

Connecting Repeating Decimals to Undergraduate Number Theory Through the Use of Calculators

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Abstract

In this paper, we will report on how we use calculators to help college students and prospective teachers to explore decimal expansions of fractions. Questions that come out of this exploration are; how many digits need to be calculated to completely determine this decimal number? Can we determine the decimal expansion by calculating finitely many digits? How can we tell if the displayed number in the calculator gives complete information about the expansion? In order to answer these and other questions, we design some specific activities which we believe will help students achieve a better understanding of decimal expansions. Our activities turn out to lead to more advanced topics in number theory.

More broadly speaking, our paper is motivated by two questions related to using technology in teaching and learning mathematics:

1. How can technology be used to teach theoretical concepts?
2. How does technology change the content of mathematics?

1. Introduction

In this paper, we will report on how we use calculators to help college students and prospective teachers to explore decimal expansions of fractions (that is, rational numbers). When we deal with real numbers we are actually dealing with decimal expansions, because real numbers are represented by decimal expansions. However the concept of decimal expansions is neglected in schools. The reason is that finding the decimal expansion with paper and pencil is tedious work. But now with the help of technology finding the decimal expansions of those real numbers that are rational is easy and fun, and there are interesting and challenging questions that can be pursued.

The author has long been interested in how technology can be used to enhance mathematics teaching and learning. We give some general background on curriculum thinking about educational use of technology. We then describe some specific activities involving decimal expansions that we believe will help students achieve a better understanding of this important topic.

2. Background

As the ICMI study in 1997 reports, there exists a list of questions on the research by use of technology in teaching and learning of mathematics (King, Hille, & Artique; 2001). This paper is trying to answer two of the proposed questions:

1. How can technology be used to teach theoretical concepts?
2. How does technology change mathematics (what is considered mathematics, how it is done)?

2.1 Technology

Within education in general, and higher education in particular, the use of technology has been purported to be the key to the future. The use of technology allows even the youngest students

a chance to encounter and conquer more difficult problems successfully and to concentrate on the rewards of doing mathematics beyond arithmetic or algebraic manipulation. Arguably, technology is also a key component of the contemporary workplace so that if courses are to adequately prepare students for the market place, then there is strong justification for inclusion of technology-rich classrooms. However, there is an equally strong argument for technology as a pedagogical tool. Technology has been seen to enhance learning through two complementary features. In the first place it reduces the tedium of boring and mindless calculations and second it allows students to explore, to predict and to summarize mathematics (NCTM, 2005; Zevenbergen, 2001).

The appropriate use of computers and calculators to enhance students' mathematical thinking is an ongoing challenge to all mathematics instructors but is especially significant for those responsible for prospective teacher training in mathematical content. Teachers need to re-examine their current practices and beliefs and also the relationship between curriculum, technology and the mathematical content to be taught. Computers and calculators have led mathematics educators to question what mathematics content is central and what is redundant, as well as, how present-day learning, teaching and assessment practices, can be and ought to be changed (Hillel, 2001, p.63).

2.2. The curriculum in undergraduate mathematics.

Research on the teaching and learning of undergraduate mathematics is becoming more visible. Mathematicians who examine mathematics education research findings and reflect on their own pedagogy have likely accepted the idea that lack of understanding mathematics is not simply a question of the "weak" or "unmotivated" students, that there are epistemological and pedagogical roots to some of the difficulties in mathematics; and that there are feasible curriculum options to ameliorate things (Hillel, 2001, p.64)

Many researchers have emphasized that in addition to studying a good deal of mathematics at the undergraduate level, prospective teachers need to develop knowledge of mathematics for teaching—an understanding of the underlying processes and structure of concepts, the relationships between different areas of mathematics, and knowledge of students' ways of thinking and mathematical backgrounds. Are we challenging prospective teachers to come up with conceptual explanations, models, and applications which are relevant to the grade level they are to teach? What good will it do them if they really understand integral domain and are able to construct satisfactory (abstract) proofs, if they have nothing to offer at the level their students can grasp?

