

Motions of Particles Described in a Three-Dimensional Space-Time Frame

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Abstract: The motion of a particle moving in space can be decomposed into three directions, but it is difficult to describe all three directions in a single three-dimensional space frame with respect to time. In mathematics, we can locate a point on a plane either using (x, y) in the rectangular Cartesian coordinate frame or using (r, θ) in the polar coordinate frame, where x equals to $r(\cos\theta)$ and y equals to $r(\sin\theta)$. In physics, we can describe the motion of a particle moving in one dimension in the rectangular Cartesian frame with the x -axis as location coordinates and the y -axis as time coordinates of the particle (x, t) . Can we describe its motion in a polar coordinate frame? The proposed 3-D S-T is constructed similar to a polar coordinate frame to solve this problem. In the proposed three-dimensional space-time frame (3-D S-T), space is represented by three perpendicular axes ($x, y,$ and z), but time is represented by spheres of different radii with the origin of the space axes as their center. The radii of the spheres representing time are set to equal the velocity of the medium used to transmit messages in the system multiplied by time. Any particle's motion in space can be described in a chosen 3-D S-T frame by selecting the proper medium for transmitting messages. This paper will illustrate how to graph the motions of particles in this 3-D S-T frame using MATLAB's graph-function. In Special Relativity, a 3-D S-T frame can be constructed to describe the motion of the other frame by choosing light as a medium for transmitting messages; then geometric lines can represent time dilation and length contraction clearly in this 3-D S-T frame. The proposed 3-D S-T frame is an alternate coordinate system that can be used to describe motions of particles, and it may provide additional understanding of space and time.

1. Introduction

In mathematics, we can locate a point on a plane by using either (x, y) in the rectangular Cartesian coordinate frame or (r, θ) in the polar coordinate frame, where x equals to $r(\cos\theta)$ and y equals to $r(\sin\theta)$. In physics, we can describe the motion of a particle moving in one dimension in the rectangular Cartesian frame with the x -axis as location coordinates and the y -axis as time coordinates of the particle by (x, t) . Can we describe its motion in a polar coordinate frame? The proposed 3-D S-T is constructed similar to a polar coordinate frame to solve this problem. In the polar coordinate frame, the coordinates consist of a directed distance and the measure of an angle between the pole and the polar axis. Since the units of the pole and polar axis must be in terms of length, time (for the pole) can be converted into distance by multiplying it with the velocity of a medium transmitting messages.

The motion of a particle moving in one dimension can be described in a two dimensional coordinate frame with x -axis as the space dimension and y -axis as the time dimension. It is difficult to visualize a four-dimensional space-time (4-D S-T) frame with a time-axis that is simultaneously perpendicular to the three space-axes. A particle's motion in a three-dimensional space can be decomposed into $x, y,$ and z directions in a traditional frame. Since it is difficult to portray a t -axis that is simultaneously perpendicular to the $x, y,$ and z axes in three-dimensional

space, three separate planes ($x-t$, $y-t$, and $z-t$) are used to describe the particle's motion. The traditional frame expresses the independence and absoluteness of space and time. The unit for the x , y , and z axes is in terms of length, and the unit for the t -axis is in terms of time. The two units are incomparable and thus independent. In construction of a traditional 4-D S-T frame, there is no restriction in choosing a unit of length for the space-axes and a unit of time for the t -axis. The absoluteness of space and time allows a stationary particle to be treated as fixed at the zero second mark from the origin of the frame, regardless of its distance from the origin. Therefore, lines can be drawn perpendicular to the space axes to represent the stationary particle in the traditional frame.

Einstein demonstrated that the simultaneity of two events is relative, not absolute, by emitting light from the center of a car of a moving train. Therefore, space and time must be treated as dependent and inseparable. In order to use the traditional 4-D S-T frame, a constraint ($x^2 + y^2 + z^2 - c^2t^2 = const.$) binding space and time must be added to the frame. Any event can only occur inside the future light cone, not outside. The further the particle is located from the origin, the longer it would take for light to travel to it. Therefore, it would take a longer time t to know if the particle remains stationary. If messages are transmitted by sound or medium other than light, the future light cone would be changed accordingly. A 3-D S-T frame can be constructed by embedding time into space without adding the constraint to show the dependence and inseparability of space and time. Any particle's motion in space can be described in a chosen 3-D S-T frame by selecting the proper medium for transmitting messages.

