How to Project Spherical Conics into the Plane

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Abstract: In this paper, we will introduce a method how to draw the orthogonal projected images of spherical conics (great circle, small circle, and conic on the sphere) in the Euclidean plane. The orthogonal projected images of spherical circles are conics in the plane. On the other hand, the orthogonal projected images of spherical conics are not conic but quartic in general. To construct these figures with basic drawing tools, stereographic projection plays an important role. The stereographic projection maps circles on the sphere to circles in the plane. Using this property, we can construct the orthogonal projected images of spherical circles in the plane. As for spherical conics, the famous Pascal's theorem (mystic hexagon) is essential. The Pascal's theorem is also valid for spherical geometry. Applying this theorem, we can also construct the orthogonal projected images of spherical conic in the plane. We will realize these constructions along with the dynamic geometry software *Cabri II Plus*. These constructions are very instructive to understand the importance of stereographic projection and also the great fun of conic.

1. Introduction

In this paper, we will introduce a method how to draw the orthogonal projected images of spherical conics (great circle, small circle, and conic on the sphere) in the Euclidean plane. Our ordinary image of the sphere is the orthogonal projected image. Under this projection, great circles of the sphere are projected to conics tangent to the boundary circle, which is the orthogonal projected image of the sphere. What is the algorithm how to draw this conic passing through any two points on the sphere? This problem is the start point of our studies. To solve this problem, we use an important property of stereographic projection of the sphere, that is, circle-to-circle correspondence (see [2], pp.74-77). To draw a conic as the orthogonal projected image, we use a circle as the stereographic projected image. In the similar way, we can construct the orthogonal projected image of the small circle passing through three points.

We will also consider the algorithm how to draw the orthogonal projected images of the spherical conics passing through five points on the sphere. This problem is solved by the spherical version of Pascal's theorem (see [1], pp.176-177) in the Euclidean plane. The orthogonal projected images of spherical conics are quartic curves in the plane and given by the locus of the stereographic projected images (see, Figure 1.1).

For these constructions, the dynamic geometry software *Cabri II Plus* can be used.

2. Orthogonal and Stereographic Projections

As mentioned in the previous section, the relation between orthogonal projection and stereographic projection is the key to solve our problem. Let us first investigate this relation. In the following argument, let $\mathbf{S}^2$ be the unit sphere centered at the origin $O(0,0,0)$ and $\mathbf{B}^2$ be the unit disk in the XY-plane ($\mathbf{R}^2$) as the orthogonal projected image of $\mathbf{S}^2$ from z-direction. Our input is a set of orthogonal projected points (two, three, or five points) in the unit disk $\mathbf{B}^2$, and the output in $\mathbf{B}^2$ is the orthogonal projected images of spherical conics (great circle, small circle, or conic) on the unit sphere $\mathbf{S}^2$. Then the orthogonal projection $\varphi_O : \mathbf{S}^2 \to \mathbf{B}^2$ is defined as follows:

$$\varphi_O(x, y, z) = (x, y).$$

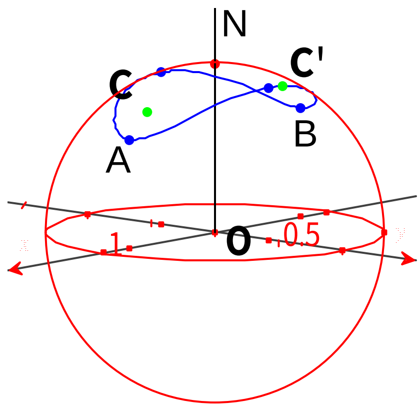

Figure 1.1 Quartic curve as the orthogonal projected image of a spherical ellipse.

Note that this map is two-to-one except for the Equator, therefore in the following argument, assume that the inputs of orthogonal projected points are regarded as the points on the northern hemisphere of $\mathbf{S}^2$.

On the other hands, stereographic projection of $\mathbf{S}^2$ from the North Pole $N(0,0,1)$ into the XY-plane $\varphi_S : \mathbf{S}^2 \backslash N \to \mathbf{R}^2$ is defined as follows:

$$\varphi_S(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right).$$

In this case, this map is one-to-one. Stereographic projection plays very important roles in the various fields of mathematics because this projection preserves angles (conformal mapping) and also has the strong property "circle-to-circle correspondence". We use the latter property in the following argument. The map φ_S projects the northern (southern) hemisphere into the outside (inside) of $\mathbf{B}^2$, respectively. The important fact is that the Equator is fixed under both projections φ_O and φ_S.