In our pre-service teacher courses, we have placed emphasis on modeling an alternative approach by presenting problems that are more problem-based in nature, problems that allow for multiple approaches, problems that may not have a unique answer, and non-routine problems that effectively utilize the power that technology can provide.

This paper presents teaching ideas designed to support the belief that students at all levels need exposure to non-routine problems that illustrate the effective use of technology in their resolution. Such use provides students with rapid and accurate data collection, leading them to sound conjectures, which is a precursor to learning mathematics concepts (Grassi & Mingus, 2002). Although we are always interested in larger issues of curriculum design and appropriate use of technology, for most of the remainder of this paper we will focus on decimal expansions.

2.3. The ICMI Proposal and the NCTM Standard on Teaching Decimals

Curriculum was considered in its widest sense by the discussion group for ICMI in 1997 to mean "matters pertaining to the purposes, goals and content of mathematics education as well as the

means for achieving curricular goals.” However, a fairly accurate picture of undergraduate mathematics is that by and large, it is still dominated by the “chalk-and talk” paradigm, a carefully selected linear ordering of course content, and assessment which is heavily based on a final examination. The curriculum for undergraduates needs to be strengthened, broadened, and designed for more activity participations by students in discovery in order for them to have a sound understanding of mathematics as a basis for life-long learning (Hillel, 2001).

A statement from NCTM’s 1989 and 2000 Standard on Teaching Decimals has concluded it is necessary for students to make sense of decimals and use them insightfully, to recognize the difficulties inherent in various representations and to grasp some sense of the “infinite” quality of decimals.

There is a need for research in how to teach our students about decimals more successfully. An interesting article (Glasgow et al. 2000) indicates that our elementary students are comfortable with fractions but perform significantly less well on questions about decimals.

Students should be able to explain what a decimal is. They should understand that a non-terminating decimal is a sequence of approximations. They should be aware of the connection between rational numbers and repeating decimals. Typical skills might include: using decimals to quickly compare the sizes of rational numbers; or being able to do such computations as 3.45×2.8 and ending up with decimal point in the right place (preferably without memorizing an algorithm). When is it useful to replace a rational fraction by a decimal? Being able to answer questions like these could be considered mastery of the topic.

By using the calculator to carry out the indicated arithmetic computations, students can learn common fraction-decimal equivalents. They can also learn that some fractions can be expressed as terminating decimals but others cannot. However, when students perform calculations of fractions on calculators, they have no way of knowing whether the decimals are terminating or non-terminating. In this paper, we are going to introduce several methods to find the period-length of the repeating decimals (Lee, 2005) and to discover the underpinning concepts in number theory. We would like to provide a way using calculator as a cognitive tool to explore and to understand the concept of repeating decimals and to make connections of that to the study of number theory. Through exploration and pattern recognition with the calculator, students can come to understand where conjectures and theorems come from which can provide a feeling of what the subject is and why it works.

3. The Teaching and Learning of Repeating Decimals

3.1 The Background.

In our designed problem-based projects, each student has one lucky number from one to nine based on his/her name and they also generated by their own name other numbers, for example, sum numbers, fractions, decimals and etc. (The author likes to give students individualized projects that have some personal significance; this seems to increase students’ interest.) When they find out their own decimals numbers, using calculator, based on their lucky numbers, they also need to categorize to what type of decimals their lucky numbers’ belong to. We want them to find patterns of the different representations of decimals. Two questions came out from this activity. First, how many digits do they need to put into this decimal number? Second, how do they know whether the displayed number in their calculator gives complete information about the decimal expansion? In order for them to better understand the notion of repeating decimals, we provide the following calculator method for students to work on. The first one is sort of like long division algorithm;

following the steps the students can find the length of decimal expansions that happen to be longer than their calculator display. The second method is based on using the constant feature in the calculators; they can multiply each consecutive number by 10 then subtract the denominator using the constant feature. The third method is based on Midy's theorem (Dickson, 2005). We will give the detailed descriptions in the next section.