In Special Relativity, a 3-D S-T frame can be constructed to describe the motion of the other frame by choosing light as a medium for transmitting messages. Geometric lines can represent time dilation and length contraction clearly in this 3-D S-T frame.

2. Construction of a 3-D S-T Frame

In this proposed 3-D S-T frame, space is represented by three perpendicular axes (x , y , and z), but time is represented by spheres of different radii with the origin of the space axes as their center. If messages are transmitted by sound of $V_m = 350m/sec$ then the radius of the sphere representing one second is equivalent to $(V_m)(1sec) = 350m$; the radius of the sphere representing two seconds is equivalent to $(V_m)(2sec) = 700m$;...; and the radius of the sphere representing n seconds is equivalent to $(V_m)(nsec) = n(350)m$.

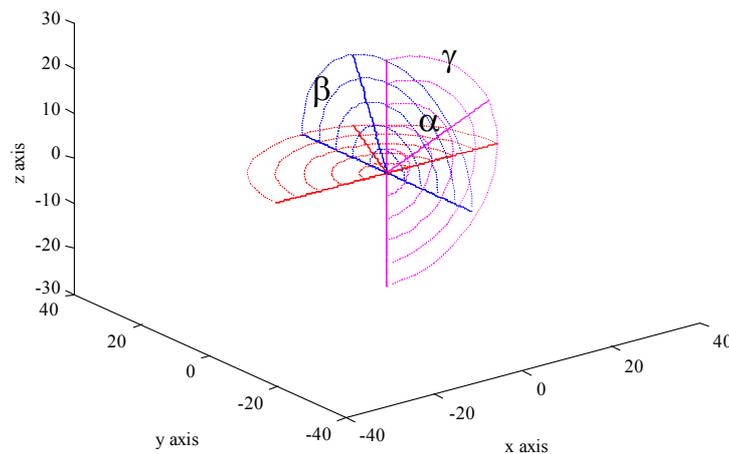


Fig.1: The construction of a 3-D S-T frame

$h=(r^2-x^2)^{1/2}$ and $\cos\alpha=x/r$ on the x - y plane represent the location of the particle along the x -axis.

$h=(r^2-y^2)^{1/2}$ and $\cos\beta=y/r$ on the y - z plane represent the location of the particle along the y -axis.

$h=(r^2-z^2)^{1/2}$ and $\cos\gamma=z/r$ on the z - x plane represent the location of the particle along the z -axis.

The motion of any particle in space can be decomposed into x , y , and z directions. The motion along the x -axis can be described as a function of time represented by circles, which are the intersections between the x - y plane and the spheres with the same center. Since $r=r(t)$ and $x=x(t)$, then $h(t) = \sqrt{r^2(t) - x^2(t)}$ and $\cos(\alpha)=x(t)/r(t)$ on the x - y plane can represent the location of the particle along the x -axis at time t . The motion along the y -axis can similarly be described as a function of time represented by circles, which are the intersections between the y - z plane and the spheres with the same center. Since $r=r(t)$ and $y=y(t)$, then $h(t) = \sqrt{r^2(t) - y^2(t)}$ and $\cos(\beta)=y(t)/r(t)$ on the y - z plane can represent the location of the particle along the y -axis at time t . The motion along the z -axis can also be described as a function of time represented by circles, which are the intersections between the z - x plane and the spheres with the same center. Since $r=r(t)$ and $z=z(t)$, then $h(t) = \sqrt{r^2(t) - z^2(t)}$ and $\cos(\gamma)=z(t)/r(t)$ on the z - y plane can represent the location of the particle along the z -axis at time t . The construction of a 3-D S-T frame is shown in Fig.1.

3. Applications of a 3-D S-T Frame

A few examples are selected to illustrate how the motions of particles can be described in this proposed 3-D S-T frame.