Our actual constructions are done in the XY-plane, hence the maps $\varphi_S \circ \varphi_O^{-1}$ and $\varphi_O \circ \varphi_S^{-1}$ are essential. The map $\varphi_S \circ \varphi_O^{-1} : \mathbf{B}^2 \to \mathbf{R}^2$ is written as follows:

$$\varphi_S \circ \varphi_O^{-1}(x,y) = \left(\frac{x}{1 \pm \sqrt{1-x^2-y^2}}, \frac{y}{1 \pm \sqrt{1-x^2-y^2}} \right), \quad (2.1)$$

and also the map $\varphi_O \circ \varphi_S^{-1} : \mathbf{R}^2 \to \mathbf{B}^2$ is written as follows:

$$\varphi_O \circ \varphi_S^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1} \right). \quad (2.2)$$

Geometrical constructions of these maps determined by the equations (2.1) and (2.2) are very simple. The algorithms are as the followings (see Figures 2.1 and 2.2):

Algorithm 2.1(Stereographic projected points from an orthogonal projected point)
Input: orthogonal projected point P in $\mathbf{B}^2$.

1. Draw the line OP, and two lines L_1 and L_2 perpendicular to OP passing through O and P.
2. Let N' be one of the intersections of L_1 and the unit circle. Let P_1 and P_2 be two intersections of L_2 and the unit circle. Assume that P_1 and N' are on the same side of OP.
3. Let Q_1 be the intersection of $N'P_1$ and OP. Let Q_2 be the intersection of $N'P_2$ and OP.

Output: Q_1 is the stereographic projected point of P on the northern hemisphere. Q_2 is that of P on the southern hemisphere.

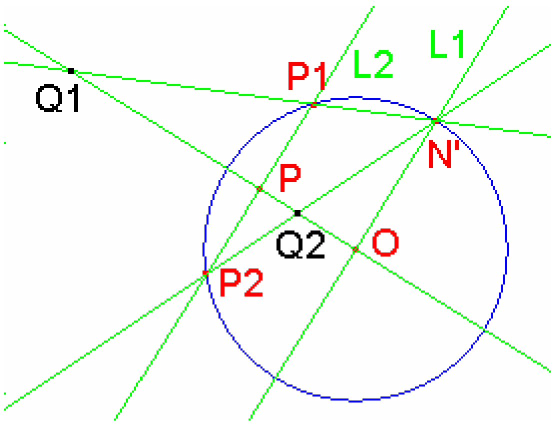

Figure 2.1 Construction of stereographic projected points from an orthogonal projected point.

Algorithm 2.2(Orthogonal projected point from a stereographic projected point)
Input: stereographic projected point Q in $\mathbf{R}^2$.
1. Draw the line OQ, and the line L_1 perpendicular to OQ passing through O.
2. Let N' be one of the intersections of L_1 and the unit circle. Let Q_1 ($\neq N'$) be another intersection of $N'Q$ and the unit circle.
3. Draw the line L_2 perpendicular to OQ passing through Q_1. Let P be the intersection of L_2 and OQ.

Output: P is the orthogonal projected point of Q.

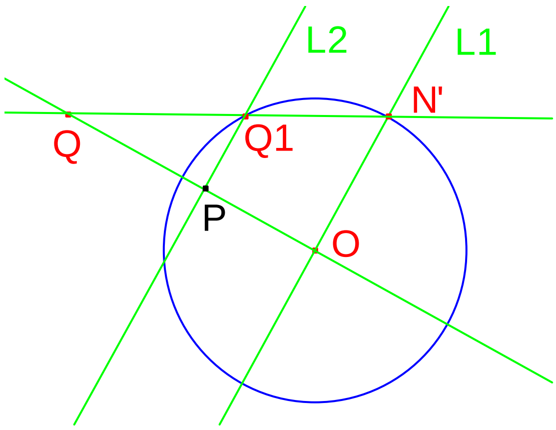

Figure 2.2 Construction of the orthogonal projected point from a stereographic projected point.

3. Great and Small Circles on $\mathbf{S}^2$

Since a spherical circle on $\mathbf{S}^2$ is the intersection of a certain plane and $\mathbf{S}^2$, the orthogonal projected image of the spherical circle is a conic in $\mathbf{B}^2$. On the other hand, its stereographic projected image is a circle in $\mathbf{R}^2$. Combining these facts, we can get the following algorithms (see Figures 3.1 and 3.2):

Algorithm 3.1(Great circle passing through two points)

Input: two orthogonal projected points P_1 and P_2 in $\mathbf{B}^2$.
1. Create two antipodal points P_1* and P_2* of P_1 and P_2 with respect to O.
2. Create three stereographic projected points Q_1, Q_2, and Q_1* of P_1, P_2, and P_1*.
3. Let E_1 be one of the intersections of the unit circle and the circle passing through Q_1, Q_2, and Q_1*.
4. Draw the conic C passing through five points P_1, P_2, P_1*, P_2*, and E_1.

