Three Visual Angles of Three Dimensional Orthogonal Axes and Their Visualization

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Abstract: Let $\angle AOB$ be an angle in the three dimensional Euclidean space $\mathbf{R}^3$. When we look at this angle from various view points, the angle $\angle AOB$ changes its appearance, which we call "visual angle". In this paper, we will discuss the relation of visual angles of three dimensional orthogonal axes and we introduce their realization technique on the plane by use of the dynamic geometry software *Cabri II Plus*.

1. Introduction

We get a lot of information from an angle. For example, whenever observer moves then $\angle AOB$ changes visual angle ($0 \leq$ visual angle $< \pi$, in radian). This visual angle is very important information, for example visual angle tells us the location of the observer (see [3] and [4]). In this paper, we think about three visual angles of three dimensional orthogonal axes. In Section 2, we will give the definition of visual angle. In Section 3, we give very important theorem of visual angles of three dimensional orthogonal axes. In Section 4, we realize these visual angles on the plane by use of the *Cabri II Plus* (Figure 4.1) and we will completely understand the law of visual angle.

2. Visual Angle

First, let us start from the definition of the visual angle.

Definition 2.1 (visual angle)
Let $\angle AOB$ be an angle in the 3-dimensional Euclidean space $\mathbf{R}^3$.
We view this $\angle AOB$ from a viewpoint $V \in \mathbf{R}^3$. Then its *visual angle* is defined as the dihedral angle of the two faces *AOV* and *BOV* of the tetrahedron *VAOB* (see Figure 2.1) (see [3]).

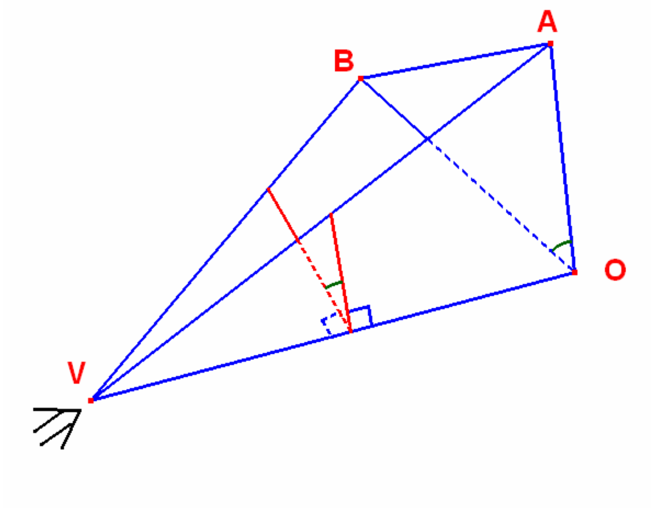

Figure 2.1 Definition of visual angle.

When we think about visual angle, the following proposition is very important.

Proposition 2.1

Let *V'*, *A'*, and *B'* be the central projected points of arbitrary points *V*, *A*, and *B* in the Euclidean space on the unit sphere centered at *O*. Then the visual angle of $\angle AOB$ is equal to the angle $\angle V'$ of the spherical triangle $\Delta V'A'B'$ (see Figure 2.2).

Proof.

Let $\vec{a}$ (resp. $\vec{b}$) be a vector tangent to the arc *V'A'* (resp. *V'B'*) at *V'* (see [2], P.49). The angle between the vector $\vec{a}$ and $\vec{b}$ is equal to the spherical angle $\angle V$ by the definition of spherical triangle. Note that two vectors $\vec{a}$ and $\vec{b}$ are perpendicular to the line *OV* where the planes *AOV* and *BOV* intersect. This is equal to the definition of visual angle. Hence the angle $\angle V'$ of the spherical triangle $\Delta V'A'B'$ is the visual angle of $\angle$ AOB. ■

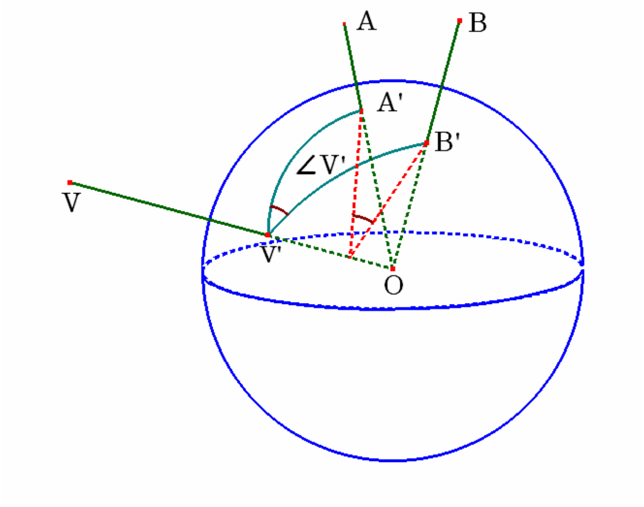

Figure 2.2 Spherical angle $\angle$ V' is equal to visual angle of $\angle$ AOB.

