Robust and soft constructions: two sides of the use of dynamic geometry environments

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Abstract
Variation is the essence of dynamic geometry environments. This talk aims at discussing two paradigms of use of variation in dynamic geometry environments: robust and soft constructions. Robust constructions are constructions for which the drag mode preserves their properties. Such constructions must be constructed by using the geometrical objects and relationships characterizing the construction to obtain. In such constructions variation is used as a verification means. In soft constructions, variation is part of the construction itself and a property becomes visible only when another one is satisfied. By means of several examples based on Cabri Geometry II and Cabri 3D, it will be shown how the soft paradigm can contribute to the learning. On the one hand, soft constructions can be part of the « private » side of the work of the students and help them identify dependency relationships between properties, on the other hand they can be used in mathematics teaching to introduce students to better understand the functioning of fundamental notions such as those of implication, valid property, hypothesis and conclusion

1. Variable and variation in mathematics

The notions of variable and variation are often attached to algebra and in particular to the notion of function. However these notions go beyond these topics and are essential in mathematics. The duality variation/invariant permeates all mathematics, including geometry. A theorem like “Any isosceles triangle has two congruent angles” expresses a relational invariant between the sides of a triangle varying in the set of the isosceles triangles, even if the varying nature of the triangle is expressed by the subtle mark “any”. A geometric property is an invariant satisfied by a variable object as soon as this object varies in a set of objects satisfying some common conditions. The variability of geometric objects is generally invisible because the formulation of a geometric property is most of the time expressed as dealing with a single static object, the quantifiers being implicit, especially in the secondary school. This is not without causing troubles for the students who do not perceive the generality of theorems or properties.

Dynamic geometry exteriorizes the duality invariant/variable in a tangible way by means of motion in the space of the plane. When a figure is constructed in order to satisfy a set of conditions, properties that derive from them are preserved in the dragging of an element of a figure. Those properties remaining invariant in the drag mode emerge from the contrast with the changing properties of the figure in the drag mode, as Mason & Heal (1995, p.301) wrote:

> “Being able to move screen objects around in space (and so over time) can add significantly to the user’s sense of the underlying concept as an object not just in itself but a something invariant amidst change”

Geometric properties are perceived in a dynamic geometry environment as invariant in the variation of the figure, exactly in the same way as an algebraic identity such as $(x + 1)^2 = x^2 + 2x + 1$ can only be perceived in the variations of x. One could say that a theorem in geometry is of same nature as an algebraic identity but from the converse point of view, an algebraic identity can be viewed as a theorem. Although dynamic geometry reveals this deep unity of mathematics as finally a science dealing with variable objects, the attention to variation as the essence of mathematics is not new.

Let us quote the French geometer Monge (1792) writing in the "Leçons données à l'Ecole Normale de l'an III":

> "Il faut que l'élève se mette en état, d'une part de pouvoir écrire tous les mouvements qu'il peut concevoir dans l'espace, et de l'autre, de se représenter perpétuellement dans l'espace le spectacle mouvant dont chacune des opérations analytiques est l'écriture."[1]

This talk is devoted to the analysis of possible uses of variation in dynamic geometry, in particular for the teaching and learning of mathematics. As often mentioned, students have great difficulties in conceiving the concept of variable and variation in algebra. In the same way, it has been often claimed that students do not perceive the generality of geometric objects and of theorems. We consider that dynamic geometry dealing with tangible variable objects may become a tool in the hands of the teacher to introduce students to this deep and essential feature of mathematical objects.

2. Two paradigms

We distinguish two paradigms in the use of variation in dynamic geometry: the paradigm of the robust constructions and the paradigm of the soft constructions.
Let us give an example illustrating this distinction in Cabri-geometry.

A robust construction

Let create a circle with center *O* and a point *A* on this circle. Construct *B* such that *AB* is a diameter of the circle. Let create a point *M* on the circle and segments *AM* and *MB*. Measure angle *AMB*. The displayed measure is 90°. When dragging *M* on the circle, it is easy to see that this measure is invariant. This is an example of a robust construction showing that for any point of the circle (except *A* and *B*) angle *AMB* is a right angle (Fig.1). When *B* is redefined as any point on the circle, the measure of *AMB* changes. It remains invariant when *M* is dragged on the same arc *AB* (Fig.2). This small experiment often carried out by teachers in secondary school may be the starting point of the formulation of two theorems:

- the often called theorem of Thales according to which an angle inscribed in a semi-circle is a right angle;
- the theorem of the inscribed angle according to which angles in the same segment of a circle are equal.

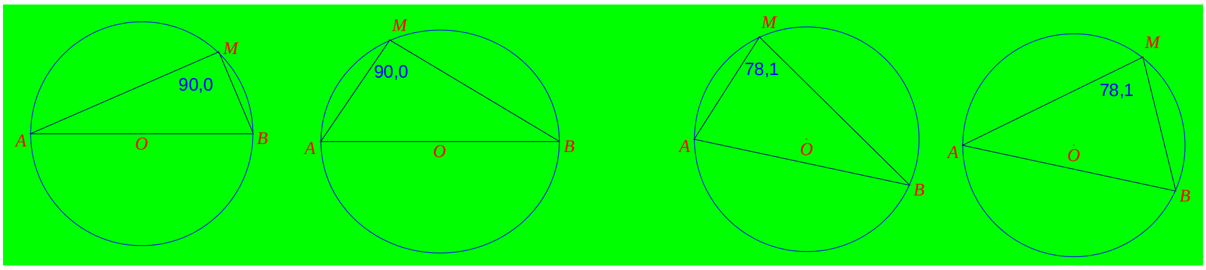

Fig.1 angle *AMB* inscribed in a semi-circle Fig.2 angle *AMB* in a given segment of circle

The robust construction contributes to a better identification in action of the elements of both theorems for several reasons:

[1] The student must be able, on the one hand to write all motions that he/she can conceive in the space, and on the other hand to perpetually imagine in the space, the moving scene transcribed by each analytical operation.

