Thought and Variational Language in the Introduction to Analysis

Ricardo Cantoral & Rosa – María Farfán

Center of Research and Advanced Studies of the National Polytechnic Institute

Cinvestav – IPN, MEXICO

rcantor@cinvestav.mx & rfarfan@cinvestav.mx

Abstract

This paper presents a novel didactic approach based on research in Mathematics Education. With this, we attempt to construct a base of meanings for mathematical analysis processes and concepts, especially that taught at a university level. We begin with activities for the construction, among students, of a universe of graphic forms, which is in turn expanded and structured; and then we continue with the development of the notion of prediction of the phenomena of flows supported by the Newton binomial. The combination of both tasks, which we sustain in this hypothesis, fosters the development of *thought and variational language*. This approach has been put into operation with humanities and physical sciences and engineering students with promising results. Naturally, the subject order of the contents of the preparatory course and the analysis course have been modified noticeably, because we now put the notions of the curve and analyticalness in the center of the design of didactic situations. This approach has made it possible to use the Taylor Series as the principal support, the mathematical objective to predict the future state of that which flows in a variational situation extracted from the knowledge of reference in order to discipline the student.

Background

In recent decades we have witnessed the appearance in the bosom of mathematics educators, mathematics teachers or mathematics education (according to the tradition of the school[1] they belong to), academic university sectors that deal with the study of the so-called advanced thought processes in math subjects in higher education. The subjects tackled are after basic algebra and they usually have subject matter that goes for analysis onwards (also called calculus in the Anglo-Saxon tradition). This amazing growth has been possible, in our opinion, thanks to two principal factors; the first is due to the growing interest of professional mathematicians in the affairs of teaching and learning and the second is the result of the stability and maturity that has been reached in research communities that are organized around academic groups with their own paradigms, as in the case of the international Psychology of Mathematics Education group or the Clame (Latin American Committee of Mathematics Education. We mention these groups because they are better known.

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[1] The name Mathematics Education gives our discipline a geographic and conceptual location; the term Mathematics Education has been used in the Anglo-Saxon world, while on continental Europe it has been called Didáctica de las Matemáticas, Didactique des Mathématiques, Didaktik der Mathematik, to mention a few of the more dynamic schools. It is now accepted as a functional premise that our discipline studies the process of the constitution, transmission and acquisition of various different mathematical contents in a school situation. It is not reduced to a search for a "good way" to teach a certain previously fixed notion, but one that allows us to assume the organization of an activity as an object of study, for example, whose declared intention is the learning of a degree of knowledge even if the goal is not reached. The purpose of research in our field is to positively affect the teaching system; improve teaching methods and contents and to propose the conditions for the stable functioning of the teaching situation so that math is not only treated as a school subject, but we want to understand how and why it is learned and why knowledge is structured for teaching purposes.

This double development process is nurtured by mathematical reflection in the bosom of teaching on one hand and on the other, by supporting the didactic explanation based on the construction of knowledge – social and individual – has, in our opinion been one of the principal and most recent contributions to our disciple: Mathematics Education. On this occasion, we intend to show some of the results of the investigation underway that we have called thought and variational language, which provides an opportunity to build a bridge between research and the classroom reality.

As has been reported in various reviews written recently (Artigue, 1998 or Tall, 1991), the studies dealing with the didactics of analysis have been based on various teaching metaphors that conserve, to a certain extent, some points in common with the central thesis provided by genetic epistemology regarding the development of thought, a thesis that even though we will not analyze it in this paper, we would like to mention that it is in the base of explanations in literature. We only mention that these studies only focus on the dealing with mathematics regarding higher education, assuming that mathematics intervenes in this level almost exclusively as the principal teaching discipline and forgetting a fundamental fact that characterizes the teaching system of higher education, also and perhaps with more force, school mathematics is at the service of other scientific domains and other reference practices, where, in turn, it takes on sense and meaning.

The line of investigation that we pursue considered, on the contrary, that it is a basic need to endow research with a scientific approach that makes is possible to incorporate the four fundamental components in the construction of knowledge; its epistemological nature, its socio-cultural nature, the cognitive planes and the forms of transmission through teaching. This multiple approximation, which in jargon is known as "the fourth dimension" we have formally named the *socio-epistemological approach*[2]. In this respect, thought and variational language shall be understood as a line of research that, located in the center of the socio-epistemological approach, makes it possible to deal with the articulation between research and social practices that give life to mathematics, variation and the changes in the didactic systems.

