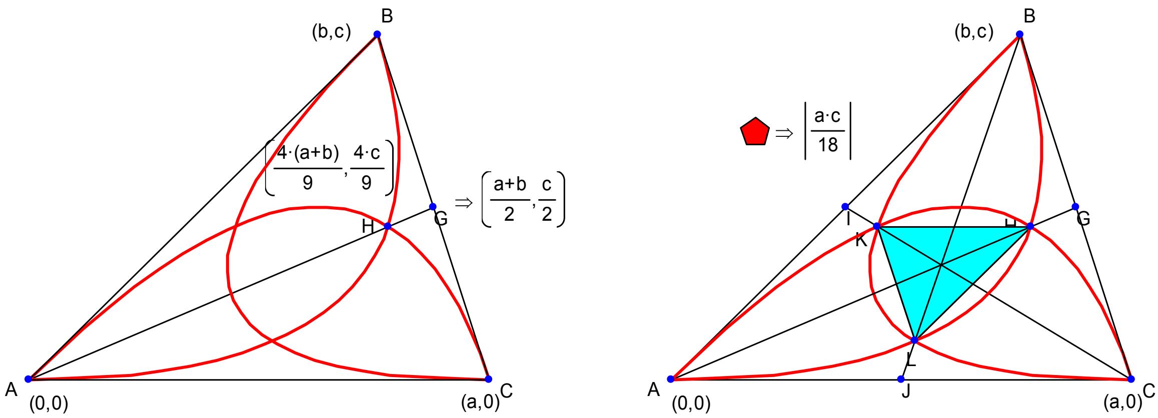


Figure 6: (a) The intersection of the parabolas is 8/9 of the way along the medians of the triangle. (b) Area KLM is 1/9 the area of ABC

As the area of the triangle ABC is ac/2, we see that the area of HKL is 1/9 the area of the original triangle. Now we can verify that K and L lie at equal distance to the median AG (fig 7).
Now let R be midpoint of the median AG. R is the point at which the parabola crosses the median. From the above, the area of the parabola segment KRL is 4/3 of the area of the triangle KLR, and hence is $\frac{ac}{81}$.

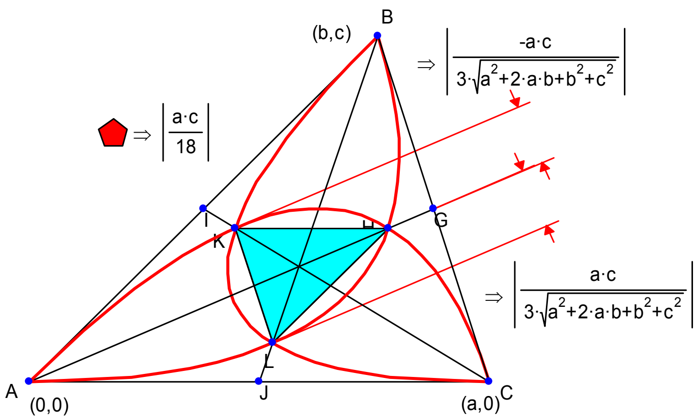


Figure 7: K and L lie at equal distances from the median AG.

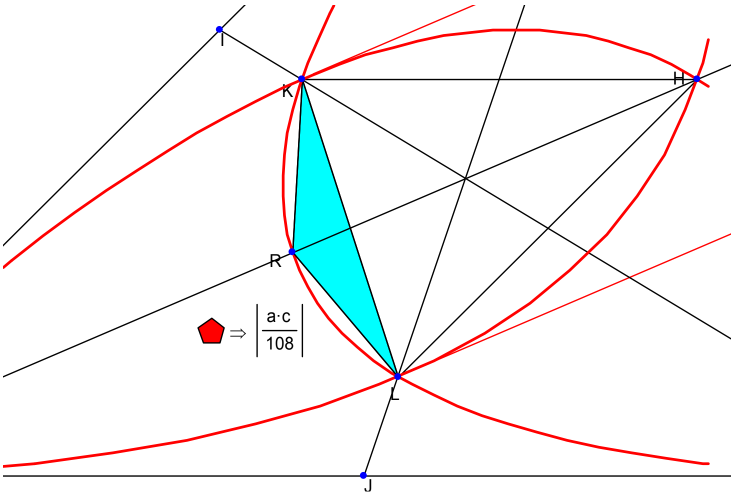


Figure 8: Segment KRL has area 1/54 that of the original triangle

Adding three such areas onto the area of the triangle HKL gives a total area for the void of:

$$V = \frac{5ac}{54}$$

or, if T is the area of the original triangle,

$$V = \frac{5T}{27}$$

3. A Ladder Problem

A classic problem, usually posed as an optimization problem in calculus, is that of finding the longest ladder which fits round a wall. Use of an envelope curve, however, can reduce this to a problem in algebra, readily solved with the aid of a CAS. It can be used to illustrate the relationship between geometric incidence between a point and a curve and the solution of an algebraic equation.

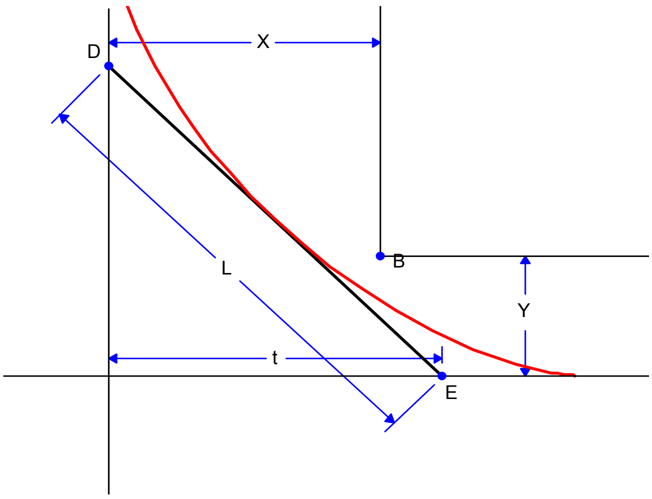


Figure 9: The envelope of a ladder of length L and the corner between corridors of width X and Y.

The problem is to determine the length of the longest ladder which will fit round a corner between two corridors of width X and Y. The situation is modeled in figure 9, and the envelope of the family of locations of the ladder is shown. A key observation is that the ladder itself stays below its envelope curve, and therefore as long as the envelope curve does not intersect the corner, then the ladder will clear. In order to find the longest ladder which clears the corner, one wants to find the value for L such that the point B lies on the envelope curve.

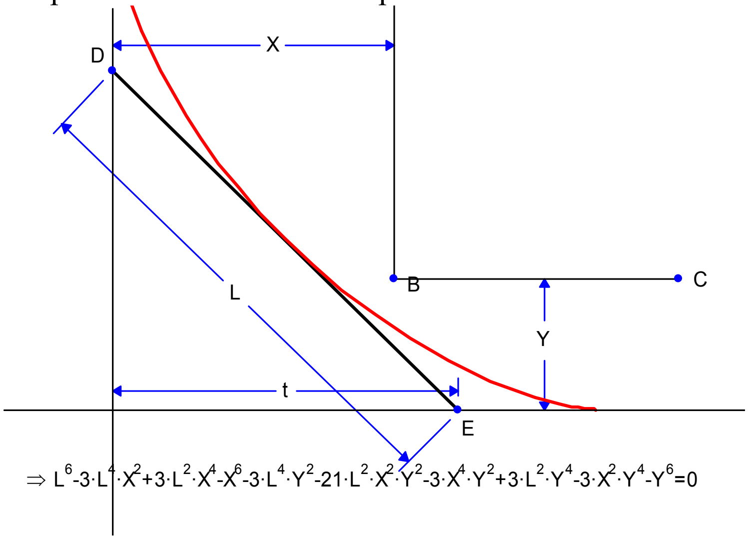


Figure 10: Equation of the envelope curve is computed by Geometry Expressions

Using Geometry Expressions, we can simply generate the equation of the envelope curve. We notice that the curve is equation is 6th order in L, but that it only includes even powers. Hence it can be considered as a cubic in L^2. We would therefore expect 6 solutions, but that they would be present as positive / negative pairs. Copying the equation from Geometry Expressions into an algebra system allows us to solve for L.

```
> solve(L^6-3*X^2*L^4+3*X^4*L^2-X^6-3*Y^2*L^4-21*Y^2*X^2*L^2-3*Y^2*X^4+3*Y^4*L^2-3*Y^4*X^2-Y^6 = 0,L);
```

➢

$$\frac{\sqrt{(Y X^2)^{(1/3)} \left(Y^2 (Y X^2)^{(1/3)} + 3\, Y (Y X^2)^{(2/3)} + 3\, Y X^2 + X^2 (Y X^2)^{(1/3)}\right)}}{(Y X^2)^{(1/3)}}, \ldots$$

Four of the resulting solutions are complex, and of the remaining two real solutions, only one is positive. This is the one to copy back into Geometry Expressions. We observe that with this length ladder, the point B lies on the envelope curve and the ladder only just makes it around the corner.