3. Law of Visual Angles of Three Dimensional Orthogonal Axes

Let us consider the visual angles as oriented. Let $\angle BAC$ be the oriented angle on the unit sphere $\mathbf{S}^2$, which is measured in the counterclockwise direction from *AB* to *AC* looking from the outside of $\mathbf{S}^2$. Note that the range of a visual angle is $[0, 2\pi)$. From now on, let us assume that the viewpoint $V(x, y, z)$ is on the unit sphere ($x^2 + y^2 + z^2 = 1$). From this view point *V*, let us look at three angles $\angle YOZ$, $\angle ZOX$, and $\angle XOY$ of orthogonal axes *O-XYZ* where O=(0,0,0), X=(1,0,0), Y=(0,1,0), and Z=(0,0,1). Let a, b and c be three visual angles of $\angle YOZ$, $\angle ZOX$ and $\angle XOY$ respectively. Then $a = \angle YVZ$, $b = \angle ZVX$ and $c = \angle XVY$ on the unit sphere (see Figure 3.1).

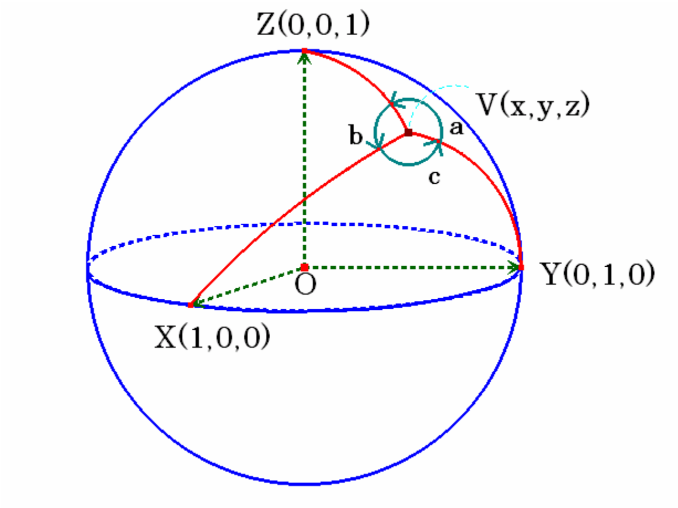

Figure 3.1 Three visual angles a, b and c on the unit sphere.

Theorem 3.1
If the viewpoint *V* is not on the axes ($x, y, z \neq \pm 1$) then three visual angles of the three dimensional orthogonal axes satisfy the following equations:

$$\tan a = -\frac{x}{yz}, \quad \tan b = -\frac{y}{zx}, \quad \tan c = -\frac{z}{xy}. \qquad (3.1)$$

In addition, if a, b nor c are not equal to 0 and π, then the view point *V* is identified as the following equation:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} sign(\sin a)\sqrt{\cot b \cot c} \\ sign(\sin b)\sqrt{\cot c \cot a} \\ sign(\sin c)\sqrt{\cot a \cot b} \end{pmatrix}, \qquad (3.2)$$

where sign(t) = ± 1 is the sign of t.

Proof.
First, it is easy to check that $x \cdot \sin a \geq 0$. We apply *the law of cosines for sides* (see [1] P.286, or [2] PP. 54-55) to the spherical triangle ΔVYZ.

$$\cos\frac{\pi}{2} = yz + \sin(\arccos y)\sin(\arccos z)\cos a\,,$$

$$\cos a = -\frac{yz}{\sqrt{1-y^2}\sqrt{1-z^2}}\,,\qquad \sin a = \frac{x}{\sqrt{1-y^2}\sqrt{1-z^2}}\,.$$

where we use the fact that $\sin a$ has the same sign as *x*. Hence

$$\tan a = -\frac{x}{yz}\,.$$

The other equations of (3.1) follow in the similar way.