3.1 A single particle remaining stationary

If a particle remains stationary at the location of $(175m, 350m, 525m)$, the distance between the particle and the origin of the frame is $d = \sqrt{(175m)^2 + (350m)^2 + (525m)^2} = 654.8m$. It takes sound $1.87sec$ to reach the particle from the origin. The description of the particle remaining stationary after $2sec$ (rounded up from $1.87sec$ for convenience) in the proposed 3-D S-T frame can be shown in Fig.2.

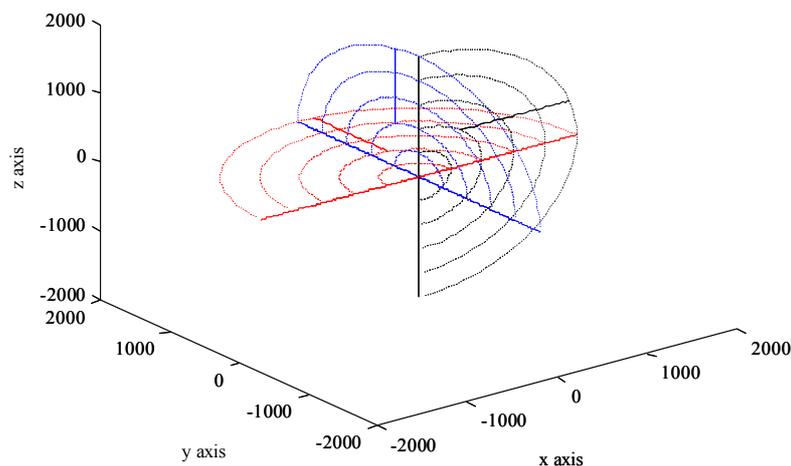


Fig.2: A single particle remains stationary

The particle remains stationary at the location of $(175m, 350m, 525m)$.

3.2 Multiple particles remaining stationary

If three particles A , B , C remain stationary at the locations of $(0, 175m, 0)$, $(0, 350m, 0)$, and $(0, 525m, 0)$ respectively, it takes sound $0.5sec$ to reach particle A , $1sec$ to reach particle B and $1.5sec$ to reach particle C from the origin. The status of each particle is shown in Fig.3.

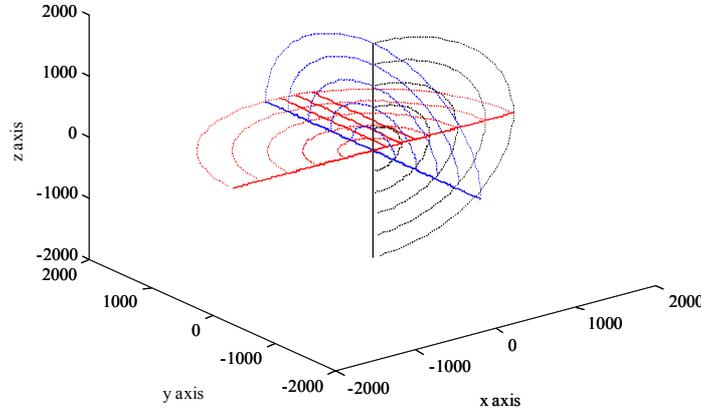


Fig3: Multiple particle remain stationary
 Three particles remain stationary at the locations of $(175m,0,0)$, $(350m,0,0)$, and $(525m,0,0)$.

3.3 A single particle moving with constant velocity from the origin

If a particle moves with a constant velocity of $v=269.26m/sec$, it can be decomposed into the x -direction: $v_x=200m/sec$, the y -direction: $v_y=150m/sec$, and the z -direction: $v_z=100m/sec$. At $3sec$, $r(t)=V_m(t)=(350m/sec)(3sec)=1050m$, for the coordinate along x -axis: $x=(v_x)(t)=(200m/sec)(3sec)=600m$, $\cos(\alpha)=600/1050=0.57$, $\alpha=55^\circ$ on the x - y plane; for the coordinate along y -axis: $y=(v_y)(t)=(150m/sec)(3sec)=450m$, $\cos(\beta)=600/1050=0.43$, $\beta=64.5^\circ$ on the y - z plane; for the coordinate along z -axis: $z=(v_z)(t)=(100m/sec)(3sec)=300m$, $\cos(\gamma)=300/1050=0.2875$, $\gamma=73^\circ$ on the z - x plane. The description of this motion is shown in Fig.4.