As a final observation, we note that the solution for L is not symmetric in X and Y. However, the geometry problem clearly is symmetric: the longest ladder which can be dragged from a corridor of width X into a corridor of width Y is obviously the same length as the longest ladder which can be dragged from a corridor of width Y into one of width X. It would therefore seem reasonable that some simplification could be done on the solution to expose its essential symmetry. This could be set as an exercise to be completed by hand, or one could coerce simplification in the algebra system by asserting that both X and Y are positive.

```
> simplify(%) assuming X>0, Y>0;
```

$$(X^{(2/3)} + Y^{(2/3)})^{(3/2)}$$

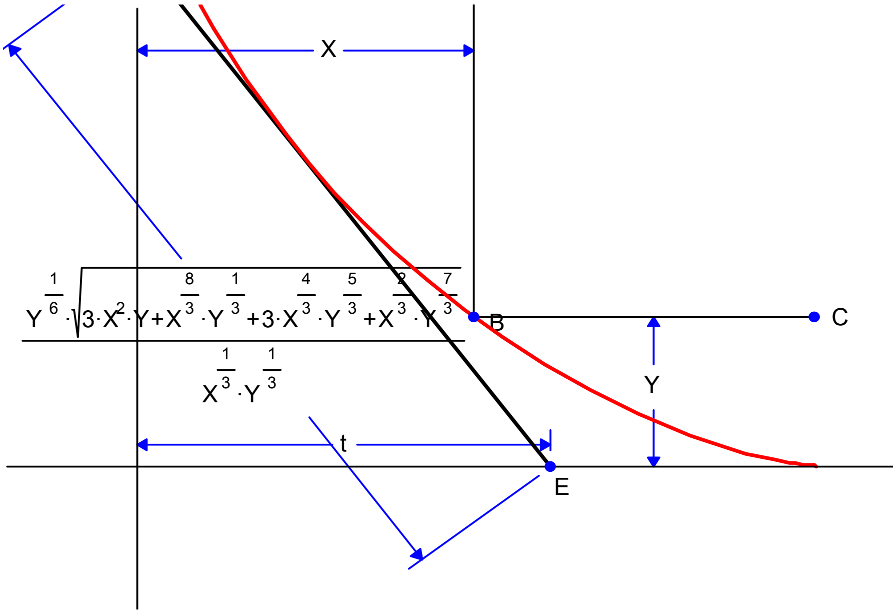


Figure 11: Solution is pasted back from Maple into Geometry Expressions, and we observe that the corner point B lies on the envelope.

4. Light Caustics

When light reflects off a convex curved surface, the reflected rays form a curve of bright light called a caustic. This curve is exactly the envelope curve of the reflected rays. Figure 12 shows the caustic curve formed by light reflecting in a wedding ring.

Figure 12: Caustic curves formed by light reflecting inside a gold ring

We will study the form of the caustic curve, when light from a point source reflects in a cylinder. To acquire experimental data against which to compare our theoretical findings, we use the apparatus shown in figure 13. A Perspex cylinder is used as our reflecting surface. A small LED lamp is used to position our light source inside the cylinder. A circular paper disk is marked in concentric rings, each ring being 1/10 of the radius of the cylinder. This allows us to observe the location of features of the curve as a proportion of the cylinder radius.

Figure 13: Apparatus for studying caustic curves with a light source inside the reflecting cylinder.

In figure 14 we see the caustic curve when the light source is at the circumference of the cylinder, and when the light source moves closer into the center.

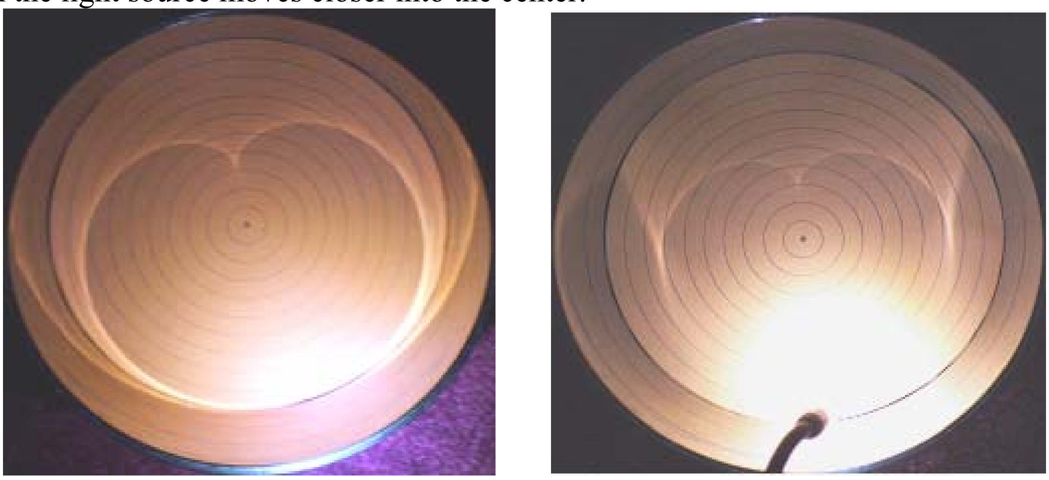


Figure 14: Caustic curves observed with (a) the light at the circumference of the cylinder and (b) the light source interior to the cylinder.

We can model the behavior of a single beam of light in Geometry Expressions. First we create a circle AB, and constrain its center to be (0,0) and its radius to be 1. We then constrain the parametric location of B on the curve to be t. This has the effect of specifying the AB to be angle t (in radians) from the x axis. Geometry Expressions allows you to reflect in a line, but not in a curve. However reflection of light in a cylinder is equivalent to reflection in the tangent to the cylinder. Hence we draw a line through B and constrain it to be tangent to the circle. We now create point C, constrain its location to be (0,-a), and draw an infinite line through C and B. We complete the model by reflecting BC in the tangent line.

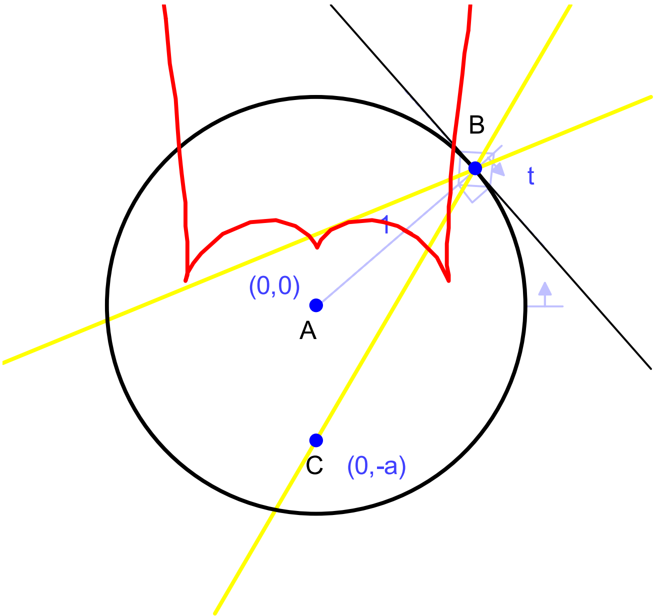
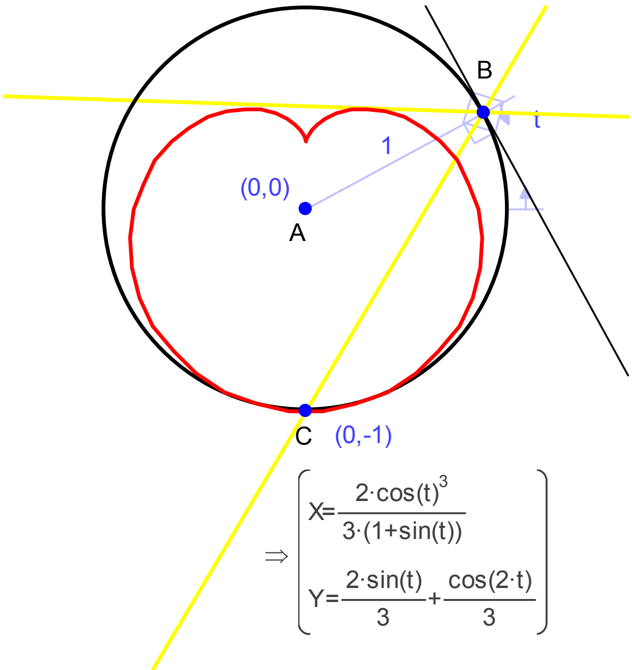


Figure 15: Geometry Expressions model of a beam of light reflected in a cylinder.

The caustic curve is the envelope of the family of reflected lines. We are interested in the location of the central cusp. First in the situation where a = 1 (fig. 16)



$$\Rightarrow \left[\begin{array}{l} X=\dfrac{2\cdot\cos(t)^3}{3\cdot(1+\sin(t))} \\ Y=\dfrac{2\cdot\sin(t)}{3}+\dfrac{\cos(2\cdot t)}{3} \end{array}\right]$$

Figure 16: Parametric equation of the caustic with light source on the circumference

The parameter t specifies the location of point B on the circle as an angle in radians. When B is at the location (1,0) t will have the value 0, when B is at the location (0,1) t will have the value $\pi/2$, etc. Clearly when B is vertically above A, the light from C will reflect straight back, and will touch the curve at the cusp.