Development of the Proposal

As can be seen, the development of thought and variational language among students calls for temporarily prolonged processes in comparison with the usual didactic times. It supposes, for example, that the mastery of basic mathematics and the associated thought process, but simultaneously demands a break with the pre-variational thought styles, as in the case of widely algebraic thought widely documented by (Artigue, 1998). Additionally, this break cannot be maintained exclusively in the center of educational issues based on a new paradigm that simply induces the construction of real numbers as the basis of the arithmatization of analysis, nor can it be based only on the idea of approximation; but it should also help in the mathematization of the prediction of the phenomena of change.

In order to accede to thought and variational language, it is necessary for the person learning to, among other things, handle the extensive universe of graphic forms that is also rich in meaning. A knowledge of the straight line and the parabola is not sufficient to develop the competence expected in the analysis courses.

[2] This approach was presented in the Research Seminar of Mathematics Education of the Higher Education Area of Cinvestav in Mexico and in a plenary of the Conference on Research in Mathematics Education in the USA in September 1997, see final quote.

To exemplify this, let's see the design of some didactic situations that we have been developing in recent years and from which we have been obtaining the results presented here. From the teaching system point of view, the pre-calculus course is traditionally a repertoire of procedures and algorithms that essentially come from algebra and analytic geometry, with greater or less emphasis on the study of function that is usually of the Dirichlet-Bourbaki definition. This teaching tends to overrate the analytical proceedings and the algorithmization, while overlooking the visual arguments, because they are not considered mathematical or because of the idea that is taken from mathematics and its teaching, without considering, for example, the cognitive structure of the students whom it is aimed at. The established didactic contract is added to this, which as part of the negotiation prevents the status of the professor from being undermined; if the professor does not satisfactorily solve the problems posed in the course, the algorithmic recourse will make it possible to correct what is established in the contract with decorum and lighten and eliminate the difficulties that are intrinsic to the mathematical content.

In the perspective of the social construction of knowledge, we can say that the nature of the concept of function is extremely complex. Its development has almost been the same as man's, that is, we find traces of the use of correspondences in ancient times and currently there is a debate about the paradigm of the function as an analytical object in mathematical circles. However, the concept of the function became important up to the point where it is conceived as a formula, that is, until integration is obtained between two fields of representation: algebra and geometry. The complexity of the concept of function is reflected in the diverse concepts and diverse representation that are faced by the students and the professors. An exhaustive list of epistemological obstacles to the concept of function can be found in Ana Sierpinska's article published in (Dubinsky and Harel, 1992).

From the pint of view of cognitive functions, the objects immersed in the conceptual field of analysis are particularly complex at this level since, as in the case we are concerned with, the usual presentation of the notion of function is presented with a procedure that is applied to certain criteria objects called numbers; this same concept, that of function, becomes the object when it is operated on under another process like differentiation or integration and so on until the most advanced notions. Thus, when starting a course of analysis, the student must conceive the function as an object, a deification and therefore subject to the operations made by another procedure with it. Put differently, what does operating a process mean? In our experience with professors in middle and upper educational service and with their students, we have seen that when visual elements are incorporated as part of the mathematic activity when confronting problems, they handle the function not just as an object but they can also transit between algebraic, geometric, numerical, iconic and verbal contexts with a degree of versatility, in other words, if there is a mastery of the geometric/visual context in the algorithms, intuition and argumentation, it will be possible to transit between various representations. Consequently, the didactic problem fundamentally lies in the cognitive difficulty in acquiring mastery in the geometric context for example; in the argumentation plane it is much easier to algebraically show the existence of a double root that geometrically, which is why, among other reasons, education seeks refuge in the algorithm with ease.

The central hypothesis from an in-depth socio-epistemological analysis as developed in (Farfán, 1997) consists of assuming that: prior to the calculus study, there is the acquisition of a graphic language that essentially facilitates the transfer of virtually alien conceptual fields, due to traditional teaching, and establishing an operational isomorphism between the basic algebra and the study of curves, better yet, between the algebraic language and the graphic language.

