A finite difference formulation for a traffic flow model

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Abstract

A simple continuum model for traffic flow, based on the law of mass conservation, is discussed. A finite difference formulation for the model is obtained and real traffic data are used for further numerical investigations. The numerical scheme is implemented and solved using the spreadsheet features and Visual Basic capabilities (VBA) of *Microsoft Excel*. Results from the numerical tests show that predictions about traffic behaviour and conditions agree well with collected data provided initial and boundary conditions are carefully treated. The modelling exercise discussed in this paper provides an excellent example of applying relevant mathematics to a relatively complex real life problem. In addition, by using a simple computer tool (namely, a spreadsheet) in the numerical solution of the model, this paper has demonstrated the value of technology in making mathematical investigations more accessible.

Introduction

Many mathematical models have been proposed to describe the behaviour of traffic flow, which invariably exhibits very complex phenomena. From as early as the 1930s, models involving the application of probability theory have been used in such scientific studies. In the decades that followed, various other approaches such as car-following models, traffic wave theory and queueing theory have been developed to study the problem. The literature on models of traffic flow is extensive and a comprehensive overview of these models can be found in [1] and [2].

In this paper, we present a simple continuum model for traffic flow and apply it to a set of real data obtained from a stretch of road in the eastern part of Singapore. In this model, traffic stream is regarded as a one-dimensional compressible fluid of certain density and velocity and its behaviour is described by a continuity equation and an equation of state (flow-density relationship). Traffic flow is thus characterized in terms of flow, speed and concentration.

For simplicity, we make two basic assumptions in this discussion. We first assume that traffic flow is conserved along the road under consideration. That is, vehicles coming into the traffic stream at one end (the inlet) of the road must leave at one other end (the outlet). In other words, there are no other side roads from which vehicles can enter or by which they can leave the road in question. This assumption, of course, may be modified later if we wish to include other entrances or exits, or even intersections. Secondly, we assume there is a relationship between traffic speed and density, or between flow and density. This relationship forms part of the overall model. The result is a model based on a conservation equation describing flow and density as a function of distance and time. Thus, unlike input-output models such as those discussed in McCartney and Carey [3], [4], a continuum model provides information on the traffic conditions (in terms of flow and density) along the entire stretch of road at any time during the period under consideration.

Traffic flow model

Continuum traffic flow models were first proposed and developed independently by Lighthill and Whitham [5] and Richards [6]. As such, the simple continuum model is often referred to as the LWR model.

Consider a stretch of unidirectional continuous road section with an inlet at A, and an outlet at B as shown in Figure 1.

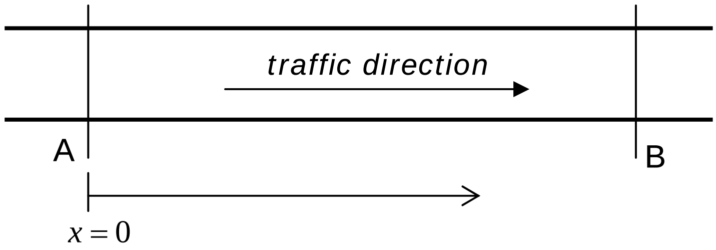

Figure 1. Section of a unidirectional continuous road

The LWR model for such a road section is given by

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = g(x,t) \qquad (1)$$

where $k(x,t)$ is the traffic density, $q(x,t)$ is the flow rate of the traffic stream and $g(x,t)$ is the rate at which vehicles are generated (and fed into the traffic stream) or dissipated (and taken off the traffic stream) per unit time per unit length. The independent variables x and t represent space and time respectively. In addition, the model is governed by the fundamental relationship

$$q = uk \qquad (2)$$

where u is the mean speed of vehicles travelling along the section of road under consideration. A more detailed and complete derivation of the model is given in [5].

While it is possible to consider a general case, for simplicity and to keep the problem tractable, we shall consider the case where $g(x,t) = 0$; that is, where there is no generation or dissipation of vehicles. In other words, we use our first assumption mentioned earlier that there are no other entrances (or "sources") or exits (or "sinks") except A and B.

We further assume that there is a speed-density relationship. This will link the mean speed u and the traffic density k. Greenshields [7] proposed a simple linear relationship,

$$u = F\left(1 - \frac{k}{K}\right) \qquad (3)$$

where F is the free-flow speed and K is the jam density. These two parameters in traffic flow studies represent the two extreme cases of road conditions. Free-flow speed is the maximum mean speed that vehicles can attain in the absence of any road congestion, and jam density is the maximum concentration of vehicles under congested traffic flow situations. Equations (1) – (3) form a set of equations for a simple continuum model for traffic flow. To complete the model, we need to specify the boundary and initial conditions. This will be discussed in subsequent sections.