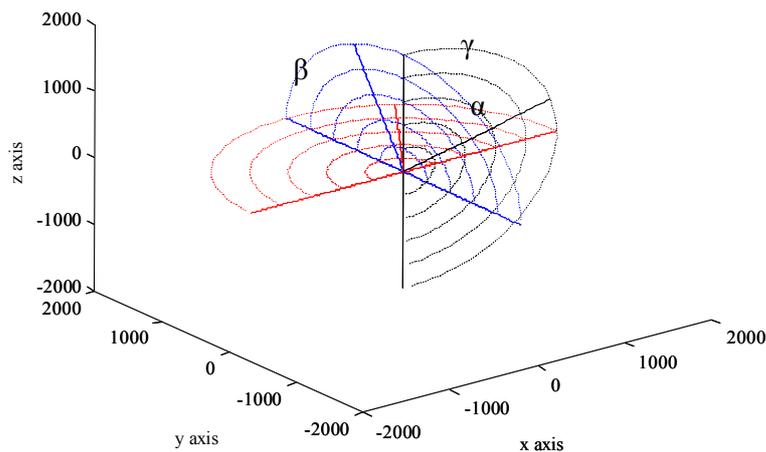


Fig.4: A single particle moves with constant velocity from the origin
 A particle moves in space with $200m/sec$ along the x -axis, $150m/sec$ along the y -axis, and $100m/sec$ along the z -axis.

3.4 A single particle moving with constant velocity from a point other than the origin

A particle located at $(200m, 250m, 300m)$ starts moving in space at $1.5sec$ with a velocity of $200m/sec$ along the x -axis, $150m/sec$ along the y -axis, and $100m/sec$ along the z -axis. The description of this motion is shown in Fig.5.

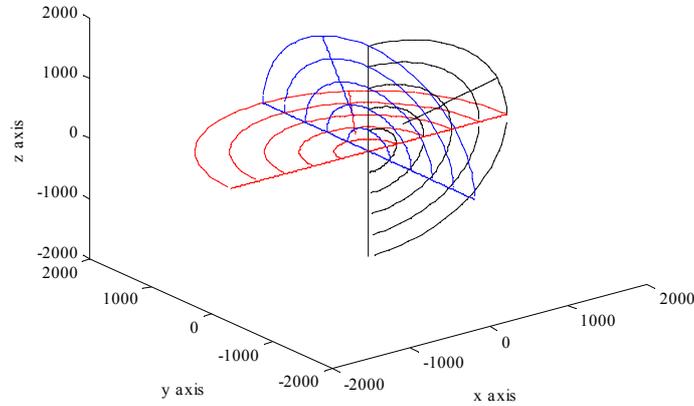


Fig.5: A single particle moves with constant velocity from a location other than the origin

A particle moves from the point of $(200m, 250m, 300m)$ with $v_x=200m/sec$, $v_y=150m/sec$, and $v_z=100m/sec$.

3.5 Multiple particles start moving with the same velocity after receiving their respective signals

Three particles located at $175m$, $350m$ and $525m$ on the x -axis start moving with the same velocity of $200m/sec$ in the y -direction after receiving their respective signals of sound from the origin O. The description of this motion is shown in Fig.6.

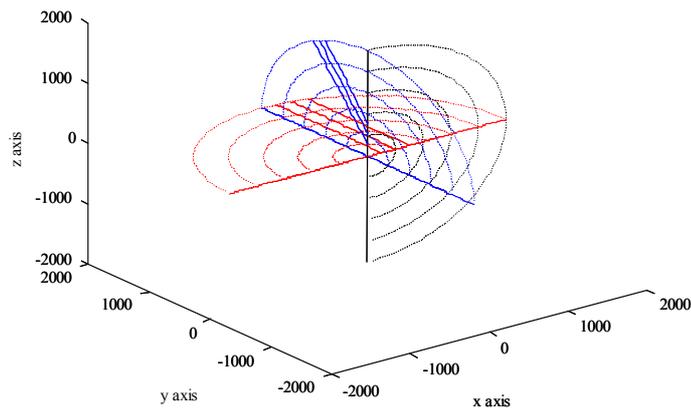


Fig.6: Multiple particles start moving with the same velocity after receiving their respective signals

Three particles located at $175m$, $350m$, and $525m$ on the x -axis start moving in the y -direction with the same velocity after receiving their respective signals.