To find the location of the cusp, we need simply substitute t= π/2 into the Y component of the curve equation in figure 16:

$$\frac{2\sin\frac{\pi}{2}}{3}+\frac{\cos\pi}{3}=\frac{1}{3}$$

Looking at the picture in figure 14a, we see that the light cusp is somewhere around the third circle or approximately 0.33 radii from the center

4.1 Cusp on the Circumference

As the light source moves towards the center of the cylinder, the caustic curve is no longer fully contained in the cylinder. However, at some point, it re-enters to give the shape in Figure 17. Using the circular graph paper, we can estimate the location of the central cusp at this point to be on the second circle.

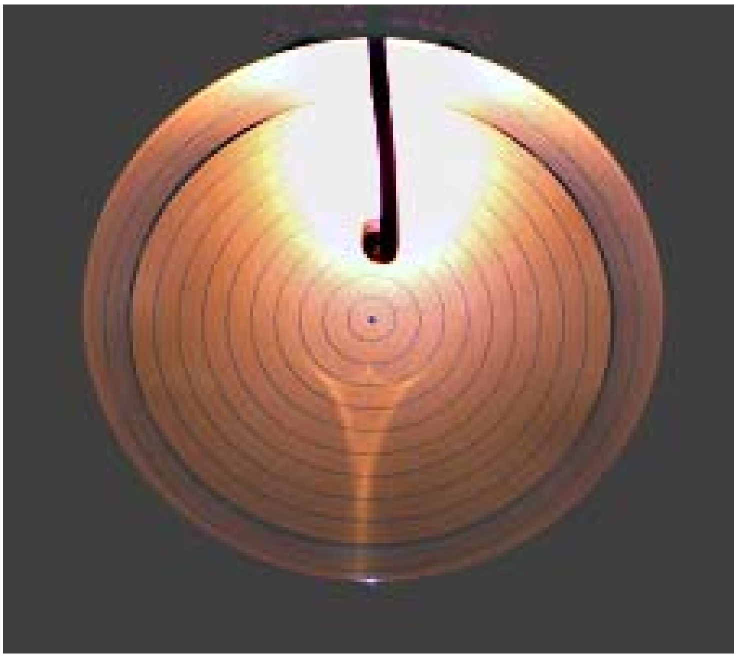


Figure 17: The caustic when the far cusp touches the circumference.

In Geometry Expressions, we can change the coordinates of C back to (0,-a) and drag C towards the center of the circle until the second central cusp appears:

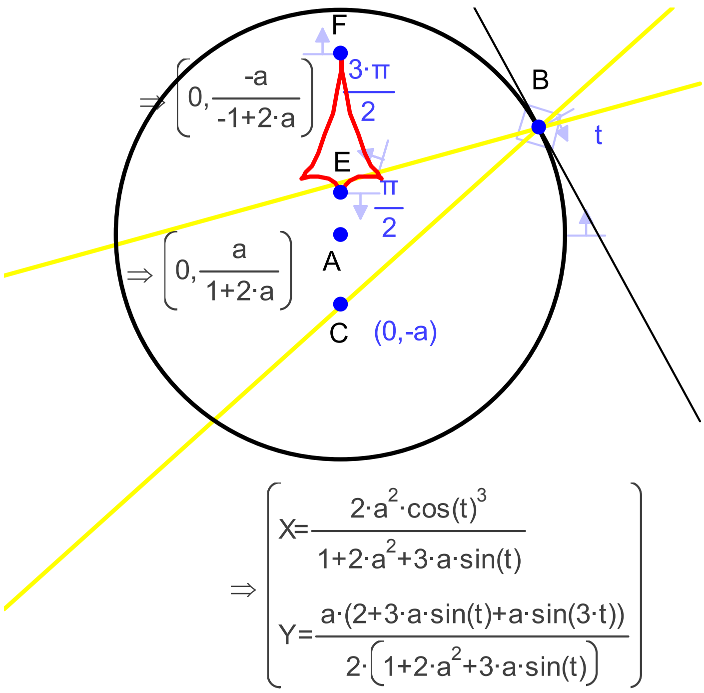


Figure 18: Curve equations for general a. Points E and F are put on the curve and constrained to be at parametric locations π/2 and 3π/2

In figure 18, points E and F are positioned at parametric locations π/2 and 3π/2 on the curve. Coordinates for these points have been computed by the software.
For F to lie on the circumference, we need:

$$\frac{-a}{-1+2a} = 1$$

Solving for a yields:

$$a = \frac{1}{3}$$

Substituting into the expression for the y coordinate of E gives

$$\frac{\frac{1}{3}}{1+\frac{2}{3}} = \frac{1}{5}$$

Examination of figure 17 shows the cusp lies on the second ring of the circular graph paper, or 1/5 of the way out from the center to the circumference, which corresponds exactly with our theoretical predictions.

4.2 Further Questions

Let's assume we move our light source a long way from the cylinder, so that the distance a is effectively infinite. Our cusp will now have location:

$$\lim_{a\to\infty} \frac{a}{1+2a} = \frac{1}{2}$$

This is the situation observed in the wedding ring (fig. 12).

Having analyzed the central cusps, we are left with the question what are the coordinates of the cusps which do not lie on the y axis?

For a curve (X(t),Y(t)), cusps occur where the derivatives of X and Y with respect to t simultaneously vanish [1]. The caustic's curve equation can be copied into Maple, differentiated and solved as follows:

$$X := \frac{2\cos(t)^3 a^2}{1 + 2a^2 + 3\sin(t)\, a}$$

$$Y := \frac{(2 + 3\sin(t)\, a + \sin(3\, t)\, a)\, a}{2\,(1 + 2a^2 + 3\sin(t)\, a)}$$

```
> solve({diff(X,t)=0,diff(Y,t)=0},t);
```

$$\left\{ t = -\frac{1}{2}\pi \right\}, \left\{ t = \frac{1}{2}\pi \right\}, \{t = \arctan(-a, \mathrm{RootOf}(a^2 - 1 + \_Z^2))\}$$

```
> allvalues(arctan(-a, RootOf(a^2-1+_Z^2)));
```

$$\arctan(-a, \sqrt{1 - a^2}),\ \arctan(-a, -\sqrt{1 - a^2})$$

Drawing points on the curve with these parametric values (copying the solutions back from Maple into Geometry Expressions), we see they lie on the cusps (fig. 19a).

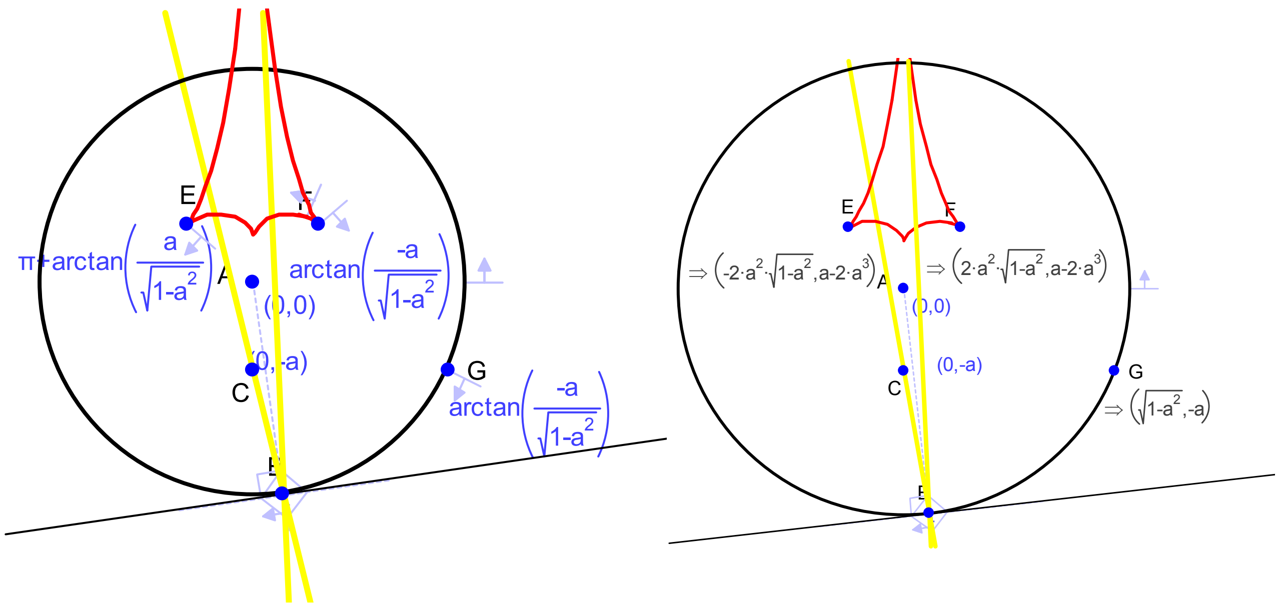


Figure 19: (a) G is the point on the circumference of the circle with the same parametric location as the right hand cusp. (b) G has the same y coordinate as C. Coordinates of the cusps are also given.