As a preliminary to these activities, the student should be aware that the expansion of N/D has a period not exceeding the number of digits of D in length, and that changing N has the simple effect of just shifting this period. Activities three and four below should be helpful in this connection. (cf. Ore, especially the chapter on decimal expansions.)

3.2 The Tools for Calculator Exploration.

3.2.1. Method One: (Long Division Algorithm)

1. Know your Calculator

i. For any number A , let $\#(A)$ represent the number of digits in the number A :

E.g. $\#(19) = 2$ and $\#(0.368421) = 6$.

ii. Let $B = \#(\text{calculator display}) - \#(D)$.

iii. To find $\#(\text{calculator display})$, do the computation simply enter 123456789012 and count the number of digits.

2. Record. Let N/D be a rational number.

i. The digits produced by algorithm are recorded in blocks of B digits, after an initial recording (digits to the left of the decimal point) that depends on whether or not $N < D$.

ii. If $N < D$, the initial record is 0.

iii. If $N > D$, convert N/D to a mixed number $a + (N'/D)$; then the initial record is a . The algorithm is then applied to N'/D . For example, the initial record for $40/19$ is 2 and the algorithm is applied to $2/19$, since $40/19 = 2 + 2/19$.

3. The Algorithm's Steps

Step 1: Write down the initial record for N/D .

Step 2: Clear calculator.

Step 3: Compute $N \div D$. For example, $7 \div 19 = 0.36842105$.

Step 4: Adjoin to the existing record the first B digits following the decimal point in the display.

Since $B = 8 - 2 = 6$, the record at this stage is 0.368421.

Step 5: Decision: If more digits in the decimal representation are desired, continue to step 6; if not, terminate the algorithm.

Step 6: Clear the calculator.

Step 7: Enter a decimal point, followed by the group of B digits last recorded in step 4.
Enter .368421.

Step 8: Multiply the entry in step 7 by the denominator D .

For example, we have $0.368421 \times 19 = 6.999999$.

Step 9: Compute the difference $N - (\text{result in step 8})$.

$7 - 6.999999 = 0.000001$.

Step 10: Convert the result in step 9 to a whole number by moving the decimal point B places to the right (or multiply 10 to the B 's power). Obtain 1.

Step 11: Repeating N in the algorithm by the whole number obtained in step 10; then return to step 2 and repeat the same sequence of steps.

- For the example $7/19$, a repetition of step 2 through 11 yields the following:
- Step 3: Note that $N = 1$ at this point, so $1 \div 19 = 0.0526315$.
- Step 4: The record becomes 0.368 421 052631.

- Step 5: Decision: Want more digits, so continue with step 6.
- Step 7: Enter .052631.
- Step 8: Multiply by D, we have $0.052631 \times 19 = 0.999989$.
- Step 9: Compute the difference N- (result in step 8). $1 - 0.999989 = 0.000011$.
- Step 10: Move decimal point 6 places to the right to obtain 11.
- Step 11: With $N=11$, return to step 2.
- Step 3: Compute $11 \div 19 = 0.5789473$.
- Step 4: The record becomes 0.368 421 052 631 578 947.
- Step 5: Decision: Once the period for $7/19$ has become clear, the algorithm is terminated. (In our example, we know that the longest possible period is 18, and we have 18 digits).

3.2.2. Method Two: Calculator' Constant Feature

The constant feature is the feature of a calculator that allows users to add (or subtract, multiply, or divide) using the same number more than once without entering it each time. For example, to add student test scores of 89, 95, 76, 76, and 76 on some calculator, users enter $89+95+76 = = =$, where each extra equal sign adds another 76. In case of $60-8 = = = = = \dots$, it suggests a repeated subtraction model for a division computation, $60 \div 8$ has quotient 7 with remainder 4 (Schultz, 2004). This illustrates how we can use a four-function calculator to build concepts, to solve problems, and to look for patterns.