3.6 Multiple particles start moving simultaneously with the same velocity

Three particles located at 175m, 350m and 525m on the x -axis remain stationary after receiving their respective signals of sound from the origin O. When the furthest particle (at 525m) receives the signal of sound at 1.5sec, all particles start moving simultaneously with the same velocity of 200m/sec in the y -direction at 1.5sec. The description of the motion is shown in Fig.7.

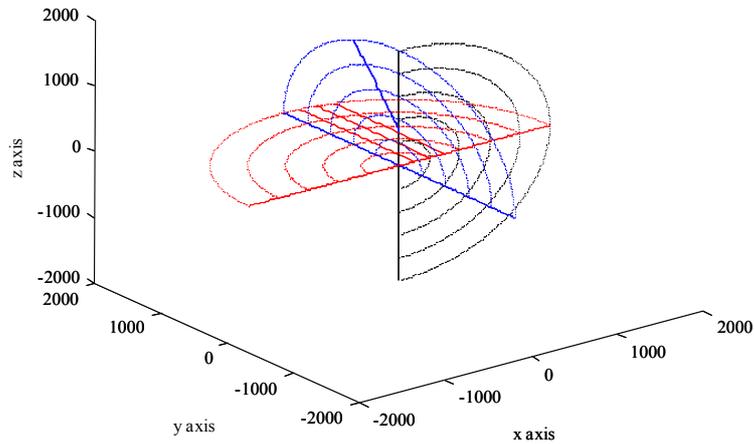


Fig.7: Multiple particles start moving simultaneously with the same velocity
Three particles located at 175m, 350m, and 525m and start moving simultaneously in the y -direction with the same velocity 200m/sec.

3.7 A single particle moving in a helix in space

A particle moves in circles on the x - y plane while also moving with constant speed along the z -axis. The description of the motion is shown in Fig.8.

More complicated motions of particles can similarly be described in a 3-D S-T frame as needed.

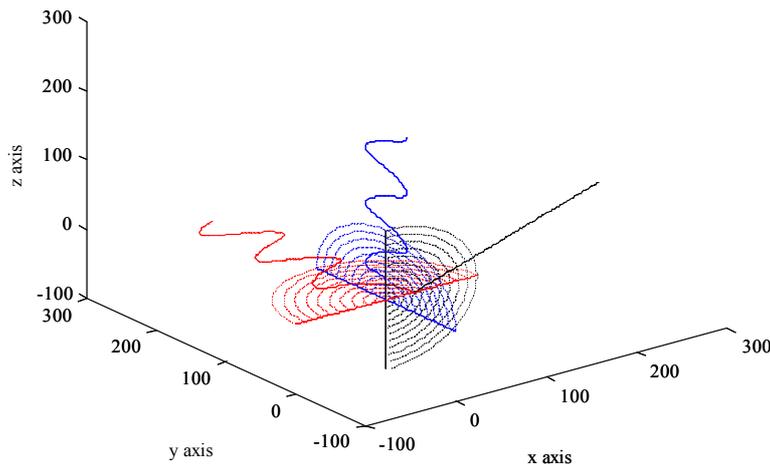


Fig. 8: A single particle moving in a helix in space
A particle moves in circles on the x - y plane while also moving with constant speed along the z -axis.