Where does point G lie on the circle when the reflected ray goes through the cusp? One way of finding out is to put a point on the circle and give it the arctan() as its value. Now, hiding the parametric locations of E,F and G, we can display their coordinate locations (fig 19b). We see that G has the same y coordinate as C.

We have discovered that cusps occur when the original ray emanates from point C either in a vertical or a horizontal direction. The vertical seems geometrically intuitive, this is the direction at which the ray reflects back on itself, and one might expect singular behavior. What, if anything, is special about the situation where the original ray is horizontal?
To answer this question, we consider the angle between the incident ray and the reflected ray. Clearly when the incident ray is vertical, this angle is 0 and hence at a minimum. We ask the question, what direction for CB yields a maximum angle?
First, we note that this is equivalent to finding a maximum value for CBA. We could, of course generate an equation for this angle in Geometry Expressions, then differentiate and solve to yield the appropriate value for t:

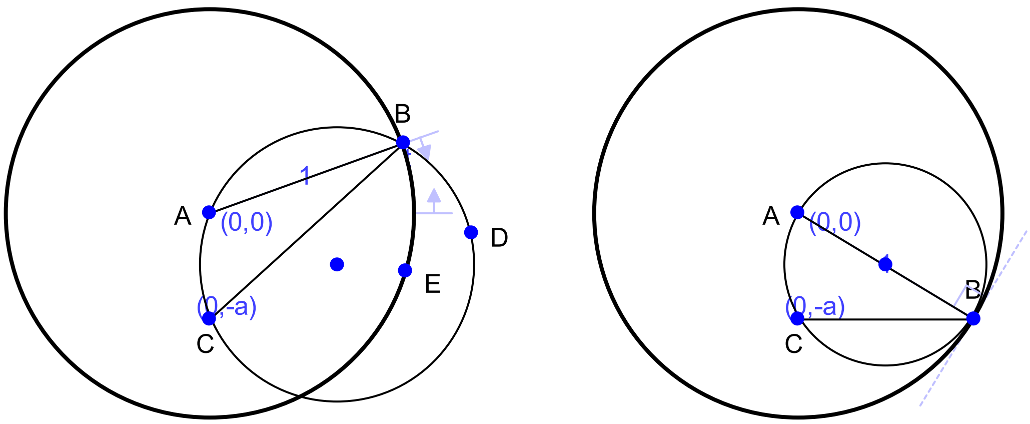


Figure 20: (a) E is inside the circumcircle of ABC, hence angle AEC < angle ABC. (b) The circumcircle of ABC is tangent to the original circle. This corresponds to a maximum of angle ABC.

However, a geometric approach to the problem can be taken by observing that all the points D such that ADC = ABC lie on the circumcircle of ABC. In figure 20a we observe that if this circumcircle intersects the original circle in two places, and if E lies on the portion of the circumference of AB cut off by it then AEC>ABC. Hence, ABC can only be maximal if the circumcircle to ABC is tangential to circle AB (fig 20b). In this case it is easy to see that AB is a diameter of the circumcircle and hence the angle ACB is right.
We have hence shown that the cusps of the caustic occur at extrema of the angle between incident and reflected rays.

5. Conclusion

We have examined three situations where the automatic derivation of envelope curves within a symbolic geometry program (Geometry Expressions) allow problems which would normally be addressed using calculus to be analyzed using a combination of geometry and algebra. In addition, results obtained symbolically, using a computer algebra system, motivated further geometrical study.

Without the use of technology, envelope curves are not accessible at a high school level. With the use of technology they can become a rich source of example problems and illustrative scenarios. In short, we have made some interesting and fun mathematics accessible through technology.

Reference

[1] DJ Struik (1950) Lectures on Classical Differential Geometry, Dover, NY

Technology and Teaching Mathematics, An Indian Perspective

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Abstract: Education system in India is "10 + 2 + 3" model, and it is very examination-oriented. Education is a state matter and there are in all 28 states and 7 union territories. There is little scope for innovation and use of technology in regular teaching. In the second part of the paper, I will discuss the role of technology in teaching and a methodology of supplementing regular teaching to make the subject matter more lively and stimulating. In the third part, I will discuss my personal experiences of conducting math lab sessions for middle school students (age 13-14) and for senior secondary students (age 16-17). The last two parts of the talk will have multimedia presentations.

1. Indian Education System

The system of education in India is "10 + 2 + 3" model. Education is a state matter and there are in all 26 states. Each state has its own Education Board which is responsible for all matters relating to school education in that state. For more details, see *Rajput (2004).*

Indian education system is very examination-oriented. The "examination mentality" has had a strong negative influence on everything connected with education in India: textbooks, style of teaching, and teacher preparation. There is little scope for innovation and use of technology in regular teaching.

2. On the role of technology and teaching mathematics

Math education is in the midst of a change driven by technological developments.
What is technology? Historically, for math education point of view, it can be summarized as

Slate $\longrightarrow$ Slide rule $\longrightarrow$ Software

From oral education to engraving on clay tablets/stones, from writing on palm leaves to invention of ink/paper and printing, to digital media, are all various phases of evolution of technology.

In the present scenario, technology in math education can be viewed as: Hand held calculators on one hand and multimedia workstations on the other. Digital class rooms on one hand and virtual class rooms on the other. Course distribution over the internet, ICT. Then there are CAS, for example: Maple, Mathematica, Mupad, Drive, and dynamic geometry softwares like GSP, Cabri, and so on. All these help to compute and simulate. Then, there are course management tools and evaluation tools.

Aspects of technology use

Technology is entering many facets of math teaching and learning. Technology is a valuable tool in the teaching and learning of mathematics, for it has the ability to empower mathematics students as well as mathematics instructors. The use of technology is prompted, in some cases, by the methods that can help a teacher to do some jobs easily and in a better way. In others, it is motivated by the belief and effort to impart instructions in a way that will help to achieve the learning goals of students.

Integration of technology in education can also be because of non-academic reasons: to build the profile of a school/college.

Positive aspects of technology use

Technology adds new components to teaching and learning of mathematics by providing:

- Tools for visualizations/ illustrations.
- Tools to do tedious computations in less time.
- Tools to recognize pattern in a problem.
- Helps to formulate conjunctures.
- Tools to develop problem solving skills.

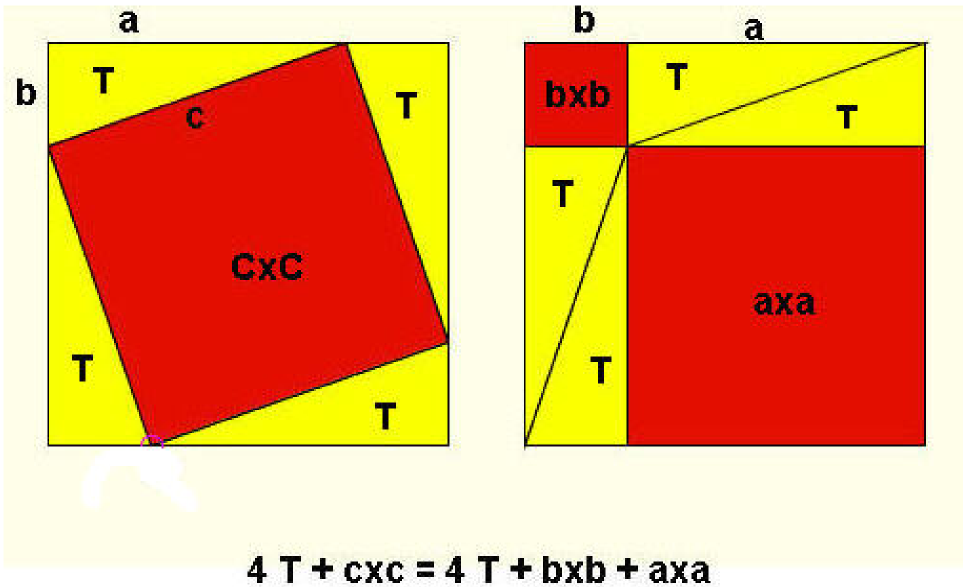
Technology Enhanced Learning, Visualizations

Visualizations can be developed manually (sometimes) or with the help of technology tools provided by calculators/computers and software. They play an important role in learning and teaching of mathematics, see *Guzman (2002).* These can also be used to make the learning process non-routine: interactive and explorative. We discuss this idea in details with two examples. The process is divided into following steps:

- Explore / Experiment
- Observe / Conjuncture
- Convince / Prove
- Extrapolate / Extend

First example

This example is taken from the middle school level mathematics: Pythagoras theorem. One of the standard visual proofs of this theorem is as given below:

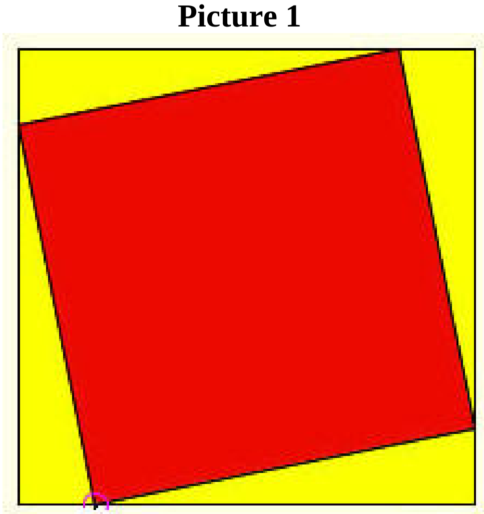


Instead of showing the above proof, students can be made to explore and discover it as follows:

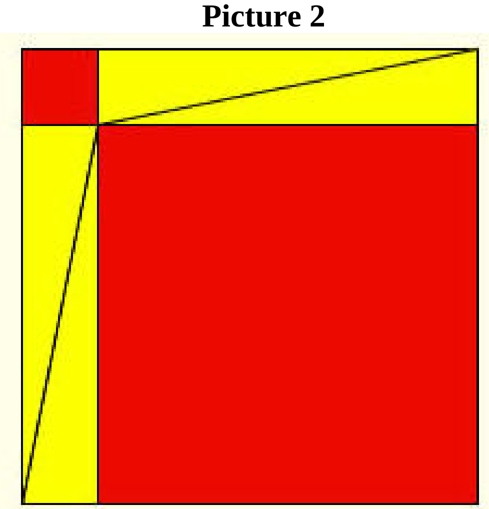
1. **Exploration:**

Students are shown an animation in which a red square is inscribed inside a yellow square, as shown in picture 1 below. And picture 2 shows a rearrangement of picture 1.

Picture 1

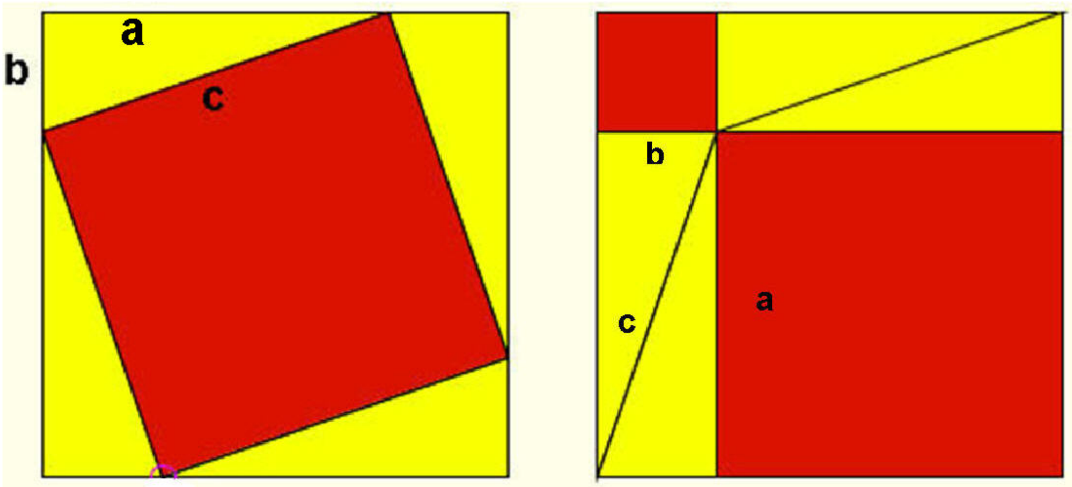


Picture 2

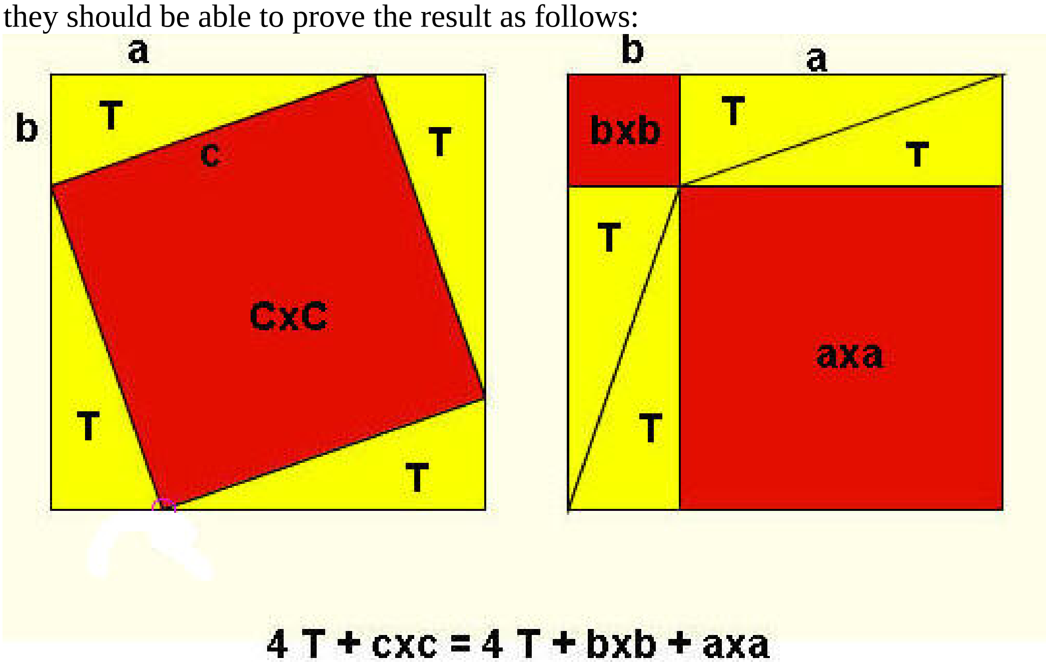


The students are asked to make some conjectures about the relations between the areas of various shapes inside the bigger square. Most probably no answers will be forthcoming. At this stage they are shown the following two pictures:

Picture 3 **Picture 4**



2. Conjecture:
Hopefully, some of the students will guess the relation $c^2 = a^2 + b^2$.
3. Convince/Prove:
Finally they should be able to prove the result as follows:



Second example

The second example is from secondary level: how the coefficients a, b and c affects the graph of a quadratic $y = ax^2 + bx + c$.
1. Exploration:
Analyze the effect of the change in the values of the coefficient a on the graph of the quadratic:

$y = ax^2 + bx + c$. Students are shown animations how the graph changes with change in the values of the coefficient *a* and are asked to give their observations.

2. Observe/Conjecture:

The, graph bends away from x-axis when a is positive, and for a negative the graph bends towards x-axis. The graph is a straight line for $a = 0$. It always passes through the point $(0, c)$, whatever is the value of a.

3. Convince/Prove:

Students should try to justify the claims that for both b and c are fixed, why does the shape of the graph of $y = ax^2 + bx + c$, depends upon the sign of a. Further, the graph is a straight line $y = bx + c$, if and only if $a = 0$. Further, why does the graph always passes through the point $(0, c)$, whatever be the value of a.

1. Exploration:

Analyze the effect of the coefficient b *alone on the graph of the quadratic:* $y = ax^2 + bx + c$.

Students are shown animations how the graph changes with change in the values of the coefficient b alone, and are asked to give their observations.

2. Observe/Conjecture:

The 'tip' of the graph traces a path similar to that of the given quadratic with shape upside down, i.e., it is also a quadratic with coefficient of second degree term has sign opposite to the given quadratic.

3. Convince/Prove:

The student verifies analytically that the point (h, k) , corresponding to the 'tip' of the graph satisfies the equation $k = -ah^2 + c$, as observed.

1. Exploration:

Analyze the effect of the coefficient c *alone on the graph of the quadratic:* $y = ax^2 + bx + c$.

Students are shown animations how the graph changes with change in the values of the coefficient c alone, and are asked to give their observations.

2. Observe/Conjecture:

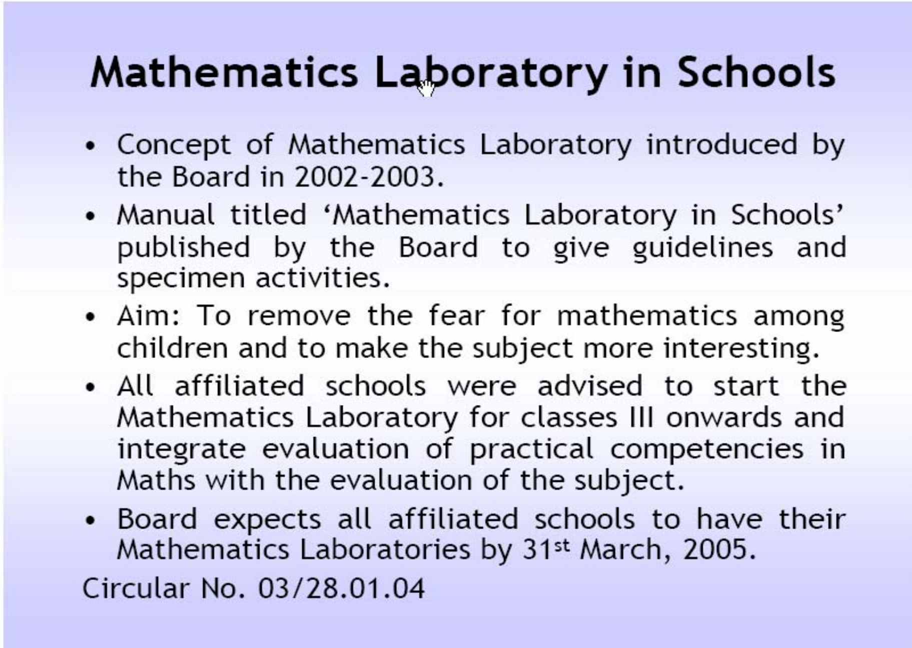
The tip (h, k) moves along a vertical line.