Formulating the finite difference scheme

The model may be solved by using a finite difference scheme. We first discretize the x-t solution space using a rectangular grid. For convenience, we choose a uniform grid spacing of Δx and Δt in the x and t directions respectively. Thus, we have $x_j = j\Delta x$ for $j = 0, 1, \ldots, J$ and $t_n = n\Delta t$ for $n = 0, 1, 2, \ldots$ if we divide the space domain into J subintervals and let time t increase to some specified time level. For convenience, we further denote the values of dependent variables at the (j,n) grid point using the subscript-superscript notation. For instance, $k(x_j, t_n)$ is denoted by k_j^n.

Using the finite difference approximations (forward-time and backward-space)

$$\left.\frac{\partial k}{\partial t}\right|_j^n \approx \frac{k_j^{n+1} - k_j^n}{\Delta t} \text{ and } \left.\frac{\partial q}{\partial x}\right|_j^n \approx \frac{q_j^n - q_{j-1}^n}{\Delta x},$$

and setting $g(x,t) = 0$, we obtain the discretized model equations

$$\frac{q_j^n - q_{j-1}^n}{\Delta x} + \frac{k_j^{n+1} - k_j^n}{\Delta t} = 0 \tag{4}$$

$$q_j^n = k_j^n u_j^n \tag{5}$$

$$u_j^n = F\left(1 - \frac{k_j^n}{K}\right) \tag{6}$$

Combining Equations (4) – (6) and further simplifying, we obtain an explicit finite difference form for the simple continuum traffic flow model given by

$$k_j^{n+1} = k_j^n + \frac{\Delta t}{\Delta x} F\left[\frac{1}{K}\left(\left(k_j^n\right)^2 - \left(k_{j-1}^n\right)^2\right) + k_{j-1}^n - k_j^n\right], \text{ for } j = 1, 2, \ldots, J \text{ and } n = 0, 1, 2, \ldots. \tag{7}$$

The initial conditions k_j^0 for $j = 0, 1, \ldots, J$, and boundary conditions k_0^n for $n = 0, 1, 2, \ldots$ must be either provided or estimated. The numerical scheme involves marching across the space domain at each time level for the time period of interest.

Stability of the numerical scheme is ensured if Δx and Δt are chosen appropriately. In this case, by choosing Δx and Δt such that $\Delta x / \Delta t > F$, we ensure that the scheme is stabile and that convergence will be reached [8].

Although Equation (7) provides the numerical scheme for the model in terms of traffic density, in practice, we often describe traffic conditions in terms of flow rate since this can be measured by counting the number of vehicles passing a point over a short period of time. Indeed, measurements made in traffic flow studies are usually of flow rate and not traffic density.

From Equations (5) and (6), we obtain the relationship between traffic density and flow rate

$$k_j^n = \frac{K}{2} - \sqrt{\frac{K^2}{4} - \frac{K}{F} q_j^n} \tag{8}$$

which will be used in the numerical scheme to handle initial and boundary conditions.

Real traffic data

Initial and boundary conditions in the numerical scheme may be obtained by actually collecting real traffic data. In the present discussion, real data have been collected by capturing traffic conditions at two points along a stretch of road in the eastern part of Singapore. This is a one-kilometre section of an expressway which has no entrances or exits, making it well suited for the model discussed here. Cameras were set up at the inlet and outlet and the traffic conditions were recorded over a period of six hours on a normal weekday. The video recordings were then played and vehicles passing the points were counted and categorized.

Vehicles are of different sizes but the continuum model assumes that vehicles are of a certain fixed and standard size. Thus, all vehicles in this study need to be categorized and then converted to its "passenger car equivalent" or PCE. In Singapore, vehicles may be generally categorized as Cars, Motocycles, Light trucks, Heavy trucks and Buses, and these have PCE values of 1.0, 0.4, 1.3, 2.6 and 2.7 respectively [9]. Using these PCE conversion values, the flow rates at the two end points of the stretch of road are calculated and tabulated in Table 1

Table 1: Flow rates (No. of cars per hour) at the inlet ($q(0,t)$) and outlet ($q(1,t)$)