Output: C is the orthogonal projected image of the great circle on $\mathbf{S}^2$.

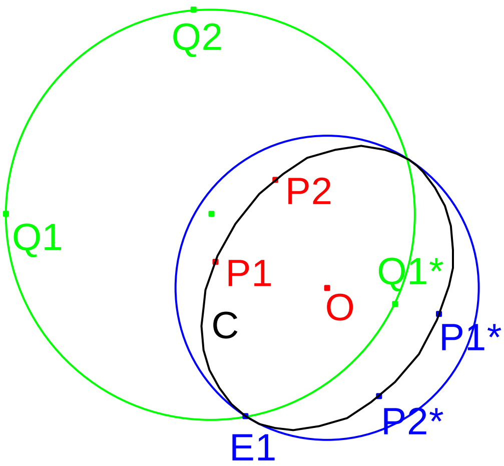

Figure 3.1 Construction of the orthogonal projected image of a great circle.

Algorithm 3.2(Small circle passing through three points)

Input: three orthogonal projected points P_1, P_2, and P_3 in $\mathbf{B}^2$.
1. Create three stereographic points Q_1, Q_2, and Q_3 of P_1, P_2, and P_3.
2. Take two points Q_4 and Q_5 on the circle passing through Q_1, Q_2, and Q_3.
3. Create two orthogonal projected points P_4 and P_5 of Q_4 and Q_5.
4. Draw the conic C passing through five points P_1, P_2, P_3, P_4, and P_5.

Output: C is the orthogonal projected image of the small circle on $\mathbf{S}^2$.

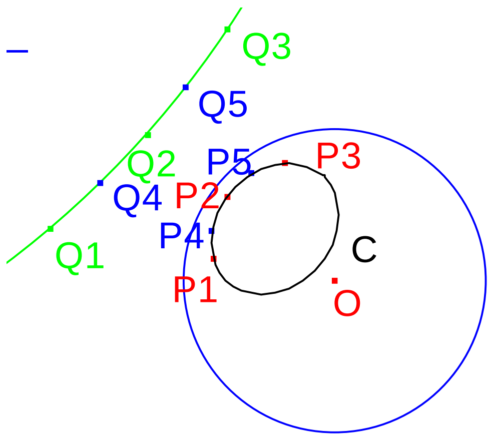

Figure 3.2 Construction of the orthogonal projected image of a small circle.

Note that in Figure 3.1 the stereographic projected image of a great circle intersects with the unit circle at two symmetric points with respect to O.

4. Pascal's Theorem on S^2

First, let us review Pascal's theorem (mystic hexagon) (see [1], pp.176-177 and [2], pp.147-148) and its application how to draw conics in the plane $\mathbf{R}^2$.

Theorem 4.1(Blaise Pascal) If a hexagon is inscribed in a smooth conic, the intersections of opposite sides of the hexagon are collinear (see Figure 4.1).

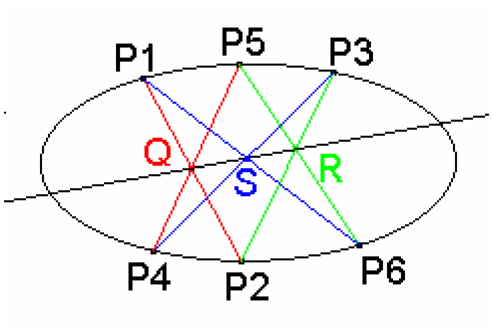

Figure 4.1 Pascal's mystic hexagon.

This theorem tells us the reason why a smooth conic is determined by five points. The algorithm to draw conics is as follows:

Algorithm 4.1(Conic in the plane)
Input: five points P_1, P_2, P_3, P_4, and P_5 in $\mathbf{R}^2$.
1. Draw any line L_{56} passing through P_5.
2. Draw four lines $L_{12}=P_1 \cup P_2$, $L_{23}=P_2 \cup P_3$, $L_{34}=P_3 \cup P_4$, and $L_{45}=P_4 \cup P_5$.
3. Draw the line L_0 passing through $L_{12} \cap L_{45}$ and $L_{23} \cap L_{56}$.
4. Draw the line L_{61} passing through $L_{34} \cap L_0$ and P_1.
5. Create the intersection P_6 of L_{56} and L_{61}.
6. Draw the trace C of P_6 rotating the line L_{56} around the point P_5.