- the construction requires to take into account two conditions to get a right angle: *AB* must be a diameter and *M* a point on a circle; when writing proofs in paper and pencil environment, students often forget mentioning the condition dealing with the diameter. As soon as one of the conditions is not satisfied, the result is not obtained.
- it allows contrasting the invariance of the angle and the varying nature of point *M*.
- it exteriorizes the variable nature of point *M* and the set in which its varies.

This explains why this construction is often used by teachers, another reason is the low cost in time of such a construction with regard to the cost of measuring angles in a paper and pencil environment, as stressed by Ruthven et al. (2005) interviewing teachers about how they use dynamic geometry in their teaching. The identification of those elements of the theorem is done in action in Cabri. The role of the teacher is then to ask students to build a formulation of the theorems.

A soft construction

Let us construct a circle, a point *A* on the circle and its opposite point *B* on the circle. Let create any point in the plane, and measure angle *AMB*. Suppose that *M* is outside the disc. The teacher asks to find a position of *M* outside the disk such that angle *AMB* is obtuse (Fig.3). The initial displayed measure of angle *AMB* is less than 90°. The students drag *M* outside the disc and cannot find such an angle. They are very convinced that angle *AMB* is acute when *M* is outside the disc. Then the teacher asks them whether it is possible to find an obtuse angle when varying *M* anywhere in the plane. The students move *M* inside the circle and observe that angle *AMB* becomes obtuse (Fig.4). They drag *M* anywhere inside the disc and observe that the angle is still obtuse. The teacher asks them to drag *M* from inside to outside back and forth and to observe the changes of the measure of angle *AMB* and asks the students to guess the measure of angle *AMB* when *M* is on the circle.

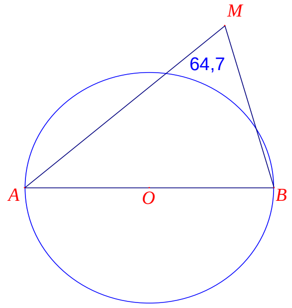

Fig.3 angle *AMB* with *M* outside the disc

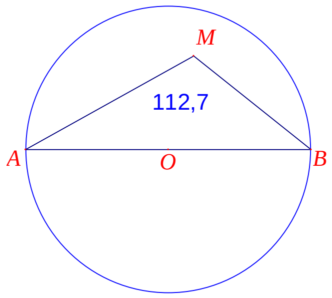

Fig.4 angle *AMB* with *M* inside the disc

We call this construction a soft construction because *M* is not constructed as a point on the circle. In this exploration, students discover a theorem different from the one of the robust construction: a theorem linking the measure of angle *AMB* to the position of *M* in the plane, outside the disc, inside the disc, on the circle. The contribution of the soft construction also differs from the contribution of the robust construction in two aspects:

- The reason why angle *AMB* is a right angle in the soft construction is made visible and the theorem of Thales gains in meaning. The circle is the border between two regions, one in which angle *AMB* is obtuse and one in which angle *AMB* is acute. Using a continuity principle, that students usually do spontaneously, it becomes clear that the measure of angle *AMB* must be 90° on the circle.

More emphasis is put on the *link* between the condition "*M* is on the circle" and the consequence "angle *AMB* is right" because it is very easy to drag back and forth point *M* across the circle.

3. Why a general adoption of robust constructions?

It is interesting to note that the distinction between robust and soft constructions was formulated only after 15 years of existence of dynamic geometry in a paper written by Healy (2000). At the beginning of dynamic geometry, the emphasis both in research in mathematics education and in mathematics teaching was put mainly on robust constructions. The contribution of tasks in which students must build robust constructions in a dynamic geometry environment was rapidly recognized as promoting the learning of geometrical properties. It is only later that research investigating solving processes of students in problems revealed the existence of soft constructions done by students. The importance of robust constructions in teaching was rapidly accepted in particular because of the students' difficulties of moving from a visual processing of diagrams to a theoretical perspective.

Spatio-graphical versus theoretical

The nature of diagrams in geometry has been analyzed by several researchers (Parzysz 1988, Sträßer 1991, Laborde 2004) who came to the common conclusion that diagrams in two-dimensional geometry play an ambiguous role: on the one hand, they refer to theoretical geometrical properties, while on the other, they offer *spatio-graphical* properties that can give rise to a student's perceptual activity. Students often conclude that it is possible to construct a geometrical diagram using only visual cues, or to deduce a property empirically by checking the diagram. When students are asked by a teacher to construct a diagram, the teacher expects them to work at the level of geometry using theoretical knowledge, whereas students very often stay at a graphical level and try only to satisfy the visual constraints. Let us explain the distinction theoretical/ spatio-graphical.