Time (h)	$q(0,t)$	$q(1,t)$	Time (h)	$q(0,t)$	$q(1,t)$
0.00	1612	1500			
0.25	1777	1678	3.25	871	799
0.50	1765	1498	3.50	808	759
0.75	1590	1498	3.75	990	794
1.00	1205	1145	4.00	834	1088
1.25	1084	989	4.25	782	1109
1.50	1004	776	4.50	718	679
1.75	1199	968	4.75	634	487
2.00	1278	834	5.00	756	559
2.25	1119	864	5.25	768	840
2.50	980	878	5.50	654	901
2.75	887	945	5.75	707	598
3.00	924	732	6.00	659	793

In this model, x is the spatial direction along the stretch of road. At the inlet $x = 0$, and at the outlet $x = 1$. Thus, we have $x_0 = 0$ and $x_J = 1$. Values of $q(0,t)$ in Table 1 may be with Equation (8) to provide information for the boundary conditions of the numerical scheme. Values of $q(1,t)$ provide the flow rates at the outlet and can be used to compare with predicted flow rates obtained from the model.

For a complete set of initial conditions, we need values of k_j^0 (or q_j^0) at $j = 0, 1, 2, \ldots, J$. That is, we need to capture traffic conditions at *many* positions along the road and not just at the end points. This is too expensive and impractical. Instead, we estimate these values using the simple linear interpolation of the available values k_0^0 and k_J^0 . That is,

$$k_j^0 = k_0^0 + x_j \left(k_J^0 - k_0^0 \right) \text{, for } j = 1, 2, \ldots, J - 1 . \qquad (9)$$

We now have a complete set of initial conditions for the numerical scheme.

Boundary conditions at $x = 0$ are needed at every time level for the finite difference scheme given by Equation (7). Real traffic data, however, are only available at specific values of t (see Table 1). While it may be tempting to choose $\Delta t = 0.25$ so that boundary values may be obtained directly from Table 1, this would result in very large errors. Moreover, the stability condition ($\Delta x / \Delta t > F$) would also require a large value for Δx, making the numerical scheme less accurate. In order to use a smaller Δt value, we will need to estimate values of the flow rate in between the collected data values. There are several ways of estimating these values and in this discussion we present two possible methods, resulting in two different models.

Model A

In Model A, we find a function whose graph approximates the data points given by $q(0,t)$ for $t = 0, 0.25, 0.75, \ldots, 6.0$. By observing the data points, we *guess* that such a function may take the form

$$q(0,t) = \left(q(0,0) + a\sin(m\pi t)\right)\exp(-kt),$$

where a, m and k are positive constants to be found. These constants may be obtained by the least squares method of minimizing the sum of errors between each data point and the corresponding approximated value. One way to do this quickly is to use the Solver tool in *MS Excel*. The Solver tool in *Excel* allows the user to specify a cell (or group of cells) whose value will be varied until the value in a target cell is as close to a pre-determined value as possible.

Using *Excel*'s Solver tool, it is found that for the set of data values in Table 1, a choice of $a = 244.55$, $m = 1.1832$ and $k = 0.1670$ will minimize the sum of residual errors in the estimation. That is, the set of data points may be approximated by the function

$$q = \left(1612 + 244.55\sin(1.1832\pi t)\right)\exp(-0.1670t). \qquad (10)$$

Model B

In Model B, a cubic spline interpolation of the data points is used to obtain the required boundary values. A set of piecewise continuous and smooth cubic functions resulting from the cubic spline interpolation of the data points may be obtained from a computer algebra system (such as Maple®). A typical set of Maple commands would look as follows:

```
> t1:=[0.00, 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00,
       2.25, 2.50, 2.75, 3.00, 3.25, 3.50, 3.75, 4.00,
       4.25, 4.50, 4.75, 5.00, 5.25, 5.50, 5.75, 6.00]:

> q0:=[1612, 1777, 1765, 1590, 1205, 1084, 1004, 1199, 1278,
       1119,  980,  887,  924,  871,  808,  990,  834,  782,
        718,  634,  756,  768,  654,  707,  659]:

> cs:=spline(t1,q0,t,cubic);

> cs:=unapply(cs,t):
```

The graph of the approximate function in Equation (10) is shown in Figure 1, while Figure 2 shows the cubic spline functions obtained from the Maple code above. As can be observed, the cubic spline interpolation obviously yields a better approximation for the set of data points.