Output: C is the conic passing through five points.

The line L_0 is called the *Pascal line*. Our main intention is to extend Algorithm 4.1 in $\mathbf{R}^2$ to that on $\mathbf{S}^2$. For this purpose, we have to go back to the definition of spherical conic.

Definition 4.1(Spherical conic) A spherical conic is the intersection of a sphere and an elliptic cone centered at the center of the sphere.

This definition is quite different from that of ordinary conic in the plane with two foci and a string with a certain length. However, the next proposition assures us that Definition 4.1 is appropriate.

Proposition 4.1 Let $C(\sin c, 0, \cos c)$ and $C'(-\sin c, 0, \cos c)$ be two points on $\mathbf{S}^2$ where $0 < c < \pi/2$. The spherical ellipse is determined as the locus of the point P on $\mathbf{S}^2$ satisfying *arc* *length*(*CP*)+*arc length*(*C'P*)=2*a* where $c < a < \pi/2$. Then this spherical ellipse is on the intersection of $\mathbf{S}^2$ and a cone centered at the origin determined by the following equation:

$$\frac{x^2}{\tan^2 a} + \frac{y^2}{\tan^2 b} = z^2 \qquad (4.1)$$

where $\cos b = \cos a / \cos c$.

Remark 4.1 Points $C(\sin c, 0, \cos c)$ and $C'(-\sin c, 0, \cos c)$ are foci of this spherical ellipse. Note that $0 < b < a$. In fact, Point $A(\sin a, 0, \cos a)$ is one of the end points of major axis of this spherical ellipse. And Point $B(0, \sin b, \cos b)$ is one of the end points of minor axis as in Figure 1.1 (see [3]).

Proof. Let $P(x, y, z)$ be a point on the spherical ellipse. Then,

$$\cos\angle COP = x\sin c + z\cos c,\ \sin\angle COP = \sqrt{1-(x\sin c + z\cos c)^2},$$

$$\cos\angle C'OP = -x\sin c + z\cos c,\ \sin\angle C'OP = \sqrt{1-(-x\sin c + z\cos c)^2}.$$

$\cos(\angle COP + \angle C'OP) = \cos 2a$ implies that

$$z^2\cos^2 c - x^2\sin^2 c - \cos 2a = \sqrt{(1-(x\sin c + z\cos c)^2)(1-(-x\sin c + z\cos c)^2)}\ .$$

Squaring the both sides of the above equation,

$$-2\cos 2a(z^2\cos^2 c - x^2\sin^2 c) + \cos^2 2a = 1 - 2(x^2\sin^2 c + z^2\cos^2 c)\ ,$$

equivalently,

$$\frac{x^2}{\sin^2 a/\sin^2 c} + \frac{z^2}{\cos^2 a/\cos^2 c} = 1\ .$$

Now, using $\cos b = \cos a/\cos c$ and $x^2 + y^2 + z^2 = 1$,

$$\frac{x^2}{\sin^2 a/(\sin^2 a - \cos^2 a\tan^2 b)} + \frac{z^2}{\cos^2 b} = x^2 + y^2 + z^2\ .$$

Consequently, we get the equation (4.1). ■

As a corollary, we obtain the following.

Corollary 4.1 For any spherical conic *C* on $\mathbf{S}^2$, there exist two points F_1, F_2 on $\mathbf{S}^2$ and a positive number $l < \pi$ such that

$$C = \{P \mid PF_1 + PF_2 = l\} \cup \{P \mid PF_1^* + PF_2^* = l\}$$
$$= \{P \mid PF_2^* - PF_1 = \pi - l\} \cup \{P \mid PF_1 - PF_2^* = \pi - l\}$$

where F_1* and F_2* are the antipodal points of F_1 and F_2, respectively.

In the next, let us show that the Pascal's theorem is valid for spherical geometry.

Theorem 4.2 If a spherical hexagon is inscribed in a spherical conic, the intersections of opposite sides of the hexagon are collinear.

Proof. Let us consider a central projection of $\mathbf{S}^2$ on a certain plane. Note that the central projected image of a great circle is a line in the plane. Since a spherical conic is the intersection of $\mathbf{S}^2$ and an elliptic cone, the central projected image of a spherical conic is a conic section in the plane. Then it is clear that the Pascal's theorem in the plane implies the Pascal's theorem on the sphere. ■

Now, we are ready to construct the spherical conic in the same way as Algorithm 4.1.

Algorithm 4.2(Conic on the sphere)
Input: five points P_1, P_2, P_3, P_4 and P_5 in $\mathbf{B}^2$.