3.2.3. Method Three: (Based on Midy's Theorem.) The following was proved in 1837 by E. Midy (Dickson, 2005, p.163):

Midy's Theorem: For a prime p , if the repetend of $1/p$ has $2n$ digits, then digit $(n + k) = 9 - \text{digit}(k)$

Suppose we needed $1/17$. Our calculator might only give us 0.0588235, and the last digit is questionable. Take that one off, and write down the rest 0.058823. Subtracting 058823 from 999999 by the Midy' Theorem, we get 941176 which is the first six digits of $16/17$. Since we already know the repetend formed a cyclic group, just multiply $1/17$ by any number a , of your choice, we will get another sequence of numbers of $a/17$ which is part of the repetend of $1/17$. Those numbers will give you more information to fill in the numbers in this cyclic repetend of $1/17$. For example, find the repetend of $1/17$:

- 1). the calculators provide 0.058823 when dividing 1 by 17, and then by Midy's Theorem, we get 941176 from $999999 - 058823$;
- 2). multiplying $1/17$ by any number a , of your choice, such as 2, we get $2/17 = 0.117647$ showing on our calculator. Based on Midy's method, we get another set of digits $882352 = 999999 - 117647$;
- 3). up to now, we combine all information, we have the numbers
0.0588235294117647;
- 4). we know the longest possible repetend of $1/17$ is 16 and we have 16 digits. We are done. So, $1/17 = 0.05882352941176470588235294117647$; and
- 5). the underlying concept of Midy's Theorem, provide the idea of limits. We can understand why $1/7 + 6/7 = 0.999999 \dots$

3.3 The designed Activities and the Number Theory

3.3.1. Activity One:

- 1). How to use a basic four-functional hand-held calculator to find the period for decimal expansion?

- 2). without a fractional key in your calculator, how to convert n/d into mixed fraction?
- 3). If we know the period of decimal expansion of $1/7$, how to find the period of $6/7$? How about $3/7$?

3.3.2. Activity Two:

Answer all the following questions by doing only one hand calculation of $1/7$:

- 1). what is the decimal expansion of $1/7$?
- 2). Divide 1000 by 7 and find the remainder.
- 3). Rewrite $1000/7$ as a mixed number.
- 4). If January first is Monday, then 1000day falls on which day?
- 5). what rational number has the decimal expansion 142.857142? (Give the answer both as improper fraction and as a mixed number.)
- 6). what rational number has the decimal expansion .857142?
- 7). If you leave out the first three steps in computing $1/7$, what problem does your calculation solve?
- 8). what are the sequences of remainders you get when calculating the decimal expansion of $1/7$?

3.3.3. Activity Three:

If we know $1/7 = .142857$,

- 1). Can we find the results of $1/7$, $2/7$, $3/7$, $4/7$, $5/7$ and $6/7$?
- 2). If we multiply $1/7$ by 2, 3, 4, 5, and 6, what do we get?
- 3). we can think of what might be called a circular pattern: If we start at different points in the circle and then continue clockwise, we get the decimal expansions for $3/7$, $2/7$, $6/7$, $4/7$, and $5/7$.

3.3.4. Activity Four:

- 1). Answer all the questions in Activity Two with $1/13$?
- 2). If you skip the first step in the calculation of $1/13$, what fraction does your answer represent?
- 3). If we use the circle pattern of $1/13$, what fractions of $a/13$ are missing from this circle pattern?
- 4). Do you think that those fractions will all have the same circular patten as each other?
- 5). what are the same and what are differences between the answers of the questions related to $1/7$ and $1/13$?