Fig.5 The diagram of a parallelogram

The diagram in Fig.5 represents a parallelogram. It shows several spatio-graphical properties: two sides are horizontal; the other two are oblique in a given direction (bottom left to top right); the opposite sides are parallel; the horizontal sides have a given length. Note that these properties are selected from a larger set of properties like color or the width of the sides. Some of these spatio-graphical properties can be interpreted in a geometrical way, while others would not be considered interesting from a geometrical point of view: for example, the position of the diagram on the sheet of paper is generally considered to be irrelevant in geometry, as is the slope of the side since it depends on the problem in which the parallelogram occurs. So some spatio-graphical properties of the diagram are *incidental* to the geometrical problem, while others are *necessary* like the parallelism properties. Further, spatio-graphical properties necessarily follow from others: there is a *necessary* link between the parallelism of opposite sides and the fact that the intersecting point of the diagonals is also their midpoint. The teaching of geometry deals with these necessary links between spatio-graphical properties, but one can understand the nature of these links if and only if one also can understand that some other links are merely incidental. *Necessity makes sense in* *opposition to contingence*. Geometry may appear useful if it allows one to predict, to produce or to explain spatio-graphical properties of diagrams because of these necessary links; but it first requires an awareness of the distinction between such properties and those that are theoretical. Such awareness is not constructed by students at the beginning of secondary school. Two kinds of tasks are often used in the teaching for helping students to construct this awareness: robust construction tasks and exploration of robust constructions.

Contribution of robust constructions

"Robust construction" tasks

Dynamic geometry environments were used from the beginning of their existence to help students construct this awareness. Students are asked to construct variable diagrams satisfying to several conditions even when one of their elements is dragged. Eye ball constructions are invalidated by the drag mode since it becomes visible that some of the conditions are not satisfied. The drag mode is a critical factor in robust construction tasks that makes the difference with a paper and pencil environment. In such construction tasks in dynamic geometry, the drag mode provides a visual feedback from the fact that the construction does not meet all the required conditions. The strength of DGE[2] lies in this possibility of showing at the spatio-graphical level the theoretical weakness of the construction. The invalidation in paper and pencil can be done only by the authority of the teacher who states that the construction is incorrect. The drag mode in DGE does not formulate a judgment but offers visual evidence to the eyes of the student. The learning potential is very different in both cases. In a DGE, students experience and draw the conclusion themselves that their construction is incorrect. In paper and pencil students may only accept the judgment of the teacher without trying to understand why it is incorrect. In terms of the theory of didactical situations (Brousseau 1997), in one case there is a "milieu" offering information and feedback to students' solutions, in the other case, there is none.

The potential of robust construction tasks in which students have to build robust constructions preserved in the drag mode was accepted universally, as shown by curricula and declarations in official texts about the teaching of mathematics. These latter often relate robust construction tasks to preparing students to proof. In a robust construction, an object characterized by a set of properties Q_j must be obtained from the conjunction of other properties P_i. The construction process involves the implication: $P_1 \wedge P_2 \wedge \ldots \wedge P_i \wedge \ldots \Rightarrow Q_1 \wedge Q_2 \wedge \ldots \wedge Q_j \wedge \ldots$.
The drag mode may invalidate the set of used properties P_i and shows that the construction, if it is correct, is not only valid for the single case of the diagram but for a set of cases. The drag mode shows the generality of the construction process.

Exploration of robust constructions

From the very beginning of DGEs, a widespread use of robust constructions in teaching, still prevailing (Ruthven et al. 2005), is to illustrate theorems. Robust constructions constructed by the teacher or to construct by students following guidelines are to be varied by the students

- either in order to recognize the taught theorem
- or to explore in order to discover a theorem.

The variation and/or exploration of a variable diagram also shows to students beginning middle school that some spatio-graphical properties are incidental and not necessary. When moving a right

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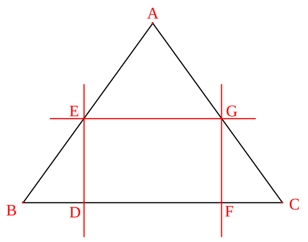
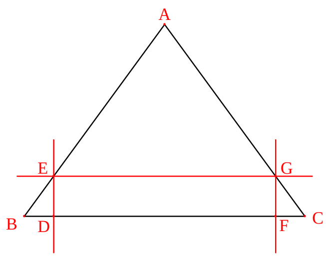
[2] In what follows, DGE denotes dynamic geometry environments.

triangle, one can observe that the sides of the right angle are not necessarily horizontal and vertical. The orthocenter of a triangle is not necessarily inside the triangle. Exploration of robust constructions contribute to a better familiarity with all possible diagrams representing the same mathematical property. Students may thus extend their visual images of a property or of an object and reject some spatio-graphical properties they used to attach to a kind of object. The drag mode is used as a tool for distinguishing between contingence and necessity.