4. Describing Slow Moving Particles in a 3-D S-T Frame

If the particles move very slowly (i.e. a few millimeters per second), the scale of the 3-D S-T frame constructed based on the message carried by sound would be too large to describe their motions. It would be difficult to distinguish between particles moving at 0.3cm/sec and at 0.5cm/sec , which are both drawn as vertical lines that seem to remain stationary in the proposed 3-D S-T frame. For example, in a system of biological particles, the maximum velocity of the particles can be chosen as the velocity of transmitting messages, $V_m = V_{max}$, to construct a 3-D S-T frame. If $V_{max} = 2\text{cm/sec}$, then let $V_m = V_{max} = 2\text{cm/sec}$. The radius of the sphere representing one second is equivalent to $(V_m)(1\text{sec}) = 2\text{cm}$; the radius sphere representing two seconds is equivalent to $(V_m)(2\text{sec}) = 4\text{cm}$; ...; the radius of the sphere representing n seconds is equivalent to $(V_m)(n\text{sec}) = (2n)\text{cm}$. The slow motion of the particles would then be distinguishable in this 3-D S-T frame.

5. Describing a Fast Moving Particle and a Slow Moving Particle in One 3-D S-T Frame

There are a canoe and a boat sailing on the lake. The boat is moving with the velocity of 250m/sec in the direction of 53.13° northeast from the center of the lake, and its velocity can be decomposed into x-direction $v_x = 150\text{m/sec}$ and y-direction $v_y = 200\text{m/sec}$. The canoe is moving with the velocity of 5m/sec in the direction of 53.13° northeast from the location of the coordinates $(360\text{m}, 480\text{m})$, and its velocity can be decomposed into x-direction $v_x = 3\text{m/sec}$ and y-direction $v_y = 4\text{m/sec}$. If there are two different mediums to transmit messages for different particles, we should choose the faster moving medium for both particles. In this case, the medium with the velocity of $v_m = 350\text{m/sec}$ (the speed of sound) can be selected. The slow moving particle nearly stays still by comparing with the fast moving particle described in one 3-D S-T frame. It takes about 2.5sec for the boat to reach the canoe and the description of motions of them can be shown in Fig.9.

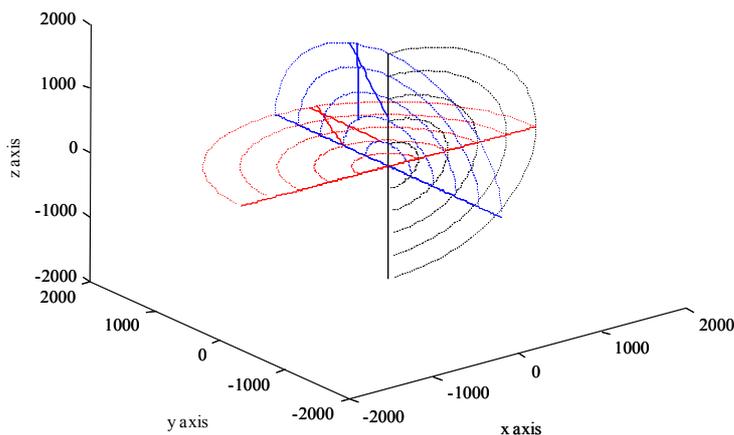


Fig.9: A fast moving boat and slow moving canoe on the lake

The canoe is moving with the velocity 5m/sec in the direction of 53.13° from the location $(360\text{m}, 480\text{m})$. The boat is moving with the velocity 250m/sec in the same direction from the origin of the frame.

6. Describing the Motion of a Particle with Velocity near to the Velocity of Light in a 3-D S-T Frame

If the message is transmitted by light of $V_m=3(10^8)m/sec$, then the radius of the sphere representing one second is equivalent to $(V_m)(1sec)=3(10^8)m$; the radius of the sphere representing two seconds is equivalent to $(V_m)(2sec)=6(10^8)m$; ...; and the radius of the sphere representing n seconds is equivalent to $(V_m)(nsec)=3n(10^8)m$. Since the velocity of light is the limiting velocity, all possible motions of a particle can be described in this 3-D S-T frame.