Next, note that $a = 0, \pi$ if and only if $x = 0$. Therefore $a, b, c \neq 0, \pi$ is equivalent to $xyz \neq 0$. Then,

$$\cot b \cot c = x^2 (> 0)\,,\ \text{so}\ x = sign(\sin a)\sqrt{\cot b \cot c}\,.$$

In the same way, we get the rest of the equation (3.2). ■

Remark 3.1
The visual angles *a, b* and *c* satisfy the following equation:

$$\tan a + \tan b + \tan c = \tan a \tan b \tan c\,.$$

which is equivalent to $a + b + c = n\pi\ (n \in \boldsymbol{Z})$.

By the way, the following corollary shows that when three dimensional orthogonal axes are drawn on the plane, we can distinguish whether it is possible or not (see Figures 3.2 and 3.3).

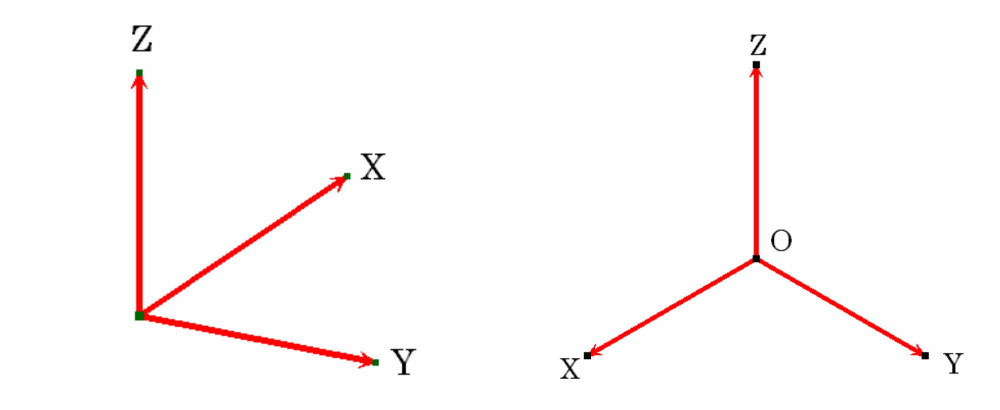

Figure 3.2 Possible cases.

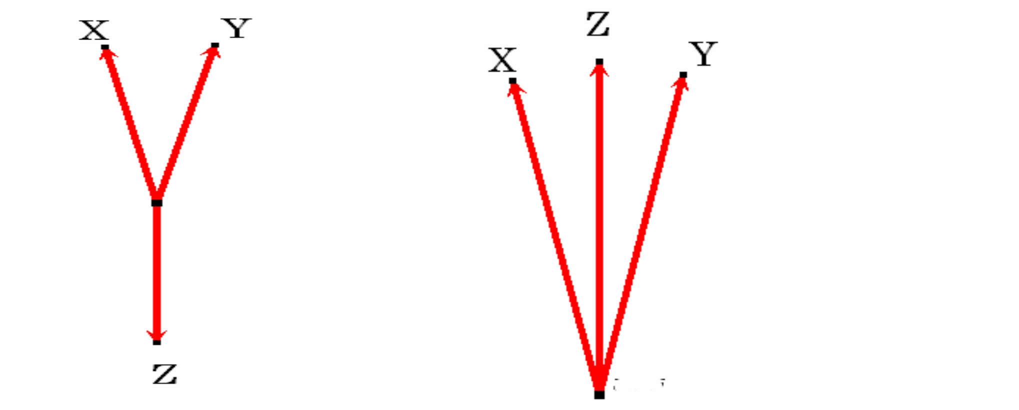

Figure 3.3 Impossible cases.

Corollary 3.1
The tangent values of three visual angles of three dimensional orthogonal axes are classified into the following four cases:

(1) All are positive.
(2) All are negative.
(3) One is equal to 0, and the others are equal to ∞, (see Figure 3.4).
(4) One is equal to ∞ and the others are indefinite, (see Figure 3.5).