In the same vein, the drag mode allows to distinguish between properties that are always true and properties sometimes true, i.e. properties satisfied in a particular case. Soury Lavergne et al. (2004) gave the following figure in Cabri to 6^{th} graders for exploration. *ABC* is any triangle, *D* a point of side *BC*, *F* the reflected point of *D* around the midpoint of *BC*, *DE* and *FG* the perpendicular lines to side *BC*. The students are given the construction ready made by the teacher but when students open the file, point *A* is in a particular position: *ABC* seems to be an isosceles triangle and as a result line *EG* is apparently parallel to line *BC* (Fig.6). Students have to formulate statements about properties of the figure by using the drag mode.
When moving *D*, the quadrilateral *DEGF* seems to stay a rectangle (Fig.7). But when moving *C*, *B* or *A*, only lines *DE* and *FG* remain parallel (Fig.8). After students have formulated these properties, the teacher asks the students to justify why lines *DE* and *FG* remain parallel even when any element of the figure is dragged whereas lines *EG* and *DF* do not. They are asked to use the text of the construction available in Cabri II plus or the Explore facility in Cabri Junior.





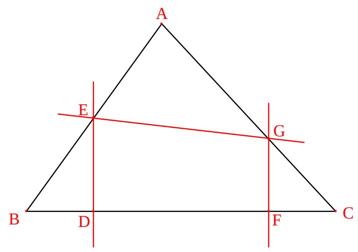


Fig.6 The figure in its initial position Fig.7 Moving D Fig.8 Moving A, B or C

This exploration task may be used to work on necessity versus contingence. As soon as some conditions are met, some other conditions are fulfilled by the figure for whatever position of the variable points. In contrast, a property true only for some positions of the variable points are not necessarily implied by the properties used to construct the figure. It may happen that they are true but not always.

4. Soft constructions in students' solving processes

Soft constructions in construction tasks

Soft constructions were observed by researchers observing students solving problems using Cabri-geometry, either in robust construction tasks or discovery and justification tasks. Some typical examples of soft constructions observed by researchers in various tasks are given in what follows.
Jones (1998, pp.79-82) describes carefully the process of elaboration of a construction by two pairs of recent mathematics graduates. They were given two intersecting straight lines *d* and *d'* and a point *P* on one of this line. They had to construct a circle tangent to two intersecting straight lines *d* and *d'* and having point *P* as its point of tangency to one of the lines. Both pairs constructed a circle with a centre chosen somewhere between the lines and with point *P* as the radius point. Then

they used the facility available in Cabri to drag the centre so that it appeared to be tangential to *d'*. The first construction was a soft construction but purely visual (Figs. 9 & 10).

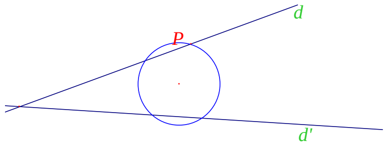


Fig.9 The first drawn circle

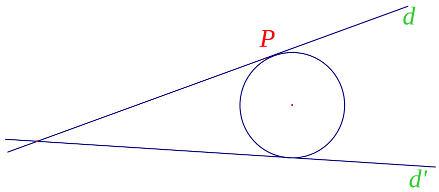


Fig. 10 Adjusting visually the circle

Despite the fact that the construction gave a satisfactory visual solution, they searched for a way of being absolutely sure. From the construction, they recognized visually that the line of the radius was perpendicular to the tangent line in *P* and this reminded them a relation between the radius and the tangent they already knew. Then they started a new construction again, drawing a perpendicular line through *P* to the given line and constrained the centre of the circle to lie on this perpendicular line (Fig.11). But again they had to adjust the circle to be tangent to line *d'*. This was a second soft construction involving the property of the tangent line to be perpendicular to the radius line (Fig.12). Then they drew a perpendicular to the lower line *d'* at a point *Q* and moved *Q* (Fig.13) until the intersecting point of two perpendicular lines was equidistant from both lines *d* and *d'* (Fig.14). The intersecting point was taken as the centre of the circle. This third construction was also a soft construction but involving another additional property: the equidistance of the centre to the points of the circumference. The construction was not robust since depending on the variable point *Q*.

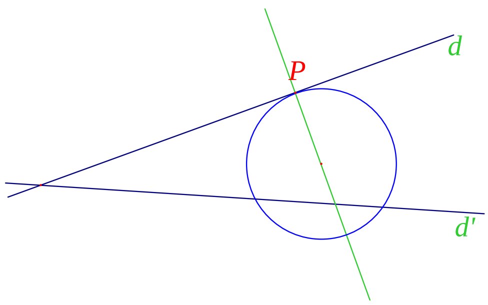


Fig.11 A circle centered on the perpendicular line in P

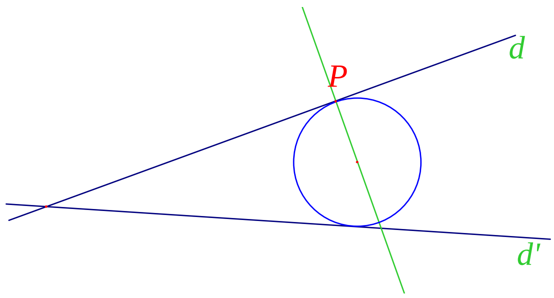


Fig.12 Adjusting the circle to be tangent to *d'*

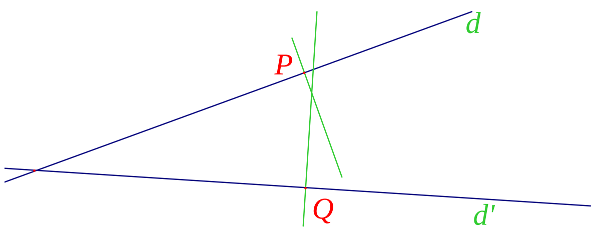


Fig.13 Drawing a perpendicular line to *d'*

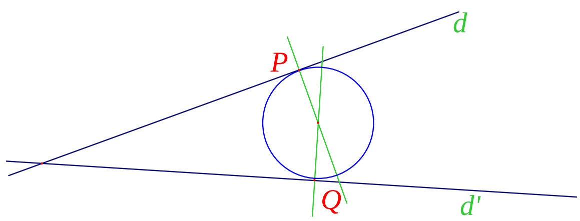


Fig.14 Moving Q to adjust the circle

This example shows very well how the sequence of successive soft constructions involving more and more geometrical properties is elaborated and how a soft construction enabled the students to recognize properties and to mobilize them in a further construction giving less room to visual adjusting.