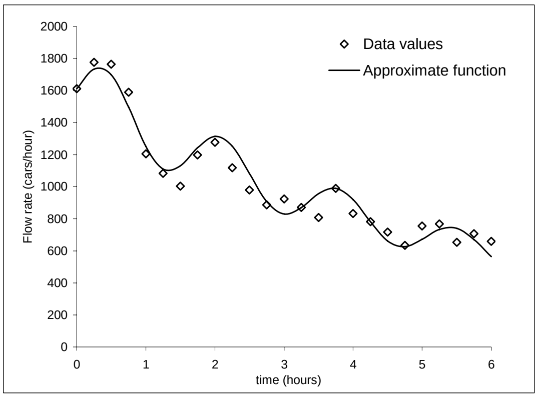

Figure 1: Graph of q and data values $q(0,t)$

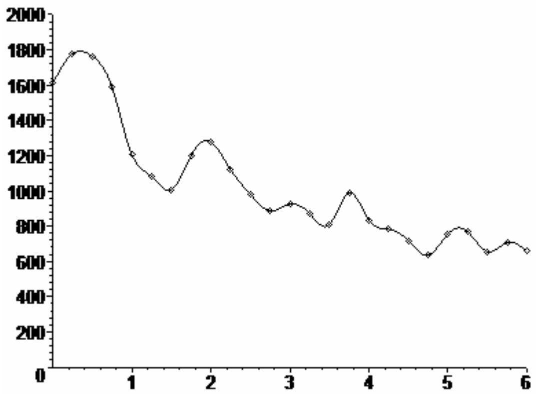

Figure 2: Cubic spline interpolation of data values $q(0,t)$

Numerical Results and Discussion

Before implementing the numerical scheme, we need to either determine various parameter values. In the present discussion, the total distance of the stretch of road is 1 km and the final time T is 6 h, as dictated by the available data. For convenience, we let $J = 25$ and thus $\Delta x = 0.04$. Choosing $\Delta t = 0.0004$ ensures stability in the numerical scheme and a relatively accurate set of results. The values of the free flow speed F and jam density K are assumed to be 77.8 km/h and 107.2 cars/km, as determined by Mak in [10].

The finite difference scheme given by (7) may be coded using any suitable programming language or environment. In this study, *Excel*'s Visual Basic Application (VBA) was used to implement the scheme based on the algorithm below:

Step		Remarks
1	Let `J=25`, `dx=0.04`, `T=6`, `dt=0.0004`, `F=77.8`, and `K=107.2`	Setting parameter values
2	Let `N=T/dt`	Find total number of time levels
3	Declare arrays `k0[0..J]` and `k1[0..J]`	To store values of k_j^n
4	Let `k0[j]=` k_j^0 for `j=0,1,...,J`	Set initial conditions
5	For `n=1,2,3,...,N` do Steps 6 to 8	
6	Set `k1[0]=` k_0^n using Equation (8), with `t=n×dt`	Set boundary conditions
7	Set `k1[j]=` k_j^{n+1} using Equation (7) for `j=0,1,...,J-1`	Use the explicit finite difference scheme
8	Set `k0[j]=k1[j]` for `j=0,1,...,J`	Update values and march on
9	Output (or store) required values in `k1[j]`	

In the above algorithm, when using Equation (8) (in Step 6), values of q_0^n are needed. These may be obtained from either Equation (10) or the cubic spline functions, depending on whether *Model A* or *Model B* is being used. It should be noted that the values obtained in this algorithm are the predicted values of traffic density across the stretch of road and over the period considered. If flow rates are needed, they may be easily computed from the corresponding traffic density values using Equations (5) and (6).

Results from Models A and B are obtained and presented graphically in terms of traffic density at the outlet ($x = 1$) against time in Figures 2 and 3. From Figure 2, we observe that the predicted traffic densities seems to follow some form of oscillatory pattern. This is only to be expected since the function used to approximate the unknown boundary conditions is of a trigonometric form. The agreement with the observed traffic condition is satisfactory in terms of general trend. The model correctly predicts a decreasing traffic density over time. However, in terms of actual density values, Model A seems to deviate quite markedly from the observed data. This could be due to the fact that the rather artificial function used to approximate the boundary condition may not have the best choice for this set of data.