3.3.5. Activity Five: Reverse Thinking

- 1). If we have a repeating decimal .142857, convert it into a fraction.
- 2). what is the relationship between 7, 142857 and 999999?
- 3). If we have $0.076923 = 1/13$, then 76923 can go into how many 9's.

3.3.6. Activity Six:

If you prefer using an 8-digit calculator to long division, you can still calculate complete decimal expansions by taking advantage of the cyclic property and the Midy' Number Theory. We demonstrate how to figure out the repetend of $1/29$.

- 1). the calculators provide 0.034482 when dividing 1 by 29, and then by Midy's Theorem, we get 965517 from 999999-034482.
- 2). Multiplying $1/29$ by any number A of your choice, such as 2, we get $2/29 = 0.068965$ showing on our calculator. Based on Midy's method, we get another set of digits 931034 = 999999 - 068965.
- 3). Try $3/29$, we get 103448 and 896551. Can not find any connection, we have to continue.

- 4). Try $4/29$, we get 137931 and 862068
- 5). Up to now, we combine all information; we have the following sets of numbers.
0.034482 965517 068965 931034 103448 896551 137931 862068
 Rearranging them based on the similar digits, we have
0.034482 862068 068965 896551 965517 137931 931034 103448.
 Eliminating the same digits, we have
0.034482 862068 965 51 7 137931.
- 6). we know the longest possible repetend of $1/29$ is 28. We only have 24 digits, so continue the process. From $5/29$, we have 172413 and 827586. From $6/29$, we have 206896 and 793103. Rearranging them, we have
0.034482 7586 206896 5517 2413 7931
- 7). Now we have 28 digits. We are done. So,
 $1/29=0.0344827586206896551724137931.$

3.3.7. Activity Seven: The days of the week

The week days repeat themselves in periods of 7 so instead of the usual names we can give each day a number.

Sunday=0, M. =1, Tu. =2, W. =3, Th. =4, F. =5, S. =6.

If we do this, each integer corresponds to a week-day, namely, the day determined by its remainder (mod 7). Try to find out what week-day corresponding to 10, 100, 1000, 10000, and 10000.

4. Closing Remarks

4.1. How can technology be used to teach theoretical concepts?

The teaching and learning of decimals in school no matter what levels still focus on arithmetic operations and comparing sizes (Glasgow et al, 2000). This paper presents a way of going beyond the basic teaching and learning of decimals. With the inexpensive hand-held calculator and some investigation algorithms, students can explore, analyze, construct, collect and represent data, present arguments, and solve problems about decimals. Not only can students better understand the notion of decimals but also acquire some basic knowledge from higher mathematics, such as the study of number theory.

The activities we presented in this paper are just the tip of icebergs above the ocean, but we have already introduced some important theorems from number theory; such as

Theorem 1: If a fraction n/p has at least one prime factor different from 2 and 5, then that fraction has a repeating decimal expansion and the length of the repeating pattern is at most $p-1$.

Theorem 2: If $1/p$ has a decimal pattern of length r , then p must divide 10^r-1 . If, in addition, p is a prime, then r must divide $p-1$ (written as $p \equiv 1 \pmod{r}$).

Theorem 3: If p is a prime, and x is the length of the circle pattern, and y is the number of unique circle found for all $1/p, 2/p, \dots, (p-1)/p$, then $x y = p-1$.

4.2. How does technology change mathematics (what is considered mathematics, how it is done)?

Mathematics teachers have needed inexpensive, easy-to-use software and technological teaching tools that allow them to create lesson plans that contain multiple representations. Teacher-

created lesson plans need to be designed to ensure deeper understanding of underlying theory and concepts, which are necessary prerequisites for our prospective teachers teaching in mathematics. Therefore, careful selection and creation of more challenging non-routine problems that lend themselves to technological dissection should be the scaffolding upon which these curricula are built. In this paper we demonstrated how technology can be used to teach theoretical concepts and how technology changes mathematical instruction and content.

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