Soft constructions are not only part of the solving process but they scaffold the path to a definite robust construction. They play an important role in moving from a purely visual solution done by adjusting to a solution entirely based on theoretical solutions but achieved by dragging. These constructions are culturally not accepted since the time of the Greek geometry rejected constructions based on motion and restricted the allowed constructions to those done with straight edge and compass. Therefore the last unsolved question in the previous example was to find a single property ensuring the equidistance of a point to two straight lines.

Soft constructions may not appear in the final solution given by the students and seem to belong to the private part of the students' work. This may explain why soft constructions were mainly observed by researchers who could do fine observations of students' work whereas they may not be seen by the teacher who has no means of observing in detail the work of a student.

Soft constructions in proof oriented exploration tasks

The role of dynamic geometry software on elaborating proof has been investigated by several researchers. A special issue of Educational Studies in Mathematics (Jones et al. 2000) is devoted to this topic. In particular, research has been done in studying dragging strategies and the general conclusion is that dragging plays a key role in forming a mathematical conjecture (Arzarello 2000, Hölzl 1996, Leung & Lopez-Real 2000, Leung & Lopez-Real 2002, Healy 2000).

Healy (2000) observed 14-15 year-old students faced with a task in which they had to "explore various methods of constructing a second triangle using different combinations of the properties (sides and angles) of an existing (and general) Cabri-triangle with the eventual aim of identifying which conditions were sufficient to ensure congruency." Healy explains her expectations:

> "In this activity were we not expecting students to construct any formal proofs, but we did want them to experience how the construction of some properties necessarily (or not) results in other geometrical by-products."

Healy discovered through this observation that, rather than constructions preserved under dragging, students preferred to investigate constructions, "in which one of the chosen properties is purposely constructed by eye, allowing the locus of permissible figures to be built up in an empirical manner under the control of the student". Healy introduced in this paper the distinction "soft/robust" and decided to call the latter constructions *soft constructions* and the former ones *robust constructions*. She illustrates the distinction between robust and soft approaches by means of the example of two pairs of students investigating whether the conditions two congruent sides and a congruent angle determined one triangle or not. Students using a soft construction, i.e. constructing two sides congruent to sides *CA* and *CB* by means of circles and moving carefully the point on the inner circle in order to obtain an angle congruent to angle *CBA* (Fig.15) immediately found a point for which the third side was not congruent and rejected the condition Side Side Angle.

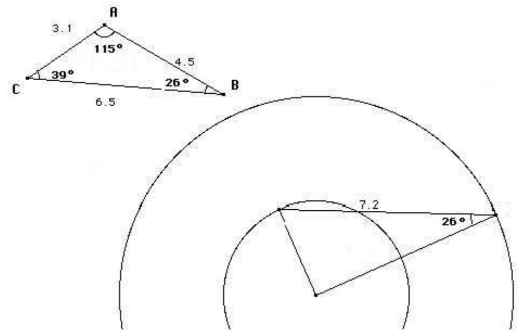


Fig.15 Fig. A soft construction providing a counter example to the condition Side Side Angle

The pair using a robust construction (using circles and carry angle option that was provided) did not notice at first the counter example and were convinced that the condition Side Side Angle was enough to determine a congruent triangle. The second intersecting point of the side with the inner circle was discovered later after (Fig.16).

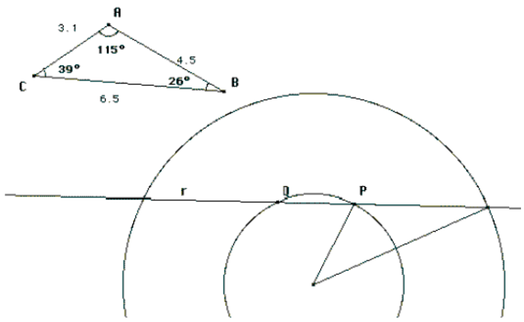


Fig.16 A robust construction using only one intersection point

Healy comments how the two kinds of construction are complementary:

> "In robust constructions, dependency is demonstrated by the fact that a relationship remains invariant through dragging. During the dragging test attention can move from general to specific as a "family" of Cabri-drawings with the same geometrical make up is produced. In soft constructions, this is not the case. Instead dragging is part of construction not verification and students observe how the dependent property becomes evident at the point in which another property is manually (and visually) satisfied. That is, the general can emerge from the specific during thorough searches for the set of loci in which the given conditions are fulfilled."