This hypothesis has been developed by taking the following two directives; firstly the possibility of operating graphic in analogy with numbers or variables is presented, which gives meaning toi basic operations, such as:

$-f(x)$ y $f(-x)$ Reflection regarding the x and y exes respectively
$f(x+a)$ y $f(x-a)$ with $a > 0$ Translation in the direction of the x axis
$f(x) + a$ y $f(x) - a$, with $a > 0$ Translation in the direction of the y axis
$af(x)$ Contraction or dilation regarding the y axis
$f^{-1}(x)$ Reflection regarding the $y = x$ straight line
$1 / f(x)$ Inverts zeros in asymptotes and vice versa, and the abscissa so that $|y| > 1$ will correspond with those where $|y| < 1$ and vice versa, leaving the points $y = 1$ y $y = -1$ on the straight lines intact.
$|f(x)|$ y $f(|x|)$ Respectively reflex the negative image of the positive symmetry regarding the x axis and reflex the substitution of the graph side with negative ordinates by reflection of the side of the graph with positive ordinates.

The second relevant aspect consists of the possibility of constructing a broad universe of functions as from three primitive reference functions: identity ($f(x) = x$), exponential $f(x) = a^x$ and the sinusoidal ($f(x) = \sin x$), all of which to construct the elemental function in the Cauchy sense. Respectively, they serve to construct by operating the graphs to algebraic, logarithmic and exponential functions and trigonometric functions graphically.

In this approach, it has been important to pose situations-problems that involved algebraic statements that favor the use of graphic language, for example, the task: solve the inequality $(|x - a| + |x - b|) / (|x + b| + |x + a|) \leq kx$, is broadly developed as a teaching strategy in (Albert and Farfán, 1997). For all of this, it is necessary to operate algebraically in order to obtain the graph of the functions involved so that they can finally be compared and to thus solve the equation systems that may arise. Likewise, seeking the end of the functions like $x / (ax^2+b)$ with positive a and b makes it possible to progress in the construction of the bridge between contexts since the task in a graphic context can be used as a guide to algebraic syntax, since it is backed by a meaning.

An example of the graphic domain is constituted by the following, we have a series of four inter-related tasks. We propose a collection of four identical graphs as shown below and then ask that each one and then we ask you to mark for each one respectively on the graph in the region in which it is satisfied that: 1) $f(x) > 0$, 2) $f'(x) > 0$, 3) $f''(x) > 0$, and finally 4) $f'''(x) > 0$. We expect that the answers to each one of them will indicate the variational strategies used and how to argument them before your classmates. The most complex question will obviously be the last one.

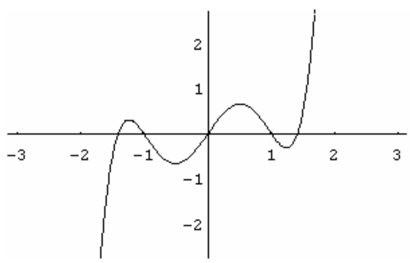

Figure. 1.- Graph of f

Question 1. $f(x) > 0$. in this case, the students usually remember that the I, II, III and IV quadrants determine the signs of the image: they are positive in the first two and negative in the rest. Thus they answer the question with relative ease.

Question 2. $f'(x) > 0$. In this case, the students frequently confuse the sign of the derivative with that of the function or they remember that the slope of the tangents of the curve determine the sign of the derivative so that you will have positive derivative for corresponding positive pending slopes. This change of register, the question posed in the symbolic context with visual support and the response is constructed in the visual context, is much more complex for the students and this is reflected in the fact that the number of correct numbers is lower and that few explanations are used.

Question 3. $f''(x) > 0$. In this case the situation is more complex. It requires progressive levels of abstraction. In this case the dominant resource is memory, because they often remember that the second derivative corresponds to the upward concave and so they are able to answer it, but there does not seem to be another argument to answer the following question.