1. Create five stereographic points Q_1, Q_2, Q_3, Q_4 and Q_5 of P_1, P_2, P_3, P_4 and P_5. And create five stereographic points Q_1^*, Q_2^*, Q_3^*, Q_4^* and Q_5^* of the antipodal points P_1^*, P_2^*, P_3^*, P_4^* and P_5^*.
2. Draw the stereographic image circle $C_{56}(\supset \{Q_5, Q_5^*\})$ of an arbitrary great circle passing through P_5.
3. Draw four stereographic image circles C_{12}, C_{23}, C_{34} and C_{45} of the great circles passing through $\{P_1, P_2\}$, $\{P_2, P_3\}$, $\{P_3, P_4\}$ and $\{P_4, P_5\}$. ($C_{ij} \supset \{Q_i, Q_i^*, Q_j, Q_j^*\}$)
4. Draw the circle C_0 passing through $C_{12} \cap C_{45}$ and $C_{23} \cap C_{56}$.
5. Draw the circle C_{61} passing through $C_{34} \cap C_0$ and Q_1.
6. Create two intersections Q_6 and Q_6^* of C_{56} and C_{61}.
7. Create the orthogonal projected points P_6 and P_6^* of Q_6 and Q_6^*.
8. Draw the trace SC of P_6 and P_6^* rotating the circle C_{56} around the point Q_5.

Output: SC is the spherical conic passing through five points.

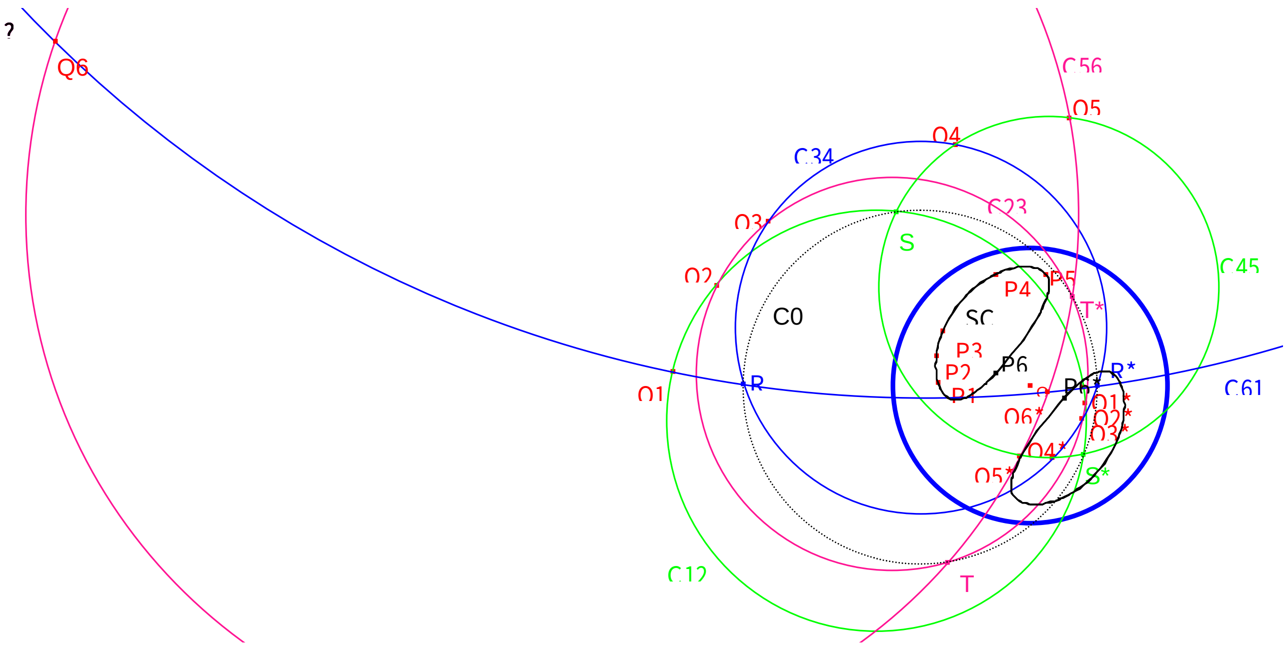

Figure 4.2 Construction of the orthogonal projected image of a spherical conic.

References

[1] Berger, M. (1977). *Geometry II.* Springer-Verlag Berlin Heidelberg.
[2] Jennings, G. (1994). *Modern Geometry with Applications.* Springer-Verlag New York.
[3] Maeda, Y. (2005). *Spherical Conic and the Fourth Parameter*. KMITL Sci. J. Vol.5 No.1 Feb. 2005 (pp.165-171).