In cosmology, the expansion velocity of the universe is very high, relative velocities of some galaxies are near 90% of the velocity of light. All galaxies are very far away from us. The interval of 1sec would be too small to meaningfully describe its motion, The units of time can be scaled up by choosing light year. The radius of the sphere representing one year is equivalent to $(V_m)(1year)=9.46(10^{15})m=1ly$; the radius of the sphere representing two years is equivalent to $(V_m)(2year)=1.89(10^{16})m=2ly$; ...; and the radius of the sphere representing n years is equivalent to $(V_m)(nyear)=9.46n(10^{15})m=nly$.

In high energy physics, if a particle's velocity approaches the velocity of light, the interval of 1sec would be too large to meaningfully describe its motion. The units can be scaled down by choosing the period (T) of any selected light as the unit of time and its corresponding wavelength (λ) as the unit of length of the space axes, since the ratio of the wavelength and wave period is equal to the velocity of light. The radius of the sphere representing $1T$ is chosen to equal λ ($V_m t=(c)(1T)=\lambda$); the radius of a sphere representing $2T$ is chosen to equal 2λ ($V_m t=(c)(2T)=2\lambda$); and the radius of the sphere representing nT is chosen to equal $n\lambda$ ($V_m t=(c)(nT)=n\lambda$). The construction of a 3-D S-T frame with light used to transmit messages is shown in Fig.10.

The relationship, $\frac{\lambda}{T} = \frac{\lambda'}{T'} = c$, can be used as the constraint for the 3-D S-T coordinate system for coordinate transformation between two traditional frames. The embedding of time into space with this new constraint illustrates the natural inseparability of space and time.

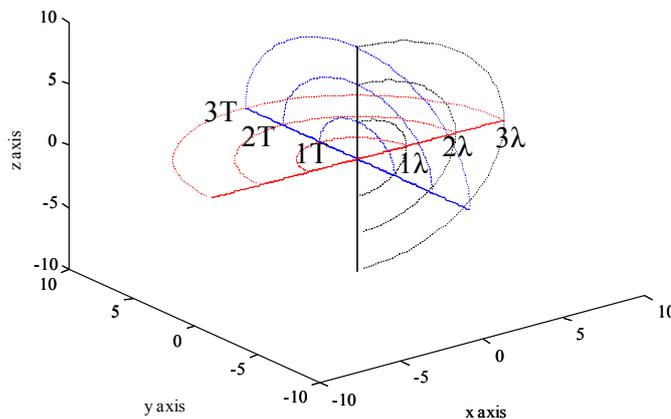


Fig.10: Fast moving particle in a 3-D S-T frame

When messages are transmitted by light, the unit of length and the unit of time can be scaled down to wavelength and wave period.

The period of an event measured in units of time is equal to $\tau=t/T$; and the coordinates of a particle's location measured in units of length are equal to $w_x=x/\lambda$, $w_y=y/\lambda$, and $w_z=z/\lambda$. With this

transformation, the coordinate (x, y, z, t) with sec and m as units in a 4-D S-T coordinate system can be converted to the coordinate (w_x, w_y, w_z, τ) with T and λ as units in a 3-D S-T coordinate system.

7. Geometric Lines Representing Time Dilation and Length Contraction

The geometric lines can represent time dilation and length contraction clearly in 3-D S-T frames. When the observer in the inertial frame of the moving train moves to the right with velocity v with respect to the observer in the reference frame of the stationary platform placed with a rod, a 3-D S-T frame can be constructed in the inertial frame with the observer as the origin O' of the frame. Another 3-D S-T frame can be constructed in the reference frame with the observer as the origin of the frame. The motion of origin O' described from an observer at origin O of the reference frame is OQ in the 3-D S-T frame shown in Fig.10. The location of origin O described from an observer at origin O' of the inertial frame is $O'Q$ in another 3-D S-T frame also shown in Fig.11.