Question 4. $f'''(x) > 0$. This question has posed a challenge for students and professors because they do understand what we are talking about, but they cannot construct a convincing answer. The difficulty is worsened if we raise the order of the derivative in the question since there is a lack of scholastic elements to construct an answer. From this point they are in a learning process because the previous points, even if mnemonic resources are used to answer the question, however the fourth question poses a question not foreseen by them, the success of the question lies in the fact that it can be deciphered by them, but the answer must be constructed. At this point, the students and teachers usually enter a learning situation. Only those who have dominated some thought and variational language processes will be able to efficiently tackle the problem. The simultaneous and coordinated handling of the successive derivates seems to be a condition without which the formation of the idea of the derivative and consequently the notion of prediction. Let's see one of the measures we have developed regarding this.

Prediction and the Taylor Series

The first idea we will use is prediction in the continuous flow phenomena in nature, with this we mean: with initial variables of a system in evolution it is possible to state its subsequent state. We will now see the clarity of this idea in two situations; the first deals with the study of the kinematics of a particle that moves in a straight line where a long-range prediction is required in continuous variation spheres and in the second, it deals with the examination pf the posing and solution of the wave equation which refers to the short-range prediction in spheres of continuous variation.

Let's suppose that we know the initial values (in the time we consider $t = 0$), for both the position $s(0) = s_0$, and the velocity $v(0) = v_0$, and acceleration $a(0) = a_0$ of a particle in movement, then any later t instant of time the position $s(t)$, velocity $v(t)$ and acceleration $a(t)$ will be given by the instrument to *Predict*: the *Taylor Series*,

$$s(t) = s(0) + s'(o)t + s''(0)t^2/2! + \ldots \qquad (1)$$

$$v(t) = v(0) + v'(0)t + v''(0)t^2/2! + \ldots \qquad (2)$$

$$a(t) = a(0) + a'(0)t + a''(0)t^2/2! + \ldots \qquad (3)$$

When dealing with an accelerated straight line movement, we have the value of t, $a(t) = a(0)$ for everything, and consequently $s^{(n)}(t) = 0$ if $n \geq 3$, therefore equations (1), (2) and (3) become the following equations:

$$s(t) = s(0) + s'(o)t + s''(0)t^2 /2! \qquad (4)$$

$$v(t) = v(0) + v'(0)t \qquad (5)$$

$$a(t) = a(0) \qquad (6)$$

Making the usual notation situations we have

$$s(t) = s_0 + v_0t + \tfrac{1}{2}at^2... \qquad (7)$$

$$v(t) = v_0 + at \qquad (8)$$

$$a(t) = a \qquad (9)$$

Let's generalize the idea of *Prediction*. The values of a parameter are known in a single spatial or temporal site, let's say in x_0, it is then necessary, with these data to state the later state of such parameter, that is, in the $x_0 + h$ value.

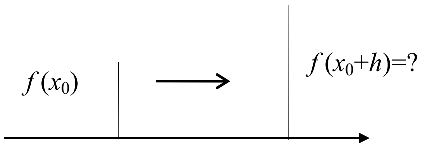

Fig. 2.- Graphical representation of *prediction idea*

In this respect, knowing the initial values: x_0, h, $f(x_0)$, $f'(x_0)$, $f''(x_0)$, etc., makes it possible to announce the posterior value of the parameter represented by $f(x)$, that is $f(x_0 + h)$:

$$f(x_0 + h) = f(x_0) + f'(x_0)h + f''(x_0)h^2 /2! + ... \qquad (10)$$

Although these ideas are intuitive, the student must accept that the Taylor Series has the aspect indicated and it is the instrument to predict. Although the second element is reached through outlined elements and some historic construction, the first however is to be found in the contemporary scholastic math discourse (Cauchy paradigm) a didactic discourse which presents it as a result of a theoretical nature, a deduction inherent to the results of mathematical analysis is required; axiom of completeness through some of the version of the mean value theorems. We now have various presentations of these results, plus the constructive nature, some of which are suggested and old didactics (Lacroix, 1797), or by historic genesis like (Taylor, 1715), (Lagrange, 1797) to mention a few, and other that are the fruit of recent research as in (Cantoral, 1995) where a collection of some of the results is obtained with the help pf the Taylor Series. On this occasion we will examine two differential equation problems. Let's start with the case of radioactive decay, which is usually presented in texts:

"The radiation disintegration law states that the velocity of disintegration is proportional to the initial amount of radium. Let's suppose that a certain point $t = 0$ *we have* R_0 *grams of radium and we want to know how much radium there will be at any instant after t".*