3. Convince/Prove:

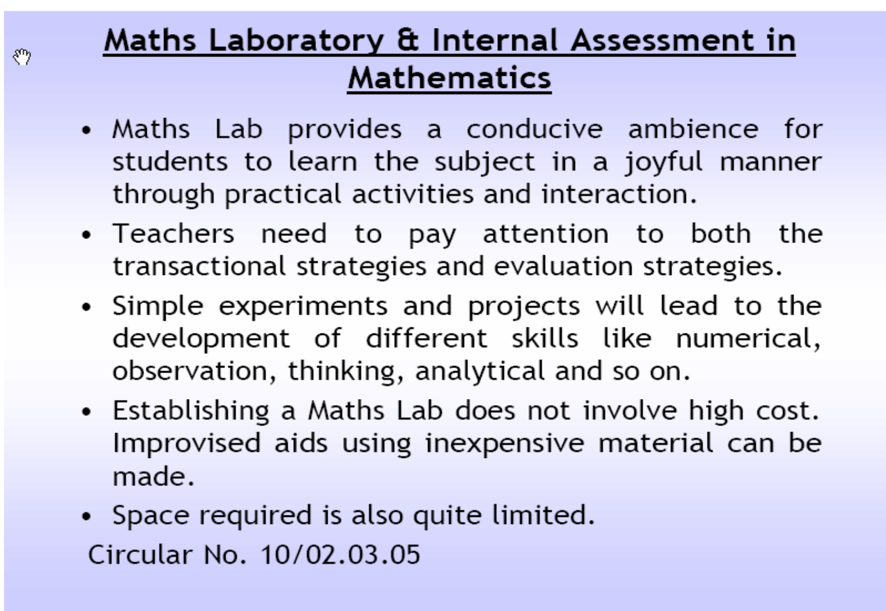
Students verify analytically that the tip (h, k) moves along a vertical line, as observed, with $k = c - ah^2$.

3. On conducting math labs

In March 2004, the "Central Board for Secondary Education" issued a directive to all the schools afflicted to it:



In March 2005 came out with some conclusions about math labs:



To help teachers of local schools to implement this, two math labs were designed.

Recipe for a math lab

- Select a topic from the curriculum
- Revise the basic concepts related to the topic
- Design some hands-on activities and give an example of the activity you want them to do.
- Activity should involve observations, analysis of observations based on the concepts revised, conclusion based on the observations and their justifications.
- Give a short quiz to test the assimilation of the concepts.

Math Lab on the concept of Areas

This lab was conducted for middle school (age 14-16 years) students (186) on the concepts of perimeter and areas. I give below an outline of the workshop:

The concepts of **rectilinear figures: t**riangle, rectangle, square, parallelogram and rhombus were revised. Computations of areas of these figures were demonstrated with flash animations.
Non rectilinear figures were introduced and the problems involved in finding the perimeter and area of a circle were discussed. With the help of animations, the computations of these were demonstrated. Students were asked to do some hands-on activities and a short quiz was given. Historical background of defining and determining pi was also demonstrated.

Feedback from students:

- Did the workshop help you in revising the concepts about Areas of figures?

YES	NO
183	3

- Were the activities interesting?

YES	NO
172	14

- How were the quiz questions?

GOOD	EASY	DIFFICULT

125	45	16

- How was the workshop?

VERY GOOD	GOOD	OK
129	43	04

- Would you like to attend more such workshops?

VERY MUCH	YES	NO
109	69	08

Math Lab on the concept of derivatives

Another Math Lab for senior secondary (age group 17-18) students (77) of K.V. I.I.T. Bombay School was conducted on the topic of derivatives and applications.

Feedback from students:

The way lab was conducted	**Contents of the lab**	**Would you like to have more such labs?**
Very interesting (46.15%)	Difficult (6.41%)	Very much (41.02%)
Interesting (42.30%)	Moderate (78.20%)	Yes (52.56%)
OK (11.53%)	Easy (15.38%)	No (6.41%)

- **Some general comments:**

1. Activities were new and interesting. Concepts are much clearer now.
2. It cleared my fundas. Would like to attend more.
3. We need such labs in Integration also.

4. Actually such labs are really good, but I need to be intelligent enough to grasp things!
5. It was time consuming, but still interesting.
6. Boring!!
7. gre8!
8. Sir, please conduct these types of labs more often.
9. This is something that I had not done before, helped me to understand graphs.
10. This kind of labs should be conducted for chemistry also.
11. Excellent, just a bit slow.
12. Cool! Expecting more such in future.
13. Amazing mental exercise! Too good, making concepts crystal clear.
14. I really liked it from the down of my heart.
15. A new look at calculus, amazing! Thanks a lot!
16. (Thank you Sir $)^2$.
17. This comment by one of the students sums it all:

4. Conclusions

Negative aspects of Technology use

One of the biggest drawbacks of technology is that it needs resources: both financial as well academic. Given that schools/colleges have limited resources, there are difficult choices to make on how to invest the resources. Also, sometimes technology gives a false sense of accomplishment to deceive the school/college and community.

For teachers, using technology increases workload to learn technology. Also it puts pressure on them since they have to change their teaching approach. In their enthusiasm, teacher may lose perspective and aim of instruction. For students, learning to use technology itself may be time consuming and frustrating. *Galbraith (2002)* states that "Results suggest, teaching demands are increased rather than decreased by the use of technology, that attitudes to mathematics and to computers occupy different dimensions, and that students adopt different preferences in the way they utilize available resources"

Using technology can lead to an easy way of getting answers for students. This can lead to weakened conceptual understanding and basic skills.

Technology is not versatile and its functions are limited. After all it is the product of human mind. Technology by itself cannot promote learning. *Olsen (1999)* discusses one of the most extensive examples of technology used to provide automated instruction: Virginia Tech's Mathematics Emporium, a 58 000 square foot (1.5-acre) computer classroom. That seems to suggest that many feel "mathematics is something primarily to be delivered and consumed". Finally, more often than not, technology needs a facilitator.

Questions:

Some of the questions one would like to ask are:

- Is the use of technology required for learning in general, or for learning mathematics in particular?
- Is there evidence that students learn mathematics better with any technology or with a specific technology?
- Is it worth spending on technology when there is a shortage of resources, both financial and academic?
- Can technology compensate for the lack of qualified instructors?
- Should the use of CAS and other software be following prior understanding of mathematical concepts and procedures, or as a means for the development of such understanding.

Some answers:

- Technology is not a universal tool and an omnipotent aid that can always help.
- Technology often provides convincing demonstration of ideas, helps to conjuncture, but does not replace 'proof'. Nor does constructing a proof rules out the use of technology. This distinction must be emphasized and the importance of both must be appreciated.
- Technology can neither replace live class room teaching nor can it compensate lack of qualified teacher. In terms of valid reasons, from a learning standpoint, there actually is no conclusive evidence that specific technology can improve students' learning of mathematics when implemented properly.
- Learning does not take place in the technology.
- Learning takes place in the interaction between motivated faculty and motivated students. Whenever possible, technology can be used to provide such environments.
- Used properly, technology can help the teacher to impart instructions in a more effective way that will enhance students understanding and motivate them to learn.

Instead of making
Kids love mathematics,
Let us create mathematics
That kids will love.

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Special Session at ATCM 2007

Connecting Dynamic Geometry Software with Computer Algebra System

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Abstract: This special session consists of five short presentations. Each author will demonstrate their examples to show why it is crucial to connect Dynamic Geometry (DG) with Computer Algebra System (CAS) in achieving the followings:

- Making Mathematics more accessible to more students.
- Motivate students to investigate more challenging problems.

Contribution from Thomas Banchoff:

For the calculus and geometry of curves and surfaces, interactive geometry software makes it possible for the instructor to illustrate phenomena in the classroom and for the student to investigate conjectures individually and in groups, and to share their experiences with the class.

Our visualization software makes it possible to enter parametric curves and surfaces as vector functions of one or two variables, and to explore the images in two- and three-dimensional linked windows as they transform under rotations and parameter changes, as well as algebraic manipulations of the vector representations.

We illustrate these features by studying the behavior of normal vectors of curves in the plane and surfaces in space and their associated curvature functions, to generate and refine conjectures and to relate algebraic proofs with geometric demonstrations. Specific examples include parallel curves and surfaces and the behavior of evolutes.