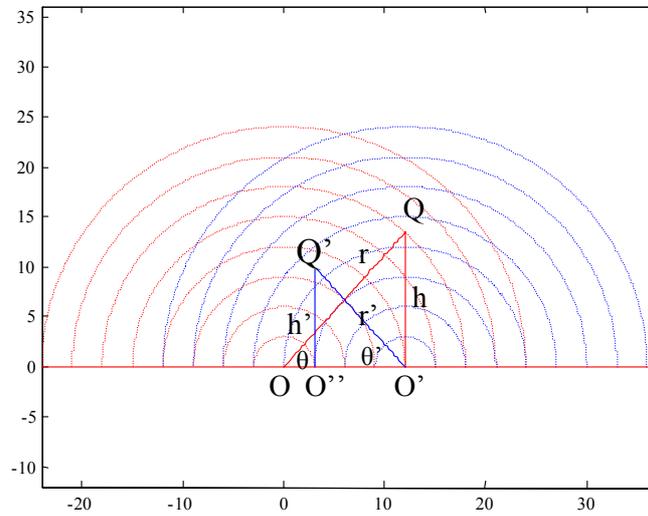


Fig.11: The geometric meaning of time dilation and the length contraction in two 3-D S-T frames

The origin O' moves to the right with the velocity v with respect to the Origin O . $l=l_0=OO'=vt$, $l'=OO'=vt'=vt'_0$ and $r=ct$, $r'=ct'=ct'_0=h$.

The two equations, $\sin \theta = h/r = \sqrt{(ct)^2 - (vt)^2} / ct = \sqrt{1 - (v/c)^2}$ and $\sin \theta' = h'/r' = \sqrt{(ct')^2 - (vt')^2} / ct' = \sqrt{1 - (v/c)^2}$, are derived from Fig.10, therefore, $\theta' = \theta$. It also shows that $\sin \theta = \sin \theta' = h/r = r'/r = ct'/ct = t'/t$, therefore, $t > t'$.

The event in which the sensor touches the left and right edges of the rod can be described by two different observers. The period of the event that occurred at the same location measured from the observer in the inertial frame of the moving train is called proper time, t'_0 , which is equal to t' and proportional to the length of r' ($r'=ct'=c t'_0=h$) in Fig.12. The period of the event that occurred at different locations measured from the observer in the reference frame of the stationary platform is called regular time, t , which is proportional to the length of r ($r=ct$) in Fig.10. Because t is larger than t'_0 , this difference is referred to as time dilation.

Since $\sin \theta = t'/t = vt'/vt = l'/l_0$, therefore $l' < l_0$. The length of a stationary rod measured from an observer in the reference frame of the stationary platform is called the proper length, l_0 ,

which is equal to $OO' = vt$ shown in Fig.10. However, the length of the same object measured from an observer in the inertial frame of the moving train is called regular length, l' , which is equal to $OO'' = vt_o'$ shown in Fig.11. Because l' is smaller than l_o , this phenomenon is called length contraction.

8. Conclusions

In the traditional frame, time is represented by a t -axis, which is simultaneously perpendicular to the x , y , and z axes of space. Thus, time and space are treated independently. Since Einstein demonstrated the inseparability of time and space, when using the traditional frame, a constraint is added for coordinate transformation between two traditional frames to show this dependence. In the proposed 3-D S-T frame, time is represented by spheres of different radii with the origin of the space axes as their center. Time is embedded into space by setting the radius of the sphere representing time t equal to $V_m t$.

In the traditional frame, the units of the time axis can be chosen as seconds, while the units of the space axes are in meters. When graphing, there is no restriction between the lengths of 1 second and 1 meter since the two are different types of units. In the proposed 3-D S-T frame, the units of radii of the spheres representing time are the same as the units of the space axes. Both units are in terms of length. The realistic difference between time and space is the one direction of time and the two directions of space. In the traditional frame, time, represented by the t -axis, can have two directions. However, in 3-D S-T frame, time, represented by spheres of different radii with the same center, can only have a single direction.

In Special Relativity, a 3-D S-T frame can be constructed to describe the motion of the other frame by choosing light as a medium for transmitting messages. The geometric meaning of time dilation of an event happening at the same location in the inertial frame for an observer in the reference frame and length contraction of a rod remaining still in the reference frame for an observer in the inertial frame can be illustrated clearly in this 3-D S-T. The proposed 3-D S-T frame is an alternate coordinate system that can be used to describe the motion of particles, and it may even provide additional understanding of space and time through further applications.

9. Reference

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