If $R(t)$ represents that amount of radium at any instant t and the velocity of disintegration is $-dR/dt$, then $kR = -dR/dt$ (with a constant k value). Using the *idea of prediction* that we presented above the problem consists of stating the posterior value in terms of the *initial data*: 0, $R(0)$, $R'(0)$, $R''(0)$, etc., hence the equation is once again expressed through the Taylor Series:

$$R(t) = R(0) + R'(0)t + R''(0)t^2 /2! + ... \qquad (11)$$

From the differential equation that regulates the behavior between the variables we have a basic relation: $R'(0) = -kR(0)$, $R''(0) = -kR'(0) = k^2R(0)$, etc. Therefore, the expression (11) acquires the following aspect:

$$R(t) = R(0) - kR(0)t + k^2 R(0)t^2 /2! - k^3 R(0)\, t^3 /3! + \dots \qquad (12)$$
$$= R(0)\{1-kt/1!+(kt)^2 /2!-(kt)^3 /3!+(kt)^4 /4!-(kt)^5 /5!+\dots\} = R(0)\{e^{-kt}\} = R_0\, e^{-kt}$$

We then examine the traditional problem of the vibrating string. Let's consider a taught, flexible and inelastic string, the ends are found on the points (0,0) and (l,0). The vibration in the direction of the y axes is the only movement to which the string is subjected. We assume that the tension force is significantly more important than the vibration and it is directed in a tangent to the string due to its flexibility. We also suppose, due to its inelasticity, that the oscillations are small enough to be considered as a differential element dx of the string taken when the string is on the horizontal axis it maintains its length even when its is vibrating.

In order to detect the regularity pattern of the oscillatory behavior, we shall study the variation in order to make a short-range prediction. We shall consider a differential element of the string dx and examine the causes of movement. At the ends of the string element, due to the degree of constant tension $T(x, t) = T(x + dx, t)$, the fundamental difference that expresses the imbalance of forces is as follows:

$$T(x, t)\,\mathrm{sen}\theta(x + dx, t) - T(x, t)\,\mathrm{sen}\theta(x, t) \qquad (13)$$

Using the Taylor Series as the short-range predictor instrument in short-range continuous variation spheres, we have,

$$T(x, t)\,[\mathrm{sen}\theta(x+dx, t) - \mathrm{sen}\theta(x, t)] = T(x, t)\left[\frac{\partial\,\mathrm{sen}\theta(x,t)}{\partial x}dx + \dots\right] \qquad (14)$$

We also have, due to the second law of movement of Newtonian mechanics, that such fundamental difference, when it expresses the net amount of the force, should satisfy the following relation:

$$T(x,t)\frac{\partial^2 f(x,t)}{\partial x^2} = \rho(x,t)\frac{\partial^2 f(x,t)}{\partial t^2} \qquad (15)$$

As we suppose the both $T(x, t)$ and $\rho(x, t)$ – the linear density of mass – remain when varying x and t, we have it that

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\rho}{T}\frac{\partial^2 f(x,t)}{\partial t^2} \qquad (16)$$

We have used the Series as an instrument for instantaneous prediction, short-range predication in continuous variation spheres, however, a long-range prediction is necessary to solve the differential equation (16), i.e., a long-range prediction with continuous variation. In such case, it is necessary to find the analytic expression that relates the variables we consider to be essential, that is, a formula for f in terms of x and t. Since the idea of prediction only states the future values of the variables and their initial variations. by using the two variables Taylor Series in an arrangement like in (Lacroix, 1797), we have the following expression for the $f(x, t)$ function only in terms of the initial values, at the point (0,0):

$$
\begin{aligned}
f(x,t) = f(0,0) &+ \frac{\partial f(0,0)}{\partial x}x + \frac{\partial^2 f(0,0)}{\partial x^2}\frac{x^2}{2!} + \frac{\partial^3 f(0,0)}{\partial x^3}\frac{x^3}{3!} + \ldots \\
&+ \frac{\partial f(0,0)}{\partial t}t + \frac{\partial^2 f(0,0)}{\partial x \partial t}\frac{xt}{1!} + \frac{\partial^3 f(0,0)}{\partial x^2 \partial t}\frac{x^2 t}{2!} + \ldots \\
&+ \frac{\partial^2 f(0,0)}{\partial t^2}\frac{t^2}{2!} + \frac{\partial^3 f(0,0)}{\partial x \partial t^2}\frac{xt^2}{2!} + \ldots \\
&+ \frac{\partial^3 f(0,0)}{\partial t^3}\frac{t^3}{3!} + \ldots
\end{aligned}
$$