The items I would choose to emphasize the way that an instructor or student can modify an existing demonstration to illustrate specific vector functions, and to define auxiliary functions in a way that is natural for mathematicians. I want to stress the fact that we can deal with several curves and their tangent-normal-binormal apparatus at the same time, as well as curves associated with surfaces. The idea of linked windows, so that selecting one image and rotating it produces the same rotation in the other, is a crucial feature of our operation, and one that distinguishes it to some extent from other 3-dimensional systems, at least the last time I checked

Some of the things I will show include defining a curve X(t) with easy to use notation, immediately being able to show its velocity and acceleration curves X'(t) and X''(t) computed by a combination of symbolic and numerical techniques, defining normalized vectors such as the unit tangent T(t) by typing T(t) = X'(t)/|X'(t)| or simply unit(X') and defining the unit normal U(t) obtained by rotating T(t) a quarter-turn in the counterclockwise direction by typing U(t) = (-T_2(t),T_1(t).

We can then observe animate the points X(t0) moving along the curve and show a vector X(t0) + U(t0) moving along with it. We can show U(t0) in separate window so that motion of X(t0) along the curve is related to motion of U(t0) along the unit circle. We can then define the signed curvature by typing kg(t) = dot(T'(t),U(t))/|X'(t)| and we can display the graph of kg(t) versus t in a separate window, with (t0,kg(t0) moving along the graph as t0 goes through the domain of X(t). We can then define a parameter r and investigate the parallel curve at distance r by typing X(t) + r*U(t) as we change the parameter r. We can observe the set of singularities of parallel curves and carry out the algebra to find the locus of singularities, presented as the evolute curve E(t) = X(t) + (1/kg(t))U(t).

We can then do the same thing for surfaces in space by typing X(u,v) for the surface and X_u(u,v) and X_v(u,v) to get the partial derivatives. We get the unit normal vector by typing N(u,v) = unit(cross(X_u(u,v),X_v(u,v))) and we can then explore the geometry of the Gauss mapping by looking at X(u0,v0) + r*N(u0,v0) on X(u,v) + r*N(u,v) in one window and N(u0,v0) on N(u,v) in another. The two surfaces move in a coordinated fashion as we rotate either in its window. As an analogue of extrinsic curvature in the case of curves, we define a curvature function K(u,v) as the dot product with N(u,v) of cross(N_u(u,v),N_v(u,v)) divided by |cross(X_u(u,v),X_v(u,v))|.

Contribution from Jen-chung Chuan:
The name ATCM contains both the words "technology" and "mathematics". The role of the mathematicians in this technological world is to check the dishonesty of the engineers. As it will be shown at the Opening Ceremony at ATCM 2007 (http://atcm.mathandtech.org) , a serious bug appears in Excel 2007, which didn't exist in the previous versions; try to enter

=850*77.1

under Excel 2007. Take a look to see what you get when enter

int(1/(1+x+x^4)^2,x=0..infinity);

under Maple 11. Furthermore, Wei-Chi Yang also found the error of

$$\frac{d}{dx}\left(-3\cdot\sqrt[3]{(2x-1)^2}\right)$$

under Maple 11; we suspect there are more errors from many software packages. We now see a warning sign of the software industry: Are we entering the Era of the Downfall of the Computer Revolution? If the public continues to tolerate the situation, then more and more similar stupidities are bound to multiply in future.

During the first week of medical school, students learn to fill out the death certificate. Visitors to the department of civil engineering of National Taiwan University are shown a two-inch wide gap due to construction irregularities. So, when we go back to classrooms, the first thing is to show students this bug in Excel 2007. All three samples will teach the students to be humble. Be humble, doctor cannot treat every disease. Be humble, engineer cannot build a perfect building. Be humble, applying computer to mathematics has its limitations.

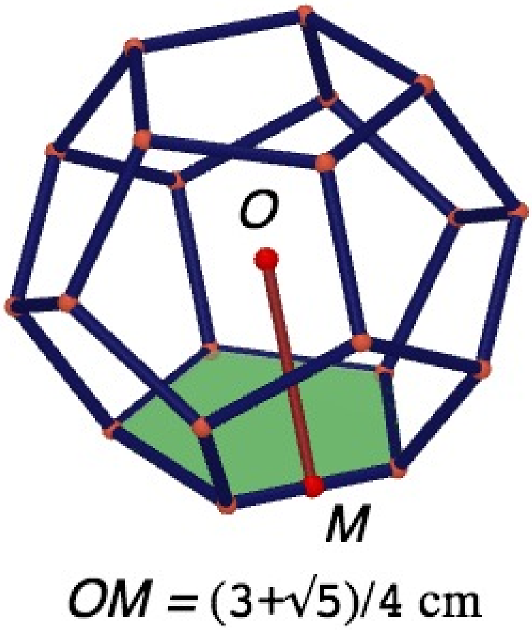
Contribution from Jean-Marie Laborde
There are various examples worldwide, relate the nature of the mathematical thinking developed by users in solving problems (perceived as open), when using powerful technologies like dynamic geometry and/or CAS. Many people report about the crucial role technology can play to help them in stating the right conjecture and or in helping scaffolding towards a proof. I do not see conflicts between the Computer Algebra and Dynamic Geometry Systems because I see them as complementary.
Consider a simple example: To express the radius of the sphere through the midpoints of the edges of a regular dodecahedron as a function of its edge length a. It can be show that

$$r = (3+V5)a/4$$

To do so one way might be to try to compute it using a regular CAS system. However, it is not so easy, because we first have to create some symbolic quantities representing the dodecahedron.
Suppose we are using now a DGS, say Cabri 3D. It is easy (in a few clicks) to create the dodecahedron. Then we can probably visualize how the Pythagorean Theorem could be applied to compute the desired radius and create a chain of intermediate computations leading to the expression $(3+V5)a/4$. Here the paper-pencil parallel computation involves some intermediate

formulas and it could be a good idea to check them at each step in comparing actual lengths displayed by the DGS and the results of the evaluation of the successive formulas for some different values of the edge length (dynamic features of DGS). This provides us with a mean to be sure about the final formula (something very important).

In addition, if we arrange for the size of the dodecahedron edge to be 1 in Cabri3D, we can turn the attribute of the display of the desired quantity, to *symbolic* and Cabri3D will then, explicitly, display the value of $(3+\sqrt{5})/4$, see the figure below:



In my short presentation I will discuss some consequences of above example and story where CAS and DGS can be viewed as a mixed compound.

Contribution from Phillip Todd

I'd like to show how to solve a number of classical problems using a combination of Symbolic Geometry (www.geometryexpressions.com) and CAS. I will finish by applying the same process to a problem which was posed to me at ATCM 2006 [1]. I'll focus on the fact that this combination of tools allows a radically different approach to problem solving, where the emphasis is on model building and solution strategy and not on the mechanics of solution, nor on geometric insight. Once the problem is solved, geometric insight may be reinserted to taste. My problems will be taken from Dorrie's, "100 Great Problems of Elementary Mathematics" [2], and will include:

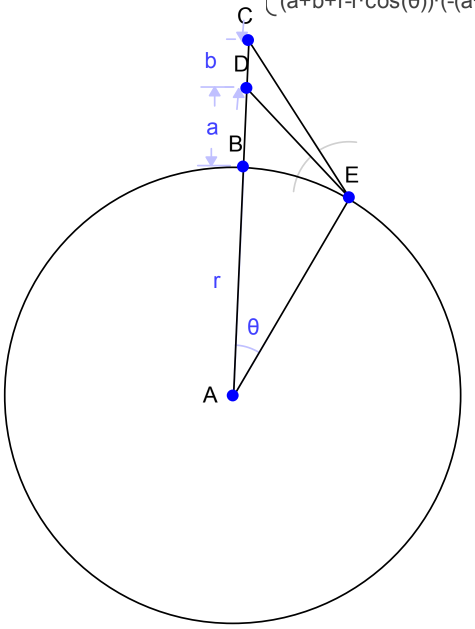
- Fergano's altitude base problem
- Regiomontanus' maximum problem
- Euler's tetrahedron problem
- Curvature of conic sections
- An inscribable circumscribable pentagon

Illustrative of the approach is the following solution of Regiomantus' Maximum Problem.

At what point on the earth's surface does a perpendicularly suspended rod appear longest?