Investigations about the way dragging is used by students in solving problems carried out by Italian researchers (Arzarello et al. 1998, Arzarello 2000, Olivero 2002) converge with the analysis of Healy. These researchers conducted a very fine analysis of the use of dragging and established a categorization of different kinds of dragging (Olivero 2002, p.98). Among all kinds, three kinds of dragging seem to play an important role:

- *Wandering dragging* is moving the points on the screen randomly in order to discover configurations,
- *Guided dragging* is done with the intention to obtain a particular shape,
- *Lieu muet dragging* is moving a point with the constraint of keeping a particular property satisfied at the initial state, the variable point follows a hidden path even without being aware of this.

Olivero could observe that “wandering dragging” and especially “guided dragging” were mainly used by students whereas “lieu muet dragging” was only sometimes used and not by all students. An example of “lieu muet dragging” is provided by a girl Tiziana investigating at what conditions the quadrilateral *HKLM* built by the perpendicular bisectors of the sides of a quadrilateral *ABCD* is a point (Fig.17). By wandering dragging, Tiziana and Bartolomeo working together discovered that *HKLM* is a point when *BCD* is a rectangle (Fig.18).

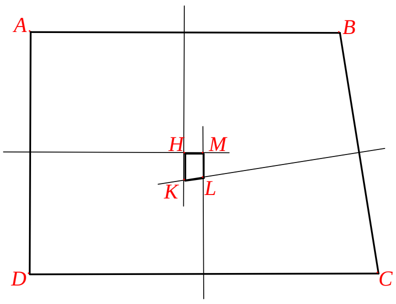


Fig.17 The quadrilateral of the midpoints *HKML*

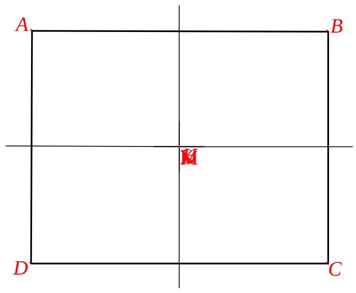


Fig.18 *HKML* reduced to a point

Bartolomeo wanted to drag the vertices of *ABCD* in order to obtain a specific quadrilateral *HKLM*, a parallelogram, a rhombus, a trapezium. Tiziana did not share this approach and tried to drag point *B* of the rectangle in order to keep *HKLM* as a point (Fig.19). She wanted then to draw a circle passing through the vertices of *ABCD*. As Bartolomeo pursued his idea, she could not do it. But later she could come back to her idea of a circle. She drew a circle and then an inscribed quadrilateral *ABCD* and could observe that *HKM*L was a point (Fig.20). She moved from a soft construction to a robust one.

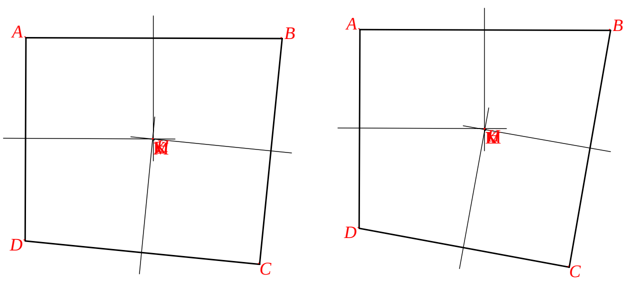


Fig.19 Dragging *C* to keep *HKML* as a point

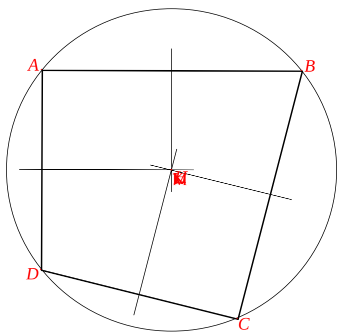


Fig.20 *ABCD* cyclic quadrilateral

The story of this pair illuminates very well the two ways of finding conditions on quadrilateral *ABCD* implying conditions on quadrilateral *HKML*. The way of Bartolomeo associated to guided dragging is oriented to find discrete cases of an implication such as “*ABCD* is a rectangle implies *HKML* is a point” and to find a common property among all quadrilaterals *ABCD* providing a point. Tiziana wants to satisfy continuously the condition “*HKML* is a point” and doing this she tries to see whether *ABCD* remains a rectangle. She finds that not and even probably discovered that the path followed by point *B* had a circular shape. Therefore she could then check whether the implication “*ABCD* is a cyclic quadrilateral implies *HKM*L is a point”. In Bartolomeo’s approach the process corresponds to an inductive process trying to generalize from several cases, whereas Tiziana’s process is closer to an abductive process. The “lieu muet dragging” offers a hint trough the shape of the hidden path.

Leung (2003) considers what is experienced in dragging is a key to concept formation because it offers a confluence of simultaneities. His claim is based on a phenomenographic approach in which learning and awareness are interpreted under a theoretical framework of variation (Marton & Booth, 1997). The central concepts of this theory are: discernment, variation and simultaneity.

> " To discern an aspect is to differentiate among the various aspects and focus on the one most relevant to the situation. Without variation there is no discernment….Learning in terms of changes in or widening in our ways of seeing the world can be understood in terms of discernment, simultaneity and variation." (Bowden and Marton, 1998, p.7)

From this perspective both guided dragging and lieu muet dragging offers opportunities of variation, discernment and simultaneity.