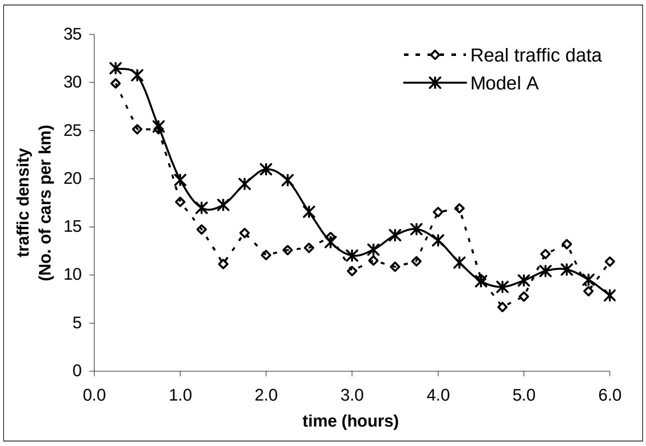

Figure 2: Predicted traffic densities for Model A (solid line) and real, observed traffic data (dotted line)

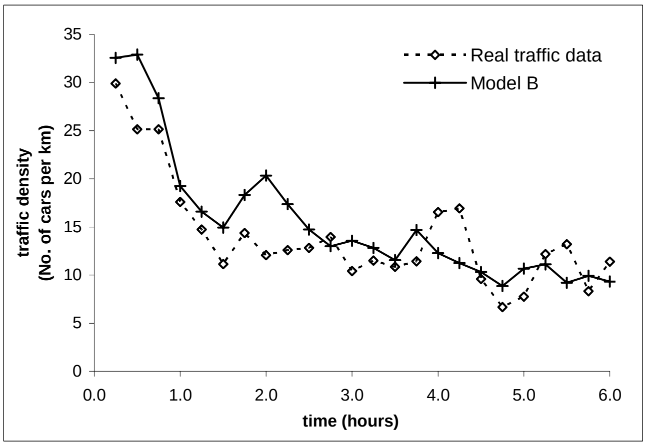

Figure 3: Predicted traffic densities for Model B (solid line) and real, observed traffic data (dotted line)

From Figure 3, we see that predicted traffic densities from Model B agrees better with observed data compared to Model A. Like in Model A, the decreasing trend is also observed. Unlike Model A, we do not observe any oscillatory behaviour in the predicted traffic density. Overall, the predicted traffic densities are in better agreement with observed data.

In both models, at around $t = 2$, the predicted values are significantly higher than the corresponding observed values. This could be explained by the fact that from $t = 0.75$ to $t = 1.5$, there is a sharp drop in traffic flow. This is well predicted by both models. Following this drop, traffic picks up and the models may have "over-compensated" the trend, and hence predicted higher values.

There are many ways to extend this study and improve the model. One could, for instance, take into account a non-zero generating function (that is, $g(x,t) \neq 0$). This would mean that the model can then be applied to a road with more than one entrance (or exit) or with intersections. Traffic conditions at these critical points may be monitored to provide a more complete set of data.

This numerical study has also demonstrated the importance of boundary conditions in such models. In such investigations, boundary conditions play an important role in determining the success of the scheme and may influence the accuracy of the results. One other possible extension of this study would be to examine how the boundary conditions at $x = 0$ may be treated or handled differently. In addition, initial conditions in the present study have been assumed to vary linearly across the stretch of road. This means that traffic density varies in a linear fashion along the 1 km stretch of expressway. This simplistic assumption can also be modified to give a more realistic model.

Other numerical techniques could be explored. Different methods of data collection can also be employed to obtain a more accurate set of observed data. Some parameter values (such as free flow speed and jam density) were obtained from the literature. These could also be further examined and better estimated.

Conclusion

In this paper, we have presented a simple continuum model for traffic flow and have demonstrated how the model could be formulated as an explicit finite difference scheme. We have also shown that boundary conditions may be treated or handled differently, and this may result in different models. Real traffic data were collected and results from numerical studies compare reasonably well with collected real data.

The model discussed is applied with real data collected in Singapore, where traffic congestion has been a major public concern for years. Many policies and schemes have been put in place to ease traffic flow so much so that the tiny island state has the dubious honour of becoming a pioneer in implementing congestion pricing systems [11]. However, despite these efforts, commuters continue to face traffic congestion on major roads everyday. Mathematical models of traffic flow may serve to shed some light on the efficacy and effectiveness of these policies, and at the same time help the authorities better understand the nature of traffic flow.

Modelling traffic flow is a complex problem; but it is this complexity that makes the problem both challenging and attractive. This paper has demonstrated that despite the complexity associated with this problem, it is possible to apply a fairly simple numerical technique, together with appropriate use of computer technology, and make a decent attempt at constructing and solving a simple continuum model for traffic flow.

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