A typical symbolic geometry approach to an optimization problem is to create a model, derive an expression for the objective quantity, differentiate and solve. In the figure, the earth has radius r, and a rod of length b is suspended in space distance a above the surface of the earth. The angle subtended by the rod at point E on the surface of the earth angle θ from the direction of the rod is calculated by Geometry Expressions:

$$\Rightarrow -\arctan\left(\frac{b\cdot r\cdot \sin(\theta)}{(a+b+r-r\cdot\cos(\theta))\cdot(-(a+r)+r\cdot\cos(\theta))-r^2\cdot\sin(\theta)^2}\right)$$



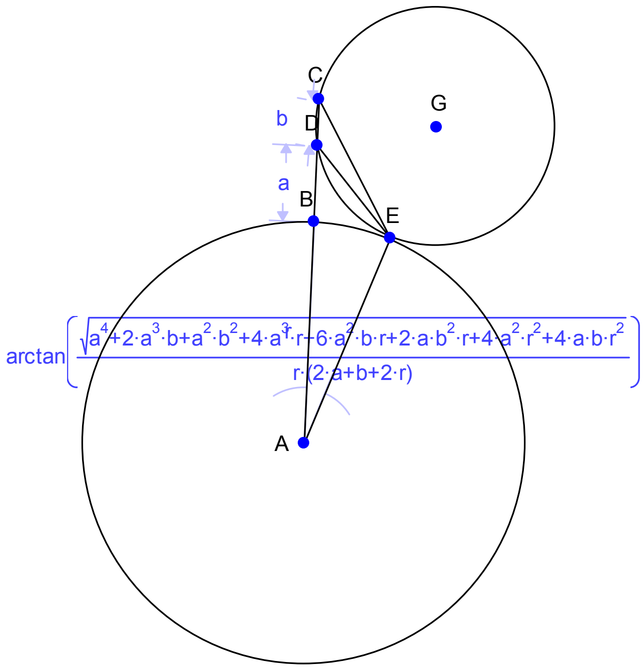
Copying into Maple, we can differentiate and solve:

$$-\arctan\left(\frac{\sin(\theta)\, r\, b}{(-a-r+\cos(\theta)\, r)\,(a+b+r-\cos(\theta)\, r)-\sin(\theta)^2\, r^2}\right)$$

> solve(diff(%,theta)=0,theta);

$$\arctan\left(\frac{\sqrt{4a^2r^2+4bar^2+2b^2ar+6a^2rb+a^4+2ba^3+4a^3r+b^2a^2}}{a^2+ba+2ar+rb+2r^2},\ \frac{r(2a+b+2r)}{a^2+ba+2ar+rb+2r^2}\right),\ \arctan\left(-\frac{\sqrt{4a^2r^2+4bar^2+2b^2ar+6a^2rb+a^4+2ba^3+4a^3r+b^2a^2}}{a^2+ba+2ar+rb+2r^2},\ \frac{r(2a+b+2r)}{a^2+ba+2ar+rb+2r^2}\right)$$

Copying a solution back into Geometry Expressions we can visualize the solution.
Geometric insight can be had by drawing the circumcircle of CDE, and observing that it is tangent to AE.



References

[1] P. Todd, "An Inscribable Pentagon", American Mathematical Monthly 114:639 August/September 2007

[2] H. Dorrie, "100 Great Problems of Elementary Mathematics", Dover, New York, 1965

Contribution from Wei-Chi Yang

One of my favorite observations is DG allows us to be coordinate free and we may use its numerical capability to approximate a solution. In the mean time, we need a CAS to verify if our conjecture is true.

My conjecture is that DG makes mathematics more accessible. There are many real-life problems and abstract concepts can be explored by students before they enter colleges. On the other hand, we need a CAS to prove results analytically, which makes mathematics challenging, and this makes some motivation for students to study more mathematics in university level or beyond.

We will see from the following example that it is to create the reflection of a curve along a line, which is to generalize the idea of finding an inverse of a function. However, it is a challenging exercise to first year university student to find the solution analytically.

Example 1. We are given a line L of the form y=m*x+b and a curve S (see Figure 1 below), find the reflection of the curve S respective to the line L.

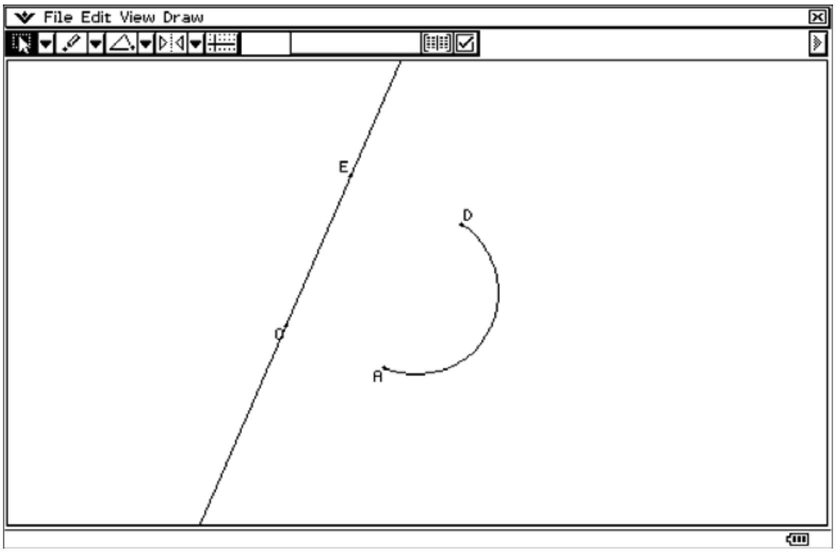


Figure 1.

The second example below shows how DG allows us to explore the concept of Riemann integral by choosing a proper coordinate system, which will simplify greatly when one wants to approximate the area bounded by curves. Again, it is a challenging exercise to prove this analytically.

Example 2. Refer to Figure 1 above, we would like to find the area bounded by the curve S, line L, line AC and line DE; here we assume AC is parallel to to DE, and AC is perpendicular to CE.

The third example below shows how we can link the idea of Lagrange Multiplers, its geometric interpretation and linear indepency of Linear Algebra together with the help of Dynamic Geometry Software packages.

Example 3. We are given three curves in the plane, see C1, C2 and C3 below in Figure 2. We need to find points A, B, and C on C1, C2 and C3 respectively so that the distance AB+AC achieves its minimum. In 3D case, we are given four surfaces in the space, represented by the orange (S1), yellow (S2), blue (S3) and purple (S4) spheres respectively in Figure 3. We want to find points A,B,C and D on S1, S2, S3 and S4 respectively so that the distance AB+AC+AD achieves its minimum.

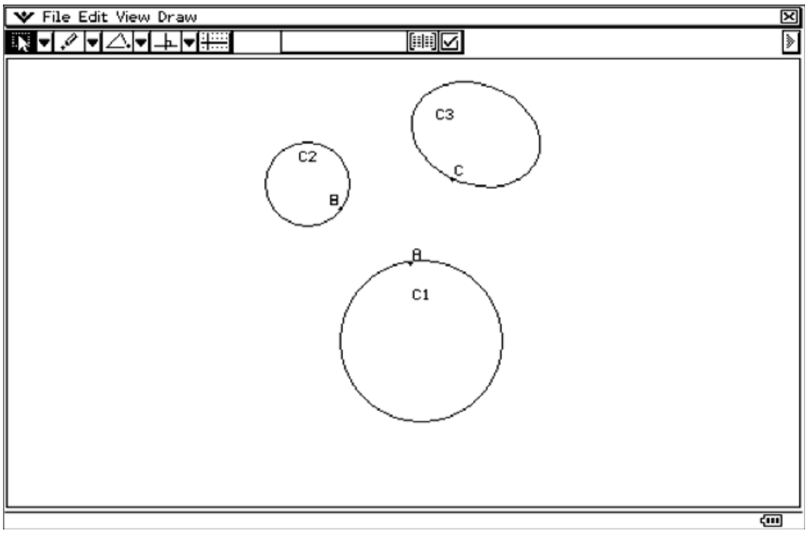


Figure 2.

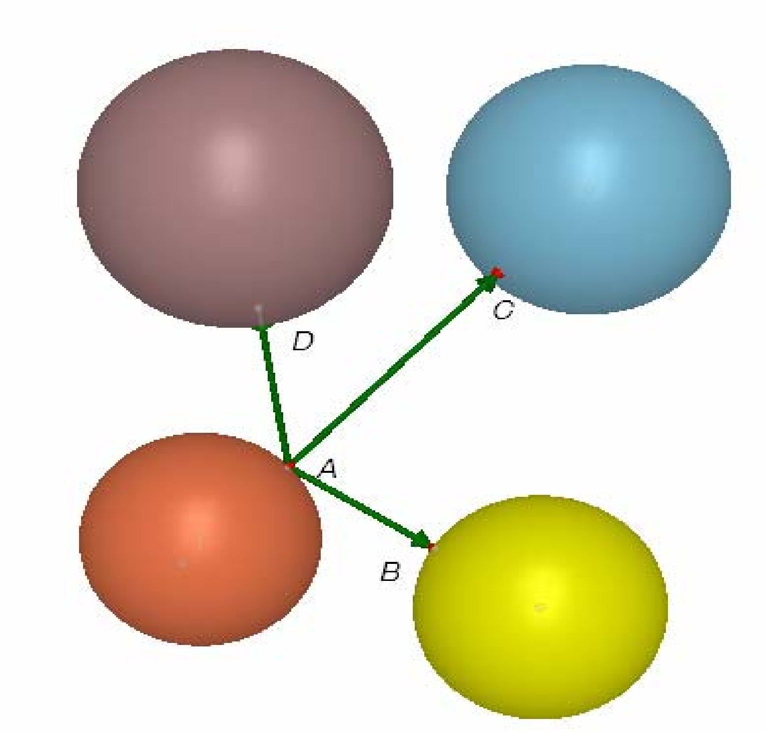


Figure 3.