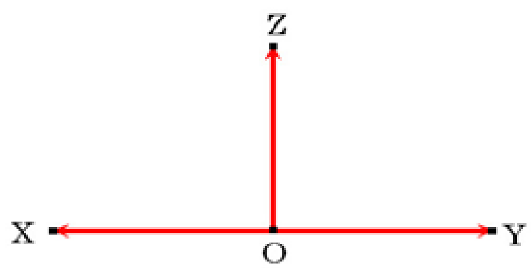

Figure 3.4

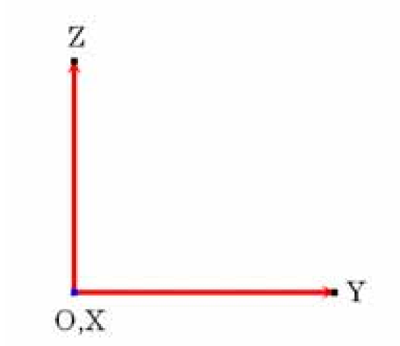

Figure 3.5

Proof.
If $xyz \neq 0$, then Case (1) and Case (2) are satisfied from the equation (3.1). Case (4) is the case that the viewpoint $V(x, y, z), (x^2 + y^2 + z^2 = 1)$ is at one of the six points X = (1, 0, 0), Y = (0, 1, 0), Z = (0, 0, 1), X^* = (-1, 0, 0), Y^* = (0, -1, 0), Z^* = (0, 0, -1). Otherwise, $xyz = 0$ and $x, y, z \neq \pm 1$, that is Case (3). ■

Hence, it is clear that Figure 3.3 is impossible for three dimensional orthogonal axes drawn on the plane.

4. Three Visual Angles of Three Dimensional Orthogonal Axes and Their Visualization

The following algorithm is a technique that we draw three visual angles of three dimensional orthogonal axes on the plane by use of the dynamic geometry software *Cabri II Plus*. We use the property of conformal map of stereographic projection. The setting of stereographic projection is as follows: let sphere $\mathbf{S}^2$ be the unit sphere centered at the origin O = (0, 0, 0). By this projection any point P on $\mathbf{S}^2\backslash N$ is mapped to the intersection of the XY-plane and the line PN where N is the North Pole N = (0, 0, 1).

Algorithm 4.1.

1. Draw the unit circle centered at the origin O = (0, 0). Let X = (1, 0), X^* = (-1, 0), Y = (0, 1) and Y^* = (0, -1) be the intersections of the unit circle and the XY-axes.
2. Let point P = (x, y) be any point in the XY-plane. (P is the stereographic projected point of V)
3. Draw the arc $A1$ passing through three points P, X and X^*.
4. Draw the arc $A2$ passing through three points P, Y and Y^*.
5. Draw the ray R from O passing through P.
6. Draw any size circle C centered at point P. (It's preferable that any size circle C is smaller than the unit circle.)
7. Let point I *(resp. J)* be the intersection of the circle C and the arc $A1$ (resp. $A2$). Let point K be the farther intersection of the circle C and the ray R from the origin O.
8. Draw the vector X' that is tangent to the arc PI at P, the vector Y' that is tangent to the arc PJ at P, and the vector Z' that is PK at P.
9. Angle $\angle Y'PZ'$, $\angle Z'PX'$, and $\angle X'PY'$ correspond to the visual angles a, b, and c of orthogonal axes defined in Section 3 (see Figure 4.1).

In the above Algorithm 4.1 the properties of circle to circle corresponding and conformal map of stereographic projection are used. More important thing is that X, X^*, Y and Y^* do not move by stereographic projection.

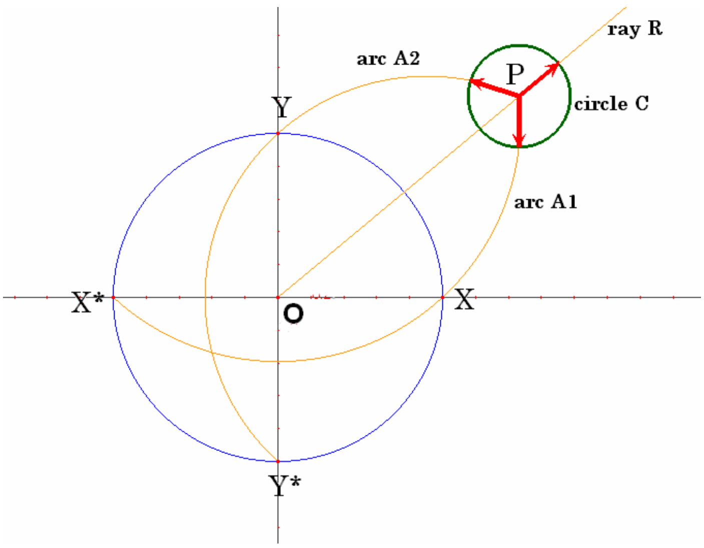

Figure 4.1 Three visual angles of three dimensional orthogonal axes and their visualization (drawn by *Cabri II plus*).

References

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[2] George A Jennings. (1994). *Modern Geometry with Applications.* Springer-Verlag New York .
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