In all those examples coming from various research pieces, the use of dragging can be linked to processes of elaboration of a proof. However both Healy (*ibidem* p.114) and Olivero (*ibidem* p.118) could observe that when students had to write formal proofs, they stopped dragging and tried to formulate proofs using a static Cabri figure. Healy concludes that this gap between the exploration phase and the writing of the proof was may be due to the way formal proof was introduced without organizing a transition from empirical explorations to formal proof. Students were introduced to formal proof as a sequence of deductive steps starting from the givens. Olivero (*ibidem* pp.240-1) claims that "even if Cabri is not directly used when proving, the influence of the previous exploration is evident and that the use of Cabri is extremely relevant for the actual proving phase, in that it allows theoretical elements to emerge."

5. Implications for teaching

The presented examples show how the two kinds of constructions may contribute to the understanding of a geometric property as an implication. The robust constructions allow checking whether conditions supposed to provide the expected consequence are satisfied. A minimal knowledge of the conditions or some ideas about them are required in such robust constructions. The soft constructions bring ideas on an implication itself or on the conditions to be met in order to obtain a consequence. The soft constructions offer thus a transition from an empirical approach to a theoretical approach.

But the teaching is focusing on robust constructions and not on soft constructions that were observed in the spontaneous exploration phase of students' work. We think that if teaching were to give room to both kinds of constructions and to moves from the one to the other one, it could improve the preparation to formal proof.

In Grenoble, the introduction of soft constructions is being experimented for some years for preparing middle school students to proof and favouring the understanding o the functioning of a property. Coutat (2003) used soft constructions in Cabri Junior to mediate the distinction between hypothesis and consequence in a theorem. Let us give an example:

Construct any quadrilateral ABCD (more general hypothesis), its diagonals and the midpoint of each diagonal. Drag any vertex A, B, C or D so that the midpoints are coinciding (variation of one element in order to obtain an additional condition) (Fig.21). The focus of the control is on the superimposition of the midpoints. The visible effect is the change of shape. The obtained shape is a parallelogram.

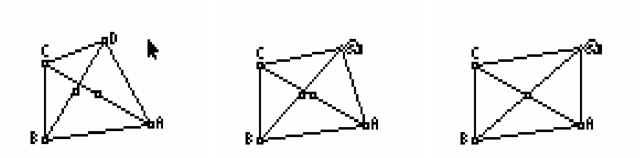


Fig.21 – Dragging vertex D in a quadrilateral until making midpoints of the diagonals coinciding

Changing the condition on an object by dragging it (here the diagonals) implies a visible change on other objects (here the parallelogram). The condition is what the student is directly changing. The visible effect is the result of the implication. The condition plays the role of the hypothesis. The effect plays the role of the conclusion. The link between condition and effect introduces a causal effect oriented from the hypothesis to the conclusion.

Soft constructions seem to be particularly useful in 3D geometry as students' intuition of space geometry is much weaker than in 2D geometry. They can provide ideas to students about relationships between elements of 3D objects and implications between these relationships.
Let us give an example with Cabri 3D.
The problem is given in two questions:

1- Do the altitudes of a tetrahedron meet in a point?
2- What are the conditions on the tetrahedron to have intersecting altitudes?

Question 1: Usually students claim that altitudes do meet when they are asked to make a prediction. Students are given a diagram in Cabri 3D (Fig.22) where apparently two altitudes meet. By changing the point of view (rotating the view) they discover that the two altitudes do not meet (Fig.23).

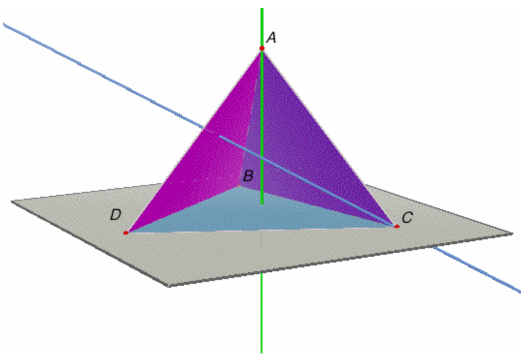


Fig. 22 Do altitudes of tetrahedron *ABCD* meet?

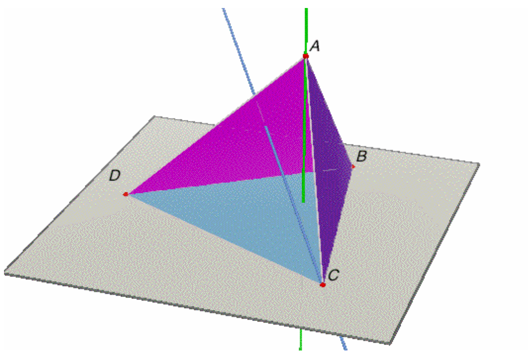


Fig.23 Changing the point of view

Question 2: Students try to move *A* to get the altitudes meeting. By changing the point of view and being exactly above the plane *BCD*, that is viewed as a square (Fig.24), one can understand that point *A* must be moved exactly onto altitude *CH*. The move from a 3D view to a 2D view allows to discover that in the guided dragging of *A* to come onto *CH*, the altitude drawn from *A* and line *CA* coincide and are perpendicular to side *BD* (Fig.25). As *CH* is already perpendicular to *BD*, a possible condition for the meeting of two altitudes could be that sides *BD* and *AC* be perpendicular. In addition, one can guess that all points *A*, *H*, *I* and *C* are coplanar.