In short, with the operational notation frequently used, we have:

$$f(x,t) = \sum_0^\infty \frac{1}{n!}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)^{(n)} f(0,0) \tag{17}$$

Note that from the differential equation that regulates the successive changes between the variables we can obtain the values of some successive derivatives provided that, naturally, we know the values before those obtained for the calculations described above. As can be seen, the following values are required:

$$f(0,0), \quad \frac{\partial f(0,0)}{\partial x}, \quad \frac{\partial f(0,0)}{\partial t}, \quad \frac{\partial^2 f(0,0)}{\partial x^2}, \quad y \quad \frac{\partial^2 f(0,0)}{\partial x \partial t};$$

given that the rest of the coefficients that appear in the Series are obtained from these and the differential equation (16)*,* we only have to obtain them in order to study the solution of the problem.

In schoolbooks, it is common to introduce initial and border values rather suddenly, our approach makes it possible to recognize what such restriction should be. Naturally, the position of the string at the origin, $f(0, t) = \alpha(t)$. The initial position just before letting it go and starting the variation, $f(x, 0) = \beta(x)$, and the initial velocity $\partial f(x, 0)/\partial t = \gamma(x)$.

With the above, we have means to recognize the nature of the solution of the differential equation: this is a series of potencies and consequently it has the form $\Sigma\Sigma\theta(x, t)$. Likewise, it can be seen that it is a product with two functions of a variable and thus the shape $f(x, t) = u(x)\, v(t)$ where each component function is expressed by means of a series of potencies. With this change of perspective in the discourse, it is possible to give naturalness to the proposal and the solution of the problem, from the determination of the need for initial and border conditions for the solution of the problem and the solution itself. Naturally, this approach can be made with other differential equations as long as the scheme can be reproduced.

Conclusions

These approaches have enabled us to obtain some research results with which we have established an incessant fruitful line of investigation and extended it to teaching. These findings favor the discussion and preparation of teaching proposals about what to teach and not just how to teach. A new line of research thus arises which takes as a study base the socio-epistemology of mathematical objects as from primary intuitions of the subject and whose purpose is to redesign the scholastic mathematical discourse. We have found that the teaching and learning of variational situations poses a large number of not so trivial questions. Each advanced concept that is to be taught is usually supported by more elemental concepts and they cannot be taught if not preceded by a sound understanding of the previous concepts. This step of the research which is fundamental to the design of didactic engineering touches on three research issues which are of interest for our group: What are the laws that regulate the variational thought teaching situations in our educational system and in the social environment? What is the nature of the regularities in the acts of understanding before situations that required variational thought? What are the forms of articulation of mathematical knowledge so that variational situations may be reached by most of the students in a school situation?

The problem of the research described has enabled us to develop Didactic Engineering projects aimed at the construction of institutional treatment. Fortunately, we are able to carry out a national research program in this direction.

In general terms, it could be said that located in variational thought and seeking the prediction of evolution of complex series of change, it is necessary for the basic functioning to have a centering on the way to vary that is even above the variable itself. This presupposes a centering on the process rather than the state, and consequently, the mechanisms of making the variables and their variations constant. The process of change is recorded in the variation of variables and it requires the recognition of the praediceir in the short-range prediction processes in discrete and continuous variation spheres. The link between the short and long range prediction processes is based on another functioning mechanism in the construction of knowledge: change has inheritance, by this we mean the subsequent state of the variation phenomenon depends completely on the circumstances that characterize the de facto state, the evolution of a system is completely determined by its primary variation. This construction process of the instrument to predict makes it possible to look at continuous variation in order to represent it in a mathematical context leads from the idea of prediction to the mathematical notion of the analytical. In this passage between notions, the analysis of the local element becomes an obliged resource, provided that the transformation takes place in the element. This construction process takes on life in the original productions of the 17th century scientists and it is reproduced in the productions of contemporary students and professors even if they have not been subject to an explicit teaching of such passages.

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