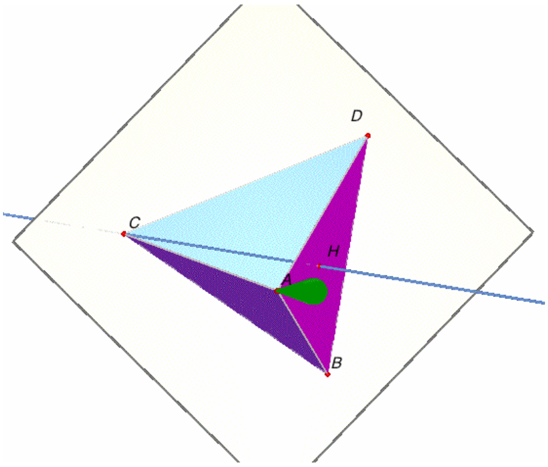


Fig. 24 Being above plane *BCD*

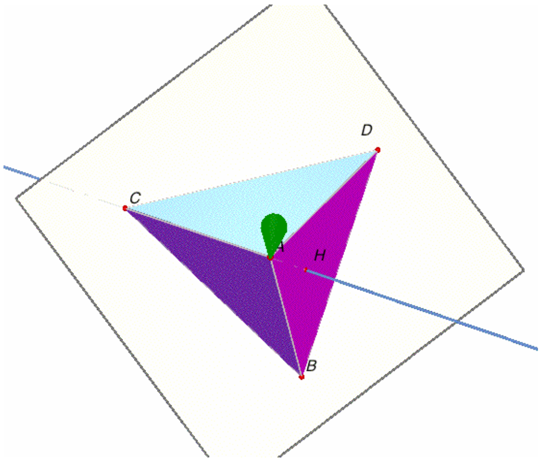


Fig. 25 Moving *A* onto altitude *CH*

A new soft construction is to experiment in order to secure the coplanarity of points *A*, *H*, *I* and *C*. Let draw the plane perpendicular to *BD* and passing through *A*. Moving *C* in a guided dragging until *C* is on the intersection line of this latter plane and plane *BCD* shows that as son as *C* is on this line, the four points are in the perpendicular plane. This explains why the altitudes *AI* and *CH* meet.

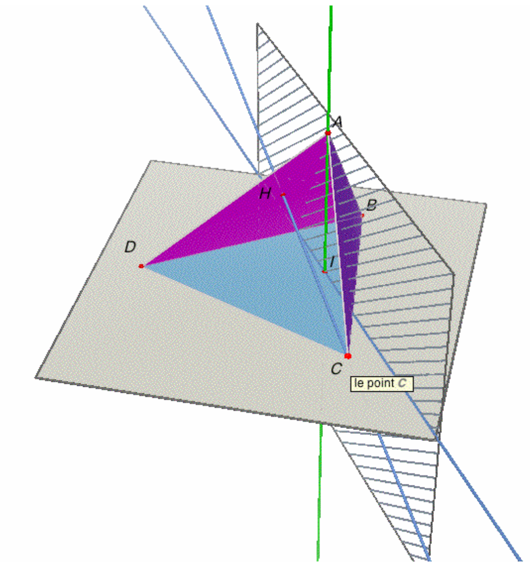


Fig.26 The plane perpendicular to BD passing through A

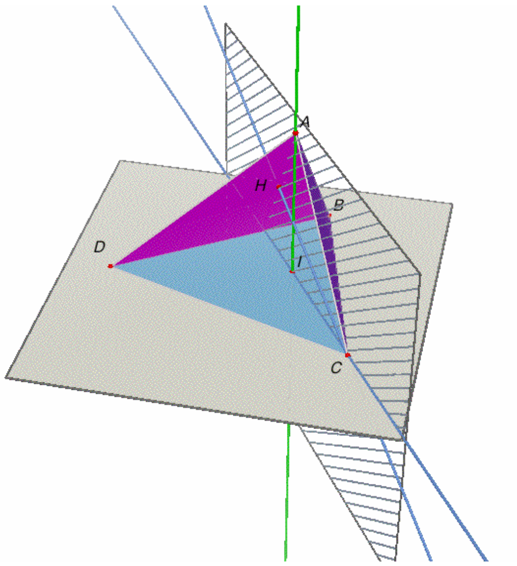


Fig.27 Dragging C onto this plane

The missing implication to explain is why points *A*, *H*, *I* and *C* belong to this plane. By definition line *CH* is perpendicular to plane *BDA*. It is then perpendicular to line *BD* and so belongs to the perpendicular plane to *BD* passing through *C*. Line AI is perpendicular by definition to plane *BCD*. Thus it is perpendicular to line *BD* and for the same reason line *AI* belongs to the perpendicular plane to *BD* passing through *A*.

The role of the soft constructions is extended in 3D geometry, as there are two kinds of dragging: the change of point of view and the dragging of an element. This new possibility should be exploited in order to improve the learning of 3D geometry.

The complementary aspects of robust and soft constructions should be exploited by the teaching. In particular, we think that soft constructions should be officially introduced by the teacher and related by him/her to the theoretical knowledge to be taught. The choice of the tasks given to students is critical in order to promote the use soft constructions. If the task is too closed, it may not give rise to such soft constructions. Tasks asking about conditions to be met to obtain a given condition are often well adapted for this.

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